Semiconjugacy to a map of constant slope II

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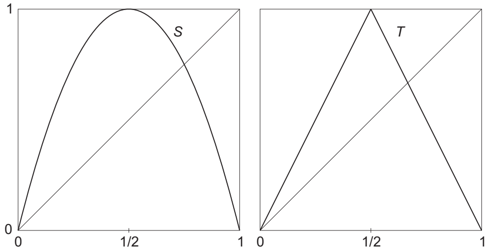
Introduction

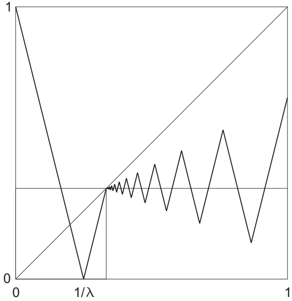
A continuous map $T : [0,1] \to [0,1]$ is said to be piecewise monotone if there are $k \in \mathbb{N}$ and points $0 = c_0 < c_1 < \cdots < c_{k-1} < c_k = 1$ such that T is monotone on each $[c_i, c_{i+1}]$, $i = 0, \dots, k-1$. A piecewise monotone map T has a *constant slope* s if $|T'(x)| = s$ for all $x \neq c_i$.
The following facts are well known for piecewise monotone interval maps:

(i) If T has a constant slope s then the topological entropy $h_{top}(T) = \max(0, \log s)$, see [Misiurewicz & Szlenk 80].

(ii) If $h_{top}(T) > 0$ then T is semiconjugate via a continuous non-decreasing map h to some map $\tilde{T}$ of constant slope $e^{h_{top}(T)}$, see [Parrry 66, Milnor & Thurston 88].



For maps with a countably infinite number of branches, the above property (i) can fail, see for example [Misiurewicz & Raith 05].



One of the few facts that remains true in the countably piecewise monotone setting is:

Proposition

If T is Lipschitz with Lipschitz constant s, then $h_{top}(T) \leq \max\{0, \log s\}$.

See for example [Katok & Hasselblatt 95]

The question we want to address is when a (continuous) countably piecewise monotone map T is conjugate to a map with constant slope $\pm\lambda$ (regardless of whether $\log \lambda = h_{top}(T)$ or not).

We want to explore what can be said if only the Markov structure (transition matrix) of a countably piecewise monotone map is known in terms of the Vere-Jones classification [Vere-Jones 67], refined by [Ruette 03].

$\mathcal{CPM}$: the class of countably piecewise monotone Markov maps

A *Markov partition* $\mathcal{P} = \{P_i\}_{i \in \mathcal{A}}$ for a map $T\colon [0,1] \to [0,1]$ consists of intervals P_i, with the following properties:

- The index set $\mathcal{A}$ is finite or countably infinite, but also if $\#\mathcal{A} = \infty$, it need not necessarily have the ordinal type as $\mathbb{N}$ or $\mathbb{Z}$.
- The intervals $P_i \subset [0,1]$ have pairwise disjoint interiors and $[0,1] \setminus \cup_{i \in \mathcal{A}} P_i$ is countable.
- If $T(P_i^\circ) \cap P_j^\circ \neq \emptyset$, then $T(P_i) \supset P_j$.
- In general there are infinitely many Markov partitions $(P_{\alpha,i})_{i \geq 1}$ such that connect-the-dots map of $Q_\alpha = [0,1] \setminus \bigcup_{i \in \mathcal{A}} P_{\alpha,i}^\circ$ equals to T. Let $\mathcal{P} = (P_i)$ be the minimal Markov partition for which $[0,1] \setminus \bigcup_{i \in \mathcal{A}} P_i^\circ = \bigcap_\alpha Q_\alpha$.

Any continuous map $T : [0,1] \to [0,1]$ is said to belong to the class $\mathcal{CPM}$ if

- it admits a Markov partition as above, $T|P_i^\circ$ is monotone (perhaps constant) for each P_i,
- and $h_{top}(T) < \infty$.

