RESONANCES IN AN AREA PRESERVING CONTINUOUS PIECEWISE LINEAR MAP

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INTRODUCTION

The study of piecewise dynamical system is mainly motivated by some applications, for instance:

- Mechanics: dry friction, impacts
- Power electronics: switching converters

The physical model of these systems is primarily time-continuous and frequently, a map is derived from it, for instance a Poincaré map

However, the map studied here is mainly inspired in theoretical considerations

THE PIECEWISE (PLANAR) MAP

In a piecewise map, the phase plane is split in several regions

The map is assumed smooth in every one of those regions

Discontinuity at their boundaries can have several degrees

1) We assume a planar map $\mathbf{F}(\mathbf{x})$, $\mathbf{x} = (x, y)$ and

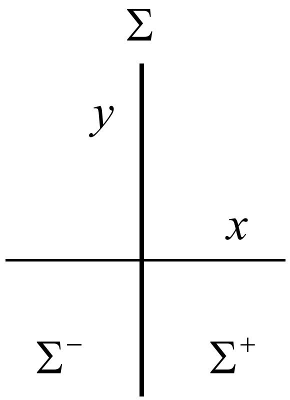
a partition of the phase plane in two regions

$\Sigma^{-} = (x, y) \in \mathbb{R}^2 : x < 0$ (left region)

$\Sigma^{+} = (x, y) \in \mathbb{R}^2 : x > 0$ (right region)

separated by the boundary line

$\Sigma = (x, y) \in \mathbb{R}^2 : x = 0$



phase plane ($\mathbb{R}^2$)

CONTINUOUS PIECEWISE LINEAR (CPWL) MAP

2) We also assume the map $\mathbf{F}$ is linear in both regions Σ^- and Σ^+

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) = \begin{cases} \mathbf{F}^-(\mathbf{x}_n) = A^-\mathbf{x}_n + B^-, & \mathbf{x}_n \in \Sigma^- \\ \mathbf{F}^+(\mathbf{x}_n) = A^+\mathbf{x}_n + B^+, & \mathbf{x}_n \in \Sigma^+ \end{cases}$$

where $\mathbf{x}_n = (x_n, y_n)$ is the state, $A^\pm = (a_{ij}^\pm)$ are 2×2 constant matrices and $B^\pm = (b_1^\pm, b_2^\pm)$ are constant vectors in $\mathbb{R}^2$

3) In addition, we assume continuity at the boundary Σ, that is

$$\begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \quad \text{so that}$$

$$A^- = \begin{pmatrix} a_{11}^- & a_{12} \\ a_{21}^- & a_{22} \end{pmatrix}, \quad A^+ = \begin{pmatrix} a_{11}^+ & a_{12} \\ a_{21}^+ & a_{22} \end{pmatrix}, \quad B^- = B^+ = B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Continuity at Σ implies a reduction of parameters from 12 to 8

A NORMAL FORM FOR THE CPWL MAP

Proposition: Assuming $a_{12} \neq 0$ (dynamics is decoupled only if $a_{12} \neq 0$), the CPWL map is conjugate to the normal form

$$\mathbf{x}_{n+1} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \mathbf{x}_n + \begin{pmatrix} 0 \\ b \end{pmatrix}, & \mathbf{x}_n \in \Sigma^- \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \mathbf{x}_n + \begin{pmatrix} 0 \\ b \end{pmatrix}, & \mathbf{x}_n \in \Sigma^+ \cup \Sigma \end{cases}$$

where T^-, T^+ and D^-, D^+ stand for trace and determinant of A^-, A^+

and $b \in \{0, 1\}$, thus dealing to two families of CPWL maps

Proof: The normal form is straightforward after the diffeomorphism

$$\tilde{\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ a_{22} & -a_{12} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ b_1 \end{pmatrix}$$

and scaling $\tilde{x} \rightarrow \big(b_1(a_{22} - 1) - a_{12}b_2\big)\tilde{x}$, provided that $b_1(a_{22} - 1) - a_{12}b_2 \neq 0$

