Algebraic entropy for birational maps in the plane

Anna Cima and Sundus Zafar

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Departament de Matemàtiques
Universitat Autònoma de Barcelona

Outline

- Algebraic entropy of a rational map in the plane.
- Algebraic entropy for birational maps.
- The zero entropy case.
- Dynamical degree of a nine parametric family of birational maps.
- Relationship between the topological and the algebraic entropy.

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1. Algebraic entropy of a rational map

Let f be a map of $\mathbb{C}^2$ with rational components. Rational maps are not defined in the whole plane. There are some points in which some denominator vanishes. And there some points where numerator and denominator of a component of f both vanishes: points where f is ill-defined.

We extend f to the projective space $P\mathbb{C}^2$, and we call $F[x_0 : x_1 : x_2]$ its extension. Then the three components of F are homogeneous polynomials of the same degree. We say that F is a minimal representative of f if the three components of F have no a common factor.

We define the degree of f as the degree of the polynomials of the components of F, if F is a minimal representative of f, and we denote by $\mathcal{I}(F)$, the indeterminacy set of F as the points

$$\mathcal{I}(F) = \{x \in P\mathbb{C}^2 : F_1(x) = F_2(x) = F_3(x) = 0\}.$$

Hence, $F : P\mathbb{C}^2 \setminus \mathcal{I}(F) \longrightarrow P\mathbb{C}^2$ and $\mathcal{I}(F)$ is a finite set of points.

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Being F a minimal representative of f of degree d consider $F \circ F$, which has as a components polynomials of degree d^2.
But $F \circ F$, is not necessarily a minimal representative of f^2 : it can occur that

$$F \circ F = [M \cdot P_1 : M \cdot P_2 : M \cdot P_3]$$

for some polynomials M, P_1, P_2, P_3.
In this case, if M is the maximal degree with this property, $deg(f^2) = deg(f)^2 - deg(M)$ and

$$F(\{M = 0\}) \subset \mathcal{I}(F).$$

We say that $V = 0$ is a degree lowering curve of F if

$$F^k(V) \subset \mathcal{I}(F)$$

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Hence we get that

$$deg(f^n) \leq deg(f)^n$$

with equality holding if and only if F has no degree lowering curves. Calling $a_n = \ln(deg(f^n))$, $a_{n+m} \leq a_n + a_m$ and hence the limit a_n/n always exists. From this: the number

$$h(f) = \lim_{n\to\infty} \frac{\ln(deg(f^n))}{n}$$

always exists and it is called the algebraic entropy of f and

$$\delta(f) = \lim_{n\to\infty} (deg(f^n))^{\frac{1}{n}}$$

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with equality holding if and only if F has no degree lowering curves. Calling $a_n = \ln(deg(f^n))$, $a_{n+m} \leq a_n + a_m$ and hence the limit a_n/n always exists. From this: the number

$$h(f) = \lim_{n\to\infty} \frac{\ln(deg(f^n))}{n}$$

always exists and it is called the algebraic entropy of f and

$$\delta(f) = \lim_{n\to\infty} (deg(f^n))^{\frac{1}{n}}$$

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1. Algebraic entropy of a rational map

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2. Algebraic entropy for birational maps

A rational map $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ is birational if there exists another rational map G and an algebraic curve V such that $F \circ G = G \circ F = id$ on $P\mathbb{C}^2 \setminus V$.

The excepcional locus of F denoted by $\mathcal{E}(F)$ is defined as follows

$$\mathcal{E}(F) = \{x \in P\mathbb{C}^2 : Det(DF)_x = 0\}.$$

Proposition.

Let F be a birational map and let F^{-1} its inverse. Then:

- Given any irreducible curve $V \in \mathcal{E}(F)$, $F(V)$ is a single point in $\mathcal{I}(F^{-1})$. For any $p \in \mathcal{I}(F^{-1})$, the preimage of p for F is an element of $\mathcal{E}(F)$.
- $\mathcal{I}(F) \subset \mathcal{E}(F)$, and every irreducible element of $\mathcal{E}(F)$ contains a point of $\mathcal{I}(F)$.
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2. Algebraic entropy for birational maps: Algebraically stable maps

Proposition.
A birational map F has a degree lowering curve V for F if and only if there exists an exceptional curve $S \in \mathcal{E}(F)$ such that $F^n(S) \in \mathcal{I}(F)$ for some $n \in \mathbb{N}$.