For a given $T \in \mathcal{CPM}$, we associate to its Markov partition $\mathcal{P}$ the transition matrix $M = M(T) = (m_{i,j})_{i,j\in\mathcal{A}}$, defined by

$$m_{ij} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j, \\ 0 & \text{otherwise} \end{cases}$$

The matrix $M = M(T)$ represents a bounded linear operator $\mathcal{M}$ on $\ell^1 = \ell^1(\mathcal{A})$, provided the supremum of the column sums is finite. In fact then,

$$\|\mathcal{M}\| = \sup_j \sum_i m_{ij}.$$

If $\|\mathcal{M}\| < \infty$, then we speak of the *operator* type (acting by right-multiplication on the Banach space ℓ^1), if $\|\mathcal{M}\| = \infty$, then we speak of the *non-operator* type.
It can be easily seen that for many $T \in \mathcal{CPM}$ the matrix $M = M(T)$ does not represent a bounded operator on ℓ^1.

For any matrix $M = M(T)$, $T \in \mathcal{CPM}$, we define the powers $M^n = (m_{ij}^{(n)})$ of M:

$$M^0 = E,\ M^n = \sum_{j \in \mathcal{A}} m_{ij} m_{jk}^{(n-1)},\ n \in \mathbb{N}.$$

$\mathcal{CPM}_\lambda$ . . . the class of all maps from $\mathcal{CPM}$ of a constant slope λ, i.e., $f \in \mathcal{CPM}_\lambda$ if $|f'(x)| = \lambda$ for all $x \in [0,1]$, possibly except at the points of $[0,1] \setminus \bigcup P_i^\circ$

The following theorem has been proved in [J.B., Studia Mathematica **208**(2012), 213–228.]

Theorem

Let $T \in \mathcal{CPM}$ *with* $M = M(T) = (m_{ij})_{i,j\in\mathcal{A}}$. *Then* T *is semiconjugate via a continuous non-decreasing map* ψ *to some map* $S \in \mathcal{CPM}_\lambda$, $\lambda > 1$, *if and only if there is a nonzero vector* $v = (v_i)_{i\in\mathcal{A}}$ *from* $\mathcal{K}^+ \subset \ell^1(\mathcal{A})$ *such that*

$$\forall\ i \in \mathcal{A}\colon\ \sum_{j\in\mathcal{A}} m_{ij} v_j = \lambda\ v_i. \tag{1}$$

If the equation (1) has a solution from $\ell^\infty(\mathcal{A}) \setminus \ell^1(\mathcal{A})$ *then* $\psi\colon\ [0,1] \to D$ *and* $S \in \mathcal{CPM}_\lambda(D)$ *for some* $D \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-\}$.

Remark

- *The theorem remains true also for natural classes of non-continuous maps.*
- *We do not assume that the entropy of T is positive.*
- *All four possibilities $\{$operator, non-operator$\} \times \{v \in \ell^1,\ v \notin \ell^1\}$ can exist.*

(i) *If T is leo (for every open set U there is an n such that $T^n(U) = [0,1]$), then any solution of the equation (1) is from ℓ^1.*

(ii) *If T is topologically mixing (for every open sets U, V there is an n such that $T^m(U) \cap V \neq \emptyset$ for all $m \geq n$), then any solution v of the equation (1) satisfies*

$$\forall\, \varepsilon > 0 \colon \quad \sum_{P_j \subset (\varepsilon, 1-\varepsilon)} v_j < \infty.$$

To show (i): If $Mv = \lambda v$ and P_i is such that the i-th row of M^n is strictly positive, then $\lambda^n v_i = \sum_j m_{ij}^{(n)} v_j \geq \sum_j v_j$, so $v \in \ell^1$.

Two easy examples, non-continuous with zero entropy

Example

Let

$$M(f) = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdot & \cdot \\ 0 & 1 & 1 & 1 & \cdot & \cdot \\ 0 & 0 & 1 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot \end{pmatrix}$$

The index set $\mathcal{A}$ has the ordinal type of $\mathbb{N}$ and the equation (1) has the solution $v(\lambda) = ((\frac{\lambda-1}{\lambda})^i)_{i\in\mathcal{A}} \in \ell^1$ for each $\lambda > 1$.

Example

Let

$$M(f) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot \end{pmatrix}$$

Again, the index set $\mathcal{A}$ has the ordinal type of $\mathbb{N}$ and the equation (1) has the solution $v(\lambda) = (\lambda^i)_{i\in\mathcal{A}} \in \ell^1$ for each $\lambda \in (0,1)$ and is positive for $\lambda > 1$.