The number of parameters has been reduced from 8 to 4 and one modal

HOMOGENEOUS AREA PRESERVING CPWL MAP

In addition to the continuity at the boundary, we assume:

4) The map is homogeneous (case $b = 0$) so that $b_1(a_{22} - 1) - a_{12}b_2 = 0$
This case happens when the fixed points for the left and the right pieces of the original map ($\mathbf{F}^-$ and $\mathbf{F}^+$) coincide and it is at the boundary Σ

5) The map is area preserving so that $D^- = D^+ = 1$

Then, the map $\mathbf{F}$ takes the normalized form $\mathbf{G}$

$$\boxed{\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n) = \begin{cases} \mathbf{G}^-(\mathbf{x}_n) = A(T^-)\mathbf{x}_n = \begin{pmatrix} T^- & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}_n, & \mathbf{x}_n \in \Sigma^- \\ \mathbf{G}^+(\mathbf{x}_n) = A(T^+)\mathbf{x}_n = \begin{pmatrix} T^+ & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}_n, & \mathbf{x}_n \in \Sigma^+ \cup \Sigma \end{cases}}$$

Therefore, the parameter space is reduced to the trace plane (T^+, T^-)

HOMOGENEOUS AREA PRESERVING LINEAR MAP

Consider the homogeneous area preserving linear map $A(T)$

$$\mathbf{x}_{n+1} = A(T)\mathbf{x}_n = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}_n$$

Then, the eigenvalues are

$$\lambda_1 = \lambda, \; \lambda_2 = \frac{1}{\lambda}, \; \text{where} \; \lambda = \frac{T - \sqrt{T^2 - 4}}{2}$$

and the origin is a fixed point of type

- *Saddle* if $|T| > 2$, and so the eigenvalues are real and one of them has a modulus greater than one
- *Centre* if $|T| < 2$, and so both eigenvalues are complex with $|\lambda| = 1$
- *Critical* $|T| = 2$, and so we have a double eigenvalue $\lambda_1 = \lambda_2 = T/2$

ASSOCIATED SLOPE MAP TO THE MAP $A(T)$

To study the angular dynamics, we define the following sets:

(a) The straigh line $\Pi_\lambda = \{(x, \lambda x)\}$

(b) The ray on the left $\Pi_\lambda^- = \Pi_\lambda \cap \Sigma^-$ and the ray on the right $\Pi_\lambda^+ = \Pi_\lambda \cap \Sigma^+$

(c) The vertical rays $\Pi_{-\infty} = \{(0, y),\ y < 0\}$ and $\Pi_\infty = \{(0, y),\ y > 0\}$

Considering the transformation of lines by means of the map $A(T)$, we get

$$A(T)\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} -y \\ 0 \end{pmatrix},$$ i.e., points in $\Pi_{-\infty}$ (Π_∞) map to Π_0^+ (Π_0^-)

$$A(T)\begin{pmatrix} x \\ Tx \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix},$$ i.e., points in Π_T^- (Π_T^+) map to $\Pi_{-\infty}$ (Π_∞)

$$A(T)\begin{pmatrix} x \\ \lambda x \end{pmatrix} = (T - \lambda)x\begin{pmatrix} 1 \\ 1/(T-\lambda) \end{pmatrix},$$ i.e., points in Π_λ map to points in $\Pi_{h(\lambda)}$