Corollary.
If for all $S \in \mathcal{E}(F)$ and for all $n \in \mathbb{N}$, $F^n(S) \notin \mathcal{I}(F)$ then $\deg(f^n) = (\deg(f))^n$ and $\delta(f) = \text{ degree}(f)$.

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2. Algebraic entropy for birational maps: The blow-up technique

Given a point $p \in \mathbb{C}^2$, the blow-up of $\mathbb{C}^2$ at p is a pair (X, π) such that if $p = (0,0)$ (if not we do a translation) it is defined by

$$X = \{((x,y),[u:v]) \in \mathbb{C}^2 \times \mathbb{P}^1 \ : \ xv = yu\}$$

and $\pi : X \to \mathbb{C}^2$ is the projection on the first component: $\pi\left((x,y),[u:v]\right) = (x,y)$. We notice that

$$\pi^{-1} p = \pi^{-1}(0,0) = \{((0,0),[u:v])\} := E_p \approx \mathbb{P}^1$$

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2. Algebraic entropy for birational maps: algebraically stable maps.

Let X be the manifold that we get after a finite number of blowing-up's.

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It can be seen that

$$\mathcal{P}\mathrm{ic}(X) = < \hat{L}, E_{p_1}, E_{p_2}, \ldots, E_{p_k} >,$$

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Consider:

$$f(x_1, x_2) = \left(x_1 - x_2, \frac{x_1 + x_2}{1 + x_2} \right),$$

$$F[x_0 : x_1 : x_2] = [x_0(x_0 + x_2) : (x_1 - x_2)(x_0 + x_2) : x_0(x_1 + x_2)].$$

Calculate the jacobian of (DF) :

$$j_F = 2x_0(x_0 + x_2)(2x_0 - x_1 + x_2).$$

So, F has three exceptional curves

$$S_0 := \{x_0 = 0\}, \, S_1 := \{x_0 + x_2 = 0\}, \, S_2 := \{2x_0 - x_1 + x_2 = 0\}$$

that colapse to

$$A_0 := [0 : 1 : 0], \, A_1 := [0 : 0 : 1], \, A_2 := [1 : 2 : 2]$$

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f is birational, being it inverse

$$f^{-1}(x,y)=\left(\frac{xy-x-y}{y-2},\frac{x-y}{y-2}\right)$$

$$F^{-1}[x_0:x_1:x_2]=[x_0(x_2-2x_0):x_1x_2+x_0x_1-x_0x_2:x_0(x_1-x_2)]$$

$$j_{F^{-1}}=(2x_0-x_1)(2x_0-x_2)x_0.$$

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$$j_{F^{-1}} = (2x_0 - x_1)(2x_0 - x_2)x_0.$$

So, F^{-1} has three exceptional curves

$$T_0 := \{2x_0 - x_1 = 0\}, \; T_1 := \{2x_0 - x_2 = 0\}, \; T_2 := \{x_0 = 0\}$$

that colapse to

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2. Algebraic entropy for birational maps: a simple example.

Now we have to follow the orbits under F_X of $A_1 = [0:0:1]$ and $A_2 = [1:2:2]$ and see if they meet $O_0 = [1:1:-1]$ or $O_1 = [0:1:1]$. Since $A_1 = [0:0:1] \in S_0$, its image belongs to E_0.We calculate F_X^2 restricted to S_0 and we find:

$$F_X^2[0 : x_1 : x_2] = [0 : x_2 : x_1 + 2x_2].$$

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2. Algebraic entropy for birational maps: a simple example.

The equalities $F_X^*(E_0) = \hat{L} - E_0$ and $F_X^*(E_0) = 2\hat{L} - E_0$ determines the matrix of F_X^* :

$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

which has as characteristic polynomial $P(\lambda) = \lambda^2 - \lambda - 1$. The sequence of the degrees of f^n satisfies

$$d_{n+2} = d_{n+1} + d_n$$

and the dynamical degree of f is the golden mean $\frac{1+\sqrt{5}}{2}$.

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3. The case of zero entropy

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational map.

Definition

f preserves a rational fibration (resp. elliptic fibration) if it exists a rational map $V : \mathbb{C}^2 \to \mathbb{C}$ such that for almost all $c \in \mathbb{C}$, the curve $V = c$ has genus equal zero (resp. genus one)and f sends $V = c$ a $V = c'$.