But we are mostly interested in continuous transitive interval maps, so from now we always assume that a map $T \in \mathcal{CPM}$ is either topologically mixing or even leo! We denote the set of all such maps in $\mathcal{CPM}$ by

$$\mathcal{CPM}_+.$$

Then all possible semiconjugacies from our Theorem (equation (1)) are in fact conjugacies!

The Vere-Jones Classification

A matrix $M = (m_{ij})_{i,j \in \mathcal{A}}$ will be called

- *irreducible*, if for each pair of indices i, j there exists a positive integer n such that $m_{ij}^{(n)} > 0$,
- and *aperiodic*, if for each pair of indices i, j there exists a positive integer n_0 such that $m_{ij}^{(n)} > 0$ for all $n \geq n_0$.

To a given $0-1$ irreducible aperiodic matrix $M = (m_{ij})_{i,j \in \mathcal{A}}$ corresponds a strongly connected oriented Markov graph $G = G(M) = (\mathcal{A}, \mathcal{E} \subset \mathcal{A} \times \mathcal{A})$, where $(i, j) \in \mathcal{E}$ if and only if $m_{ij} = 1$, and vice versa.

Remark

If $T \in \mathcal{CPM}_+$ is topologically mixing then its transition matrix $M = M(T)$ is irreducible and aperiodic.

Proposition (Vere-Jones 67)

(i) Let $M = (m_{ij})_{i,j\in\mathcal{A}}$ *be an infinite nonnegative irreducible aperiodic matrix. There exists a common value* λ_M *such that for each* i, j

$$\lim_{n\to\infty} (m_{ij}^{(n)})^{\frac{1}{n}} = \lambda_M.$$

(ii) For any value $r > 0$ *and all* i, j

- *the series* $\sum_n m_{ij}^{(n)} r^n$ *are either all convergent or all divergent;*
- *as* $n \to \infty$*, either all or none of the sequences* $\{m_{ij}^{(n)} r^n\}_n$ *tend to zero.*

By definition, the value $R = \frac{1}{\lambda_M}$ is a common radius of convergence of the power series $M_{ij}(z) = \sum_{n\geq 0} m_{ij}^{(n)} z^n$. Thus, $M_{ij}(r) < \infty$ for $0 < r < R$ and $M_{ij}(r) = \infty$ if $r > R$.

Let $T \in \mathcal{CPM}_+$, consider its transition matrix $M(T)$ and the associated oriented Markov graph $G = G(T)$.
In G:

- $m_{ij}^{(n)}$ = the number of paths of length n connecting i to j.

Let

- $f_{ij}^{(n)}$ = the number of paths of length n connecting i to j, without appearance of j before the final j (first entrance), in particular $f_{ij}^{(1)} = m_{ij}$.

We can consider radia of convergence (depending on i, j) of power series $F_{ij}(z) = \sum_{n \geq 1} f_{ij}^{(n)} z^n$.

The behaviour of the series $M_{ij}(z)$, $F_{ij}(z)$ for $z = R$ is used for classification of irreducible aperiodic matrices. This is summarized in the following table which applies independently of the sites $i, j \in \mathcal{A}$ if M is irreducible [Vere-Jones 67], [Ruette 03]:

	transient	n-recurrent	weakly p-recurrent	strongly p-recurrent
$\sum_{n\geq 0} f_{ii}^{(n)} R^n$	< 1	$= 1$	$= 1$	$= 1$
$\sum_{n\geq 0} n f_{ii}^{(n)} R^n$	$\leq \infty$	∞	$< \infty$	$< \infty$
$\sum_{n\geq 0} m_{ij}^{(n)} R^n$	$< \infty$	$= \infty$	$= \infty$	$= \infty$
$\lim_{n\to\infty} m_{ij}^{(n)} R^n$	$= 0$	$= 0$	$= \lambda_{ij} > 0$	$= \lambda_{ij} > 0$
for all i	$R = L$	$R = L$	$R = L$	$R < L_{ii}$

We say that a map $T \in \mathcal{CPM}_+$ is R-transient, R-null or R-positive if it is the case for its matrix M. **The above table describes various possibilites of covering properties of maps from $\mathcal{CPM}_+$.**

Gurevich entropy

Proposition

Let $T \in \mathcal{CPM}_+$ and $M = M(T)$. The following is true.

(i) $\lambda_M = e^{h_{top}(T)}$.