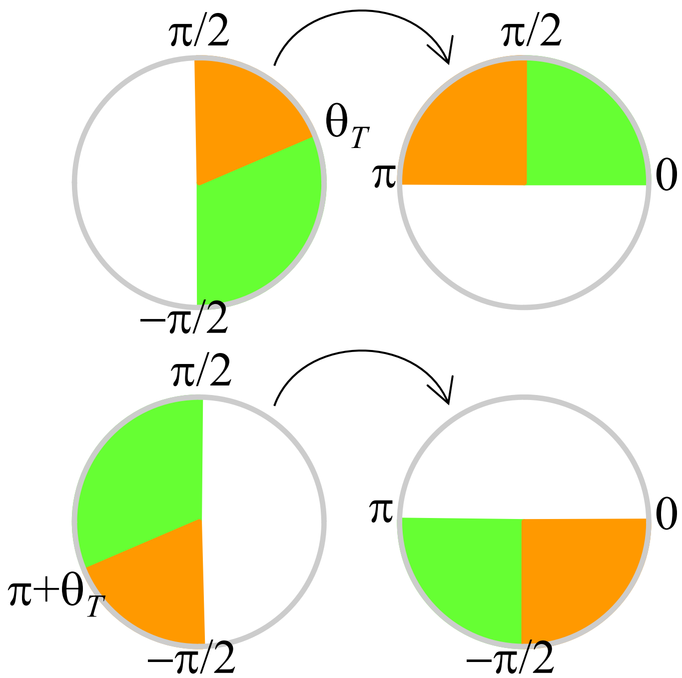
where the slope map h is defined as $\boxed{h(\lambda) = \dfrac{1}{T - \lambda}}$

Important property: The slope map h is increasing

ASSOCIATED CIRCLE MAP TO THE MAP $A(T)$

If we put $\lambda = \tan(\theta)$, $\theta \in S^1$ and define $\theta_T = \tan^{-1}(T)$, we can define the map $g : S^1 \to S^1$

$$g(\theta) = \begin{cases} 0 & \text{if } \theta = -\pi/2 \\ \tan^{-1}\left(h\left(\tan(\theta)\right)\right) & \text{if } -\pi/2 < \theta < \theta_T \\ \pi/2 & \text{if } \lambda = \theta_T \\ \pi + \tan^{-1}\left(h\left(\tan(\theta)\right)\right) & \text{if } \theta > \theta_T \\ \pi & \text{if } \theta = \pi/2 \\ \pi + \tan^{-1}\left(h\left(\tan(\theta)\right)\right) & \text{if } \pi/2 < \theta < \pi + \theta_T \\ -\pi/2 & \text{if } \theta = \pi + \theta_T \\ \tan^{-1}\left(h\left(\tan(\theta)\right)\right) & \text{if } \theta > \pi + \theta_T \end{cases}$$



As $h(\bar{\lambda}) = \bar{\lambda}$ implies $g(\bar{\theta}) = \bar{\theta}$ and $g(\pi + \bar{\theta}) = \pi + \bar{\theta}$, where $\bar{\lambda} = \tan(\bar{\theta})$, a fixed point of the slope map h determines two fixed points of the associate circle map g

Note: if $\tan(\theta) < T$, rays in Σ^+ (Σ^-) map to Σ^+ (Σ^-); if $\tan(\theta) > T$, rays in Σ^+ (Σ^-) map to Σ^- (Σ^+)

THE LINEAR MAP $A(T)$. THE SADDLE CASE $|T| > 2$

The fixed points of the slope map h are $\lambda_1 = \dfrac{T - \sqrt{T^2 - 4}}{2}$, $\lambda_2 = \dfrac{T + \sqrt{T^2 - 4}}{2}$

They exist only if $|T| \geq 2$ and they satisfy the condition $\lambda_1 \lambda_2 = 1$

The fixed points of map g are $\theta_1 = k\pi + \tan^{-1}(\lambda_1)$, $\theta_2 = k\pi + \tan^{-1}(\lambda_2)$ for $k = 0, 1$ and their stability is determined by the stability of the fixed points of the map h

Since $\dfrac{\mathrm{d}h}{\mathrm{d}\lambda}(\lambda_1) = \dfrac{1}{\lambda_2^2}$ and $\dfrac{\mathrm{d}h}{\mathrm{d}\lambda}(\lambda_2) = \dfrac{1}{\lambda_1^2}$,

we have $\begin{cases} \text{if } T > 2: & 0 < \lambda_1 < +1 < \lambda_2 < T, \text{ so } \theta_1 \text{ is attractor and } \theta_2 \text{ is repellor} \\ \text{if } T < -2: & T < \lambda_1 < -1 < \lambda_2 < 0, \text{ so } \theta_1 \text{ is repellor and } \theta_2 \text{ is attractor} \end{cases}$