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Theorem. Let f be a birational map and let d_n be the degree of f^n. Assume that f has zero entropy. Then exactly one of the following holds:

- The sequence of degrees d_n grows quadratically and f preserves an elliptic fibration. In this case there is a regularization of F which is an automorphism.
- The sequence of degrees d_n grows linearly and f preserves a rational fibration. In this case there does not exist any regularization of F being an automorphism.
- The sequence of degrees d_n is bounded and f preserves two generically transverse fibrations. In this case there is a regularization of F which is an automorphism.

Furthermore in the first and second, the invariant fibrations are unique.

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4. Dynamical degree of a nine parametric family of birational maps.

We consider the family of fractional maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form:

$$f(x,y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right), \tag{1}$$

where the parameters are complex numbers. The indeterminacy set of F is $\mathcal{I}(F) = \{O_0, O_1, O_2\}$, with

$$O_0 = [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01}], O_1 = [0 : \alpha_2 : -\alpha_1], O_2 = [0 : \gamma_2 : -\gamma_1],$$

(here $(\beta\gamma)_{ij} := \beta_i\gamma_j - \beta_j\gamma_i$ for $i,j \in \{0,1,2\}$). The indeterminacy set of F^{-1} is $\mathcal{I}(F^{-1}) = \{A_0, A_1, A_2\}$, with

$$\begin{array}{l} A_0 = [0 : 1 : 0], \\ A_1 = [0 : 0 : 1], \\ A_2 = [(\beta\gamma)_{12}\,(\alpha\gamma)_{12} : \\ (\alpha_0\,(\beta\gamma)_{12} - \alpha_1\,(\beta\gamma)_{02} + \alpha_2\,(\beta\gamma)_{01})\,(\alpha\gamma)_{12} : (\alpha\beta)_{12}\,(\beta\gamma)_{12}]. \end{array}$$

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We divide the family in 5 subfamilies.

Theorem 1. Suppose that $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are all non zero and that $(\beta\gamma)_{12} \neq 0$ and $(\alpha\gamma)_{12} \neq 0$. Then either,

(i) If it exists $p \in \mathbb{N}$ such that $F^p(A_2) = O_0$, then the dynamical degree of F is given by the largest root of the polynomial

$$x^{p+2} - 2x^{p+1} + x - 1.$$

(ii) If no such p exists then dynamical degree of F is 2.

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$$x^{p+2} - 2x^{p+1} + x - 1.$$

(ii) If no such p exists then dynamical degree of F is 2.

4. Dynamical degree of a nine parametric family of birational maps.

We divide the family in 5 subfamilies.

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Theorem 2. Assume that $\gamma_1 = 0$, $\alpha_2 \neq 0$, $(\beta\gamma)_{12} \neq 0$ and $(\alpha\gamma)_{12} \neq 0$. Let $\tilde{F}$ be the induced map after blowing up the point A_0. Then the following hold:

- If $\tilde{F}^p(A_2) = O_0$ for some $p \in \mathbb{N}$ and $\tilde{F}^{2k}(A_1) \neq O_1$ for all $k \in \mathbb{N}$ then the characteristic polynomial associated with F is given by
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4. A subfamily with zero entropy

$$\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1.$$

- for $p > \frac{2(1+k)}{k}$ the sequence of degrees d_n grows exponentially for all $p,\ k \in \mathbb{N}$;
- for $(p, k) \in \{(3, 2), (4, 1)\}$ the sequence of degrees d_n grows quadratically;
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$$(p,k)\in\{(0,\,k),\,(1,\,k),\,(2,\,k),\,(3,\,1)\}$$

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5. Relationship between the topological and the algebraic entropy.

When it is defined, the topological entropy of a birational map coincides with the algebraic entropy.
The proof is based on the following result of Y. Yomdin in 1987:

For $f : N \longrightarrow N$ a continuous mapping defined in the compact m-dimensional $\mathcal{C}^\infty$-smooth manifold N, the topological entropy of f is greater or equal to $S(f)$, where $S(f) = max_l S_l(f)$ and $S_l(f)$ is the logarithm of the spectral radius of $f^* : H_l(N, \mathbb{R}) \longrightarrow H_l(N, \mathbb{R})$.

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