(ii) *If M represents an operator $\mathcal{M}$ on ℓ^1 with the spectral radius $r(\mathcal{M})$ then $e^{h_{top}(T)} = r(\mathcal{M})$.*

Following [Gurevich 69] we can define the entropy of an irreducible aperiodic infinite matrix M, and also its associated oriented Markov graph G as

$$h(M) = h(G) = -\log R = \log \lambda_M > 0.$$

By the previous Proposition for $T \in \mathcal{CPM}_+$,
$h(M(T)) = h(G(T)) = h_{top}(T)$.

n-recurrent and p-recurrent (leo) maps

For the following theorem, see [Vere-Jones 67]:

Theorem

For every recurrent irreducible aperiodic matrix, the equation (1) is uniquely λ_M-solvable with v strictly positive.

Applying the above result we obtain

Theorem

For every recurrent leo map $T \in \mathcal{CPM}_+$, the equation (1) is uniquely λ_M-solvable with $v \in \ell^1$.

Salama's criteria

There are geometrical criteria - see [Salama 88] and also [Ruette 03] - for each case of the Vere-Jones classification to apply in terms of whether the underlying strongly connected Markov graph can be enlarged/reduced (in the class of strongly connected Markov graphs) without changing the entropy.

Theorem

Assume that M is an irreducible aperiodic $0-1$ matrix with $h(M) < \infty$, denote G its associated strongly connected Markov graph. Then

- *If there are strongly connected Markov graphs G_0 and G_1 such that $G_0 \subsetneq G \subsetneq G_1$ and $h(G_0) = h(G) = h(G_1)$, then M is transient.*
- *M is strongly positive recurrent if and only if $h(G_0) < h(G)$ for any $G_0 \subsetneq G$.*
- *M is null-recurrent if and only if there exists $G_0 \subsetneq G$ with $h(G_0) = h(G)$, but $h(G) < h(G_1)$ for every $G_1 \supsetneq G$.*

transient operator type

Theorem

Let $T \in \mathcal{CPM}$ be transient with $M = M(T)$ representing an operator on ℓ^1. Then the equation (1) does not have any λ_M-solution in ℓ^1.

Sketch of the proof. The limit $\lim_{\lambda \searrow \lambda_M} \frac{1}{\lambda}(M_{ij}(\frac{1}{\lambda}))$ exists hence the matrix $\frac{1}{\lambda_M}(M_{ij}(\frac{1}{\lambda_M}))$ equals to the non-continuous resolvent operator $(\lambda_M - M)^{-1}$ hence the spectral radius λ_M od M is in the continuous part of the spectrum $\sigma(M)$.

example

transient non-operator type

In this case we know several examples of maps from $\mathcal{CPM}_+$ for that the equation (1) is λ-solvable in ℓ^1 for each $\lambda \geq \lambda_M$.

Example

$V = \{v_i\}_{i \geq -1}$, $X = \{x_i\}_{i \geq 1}$ V, X converge to $1/2$ and $0 = v_{-1} = x_0 = v_0 < x_1 < v_1 < x_2 < v_2 < x_3 < v_3 < \cdots$

$$f = f(V, X) : [0, 1] \to [0, 1]$$

(a) $f(v_{2i-1}) = 1 - v_{2i-1}$, $i \geq 1$, $f(v_{2i}) = v_{2i}$, $i \geq 0$,

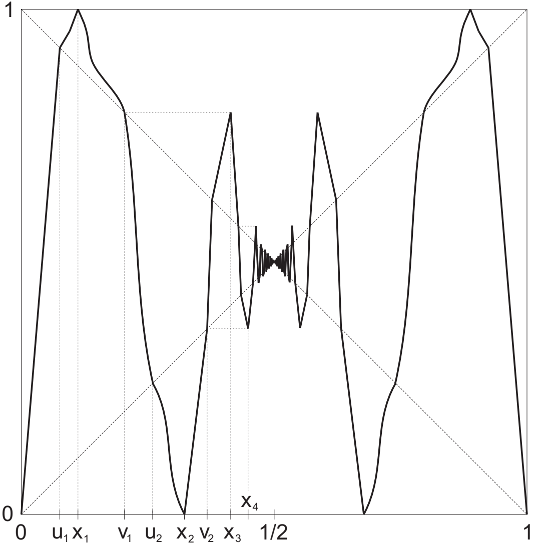
(b) $f(x_{2i-1}) = 1 - v_{2i-3}$, $i \geq 1$, $f(x_{2i}) = v_{2i-2}$, $i \geq 1$,

(c) $f_{u,v} = \left| \frac{f(u)-f(v)}{u-v} \right| > 1$ for each interval $[u, v] \subset [x_i, x_{i+1}]$,

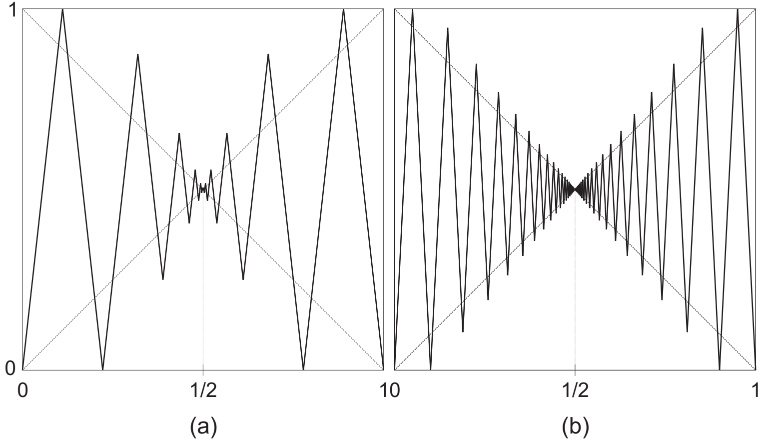
(d) $f(1/2) = 1/2$ and $f(t) = f(1-t)$ for each $t \in [1/2, 1]$.

(the property (c) can be satisfied since for our V, X by (a),(b), $f_{x_i, x_{i+1}} > 2$ for each $i \geq 0$)
We denote by $\mathcal{F}(V, X)$ the set of all continuous interval maps fulfilling (a)-(d) for a fixed pair V, X and

$$\mathcal{F} := \bigcup_{V,X} \mathcal{F}(V, X).$$



The map $T \in \mathcal{CPM}_+$. The equation (1) is λ-solvable in ℓ^1 for each $\lambda \geq 9 = e^{h_{top}(T)}$.



(a) The map T_9; (b) the map T_{20}.

n-recurrent operator and non-operator type

Let us repeat that by [Vere-Jones 67]

Theorem

For every recurrent map $T \in \mathcal{CPM}_+$, the equation (1) is uniquely λ_M-solvable with v positive.

Using the above theorem, Salama's criteria and the properties from our Table, one can construct examples of *n*-recurrent (operator, non-operator type) maps from $\mathcal{CPM}_+$ for which the equation (1) is solvable in $\ell^\infty \setminus \ell^1$, so not in ℓ^1.

random walk on $\mathbb{Z}$

There exists a (non-operator type) n-recurrent map $T \in \mathcal{CPM}_+$ for which the equation (1) is λ_M-solvable in ℓ^1.

weakly p-recurrent operator and non-operator type

Again the same theorem can be applied:

Theorem

For every recurrent map $T \in \mathcal{CPM}_+$, the equation (1) is uniquely λ_M-solvable with v positive.

There exists a (non-operator type) weakly p-recurrent map $T \in \mathcal{CPM}_+$ for which the equation (1) is λ_M-solvable in ℓ^1.

strongly p-recurrent operator type

Theorem

For every strongly p-recurrent (operator type) map $T \in \mathcal{CPM}_+$, the equation (1) is uniquely λ_M-solvable in ℓ_1.

As a concrete example of such maps we introduce the following.

Definition*

For an integer $m > 1$, we say that a map $T \in \mathcal{CPM}_+$ is m-ruled, if there are $\mathcal{P}$-basic intervals $I_1, \dots, I_m$ such that

- T has an m-horseshoe created by the intervals $I_1, \dots, I_m$
- $\forall j \in \mathcal{A}\ \forall y \in P_j^\circ$: $\operatorname{card}[T^{-1}(y) \cap ([0,1] \setminus \bigcup_{i=1}^m I_i)] < m$.

Theorem

If $T \in \mathcal{CPM}_+$ is m-ruled and of operator type then it is strongly positive recurrent.

example

strongly p-recurrent non-operator type

example

Thank you for your attention!