Note that λ_1 and λ_2 are also the eigenvalues of the linear map $\mathbf{x}_{n+1} = A(T)\,\mathbf{x}_n$, corresponding λ_1 to the line Π_{λ_2} and λ_2 to the line Π_{λ_1}

If $T > 2$ $(T < -2)$, the orbits in the invariant line Π_{λ_2} (Π_{λ_1}) converge to the origin and the remaining orbits are unbounded

THE LINEAR MAP $A(T)$. THE CENTRE CASE $|T| < 2$

If $|T| < 2$, the eigenvalues are complex and every orbit is bounded

If we put $T = 2\cos(\beta)$, where $0 < \beta < \pi$, the dynamics of the circle map g is then classified according to the rotation number $\rho = \beta/(2\pi)$

- If ρ is rational that is $\rho = p/q$, the dynamics is periodic with period q
- If ρ is irrational the dynamics is quasiperiodic

Proof : By means of the change $\mathbf{y}_n = \begin{pmatrix} 1/\sin\beta & -\cos\beta/\sin\beta \\ 0 & 1 \end{pmatrix} \mathbf{x}_n$,

we get $\mathbf{y}_{n+1} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \mathbf{y}_n$, so $A(T)$ is equivalent to a rotation

THE LINEAR MAP. RECAP AND CASES $T = \pm 2$

- *Centre (focus)* if trace $|T| < 2$. The orbits are bounded. Also, the orbits are periodic if the rotation number is rational or quasiperiodic otherwise

- *Saddle* if trace $|T| > 2$. The orbits are unbounded except those contained in the line corresponding to the repellor in the circle map. We can distinguish:

Case $T > 2$ ($\lambda_{1,2} > 0$). The orbits escape along one direction of a straight line

Case $T < -2$ ($\lambda_{1,2} < 0$). The orbits escape alternating direction of a straight line

- For $T = \pm 2$, the eigenvalues are $\lambda_1 = \lambda_2 = T/2 = \pm 1$

Then the line $\Pi_{T/2}$ is invariant. If $T = 2$ the line Π_1 is plenty of equilibrium points and if $T = -2$, every point in line Π_{-1} is two periodic

The pair of fixed points of the circle map g existing for $|T| > 2$ collapses at $|T| = 2$ and disappear for $|T| < 2$. This is a saddle-node bifurcation for g

SLOPE AND CIRCLE MAPS ASSOCIATED TO THE CPWL MAP **G**

Let first recall the definition of the area preserving CPWL map **G**

$$\mathbf{x}_{n+1}=\mathbf{G}\left(\mathbf{x}_n\right)=\begin{cases}\mathbf{G}^-\left(\mathbf{x}_n\right)=A\left(T^-\right)\mathbf{x}_n=\begin{pmatrix}T^- & -1\\ 1 & 0\end{pmatrix}\mathbf{x}_n, & \mathbf{x}_n\in\Sigma^-\\[2ex] \mathbf{G}^+\left(\mathbf{x}_n\right)=A\left(T^+\right)\mathbf{x}_n=\begin{pmatrix}T^+ & -1\\ 1 & 0\end{pmatrix}\mathbf{x}_n, & \mathbf{x}_n\in\Sigma^+\cup\Sigma\end{cases}$$

We define the slope maps h^-, h^+ associated to $A(T^-)$, $A(T^+)$ respectively

$$h^-\left(\lambda\right)=\frac{1}{T^--\lambda},\quad h^+\left(\lambda\right)=\frac{1}{T^+-\lambda}$$

We also denote the circle maps associated to h^-, h^+ as g^-, g^+ respectively

The circle map g: $S^1\rightarrow S^1$ associated to the CPWL map **G** is then defined

$$\boxed{g\left(\theta\right)=\begin{cases}\mathsf{g}^-\left(\theta\right) & \text{if}\quad \pi/2<\theta<3\pi/2\\ \mathsf{g}^+\left(\theta\right) & \text{if}\quad -\pi/2\le\theta\le\pi/2\end{cases}}$$

THE CPWL MAP **G**. SYMMETRY PROPERTIES

1) Map **G** is invariant under the change of variables and parameters

$$(x, y, T^-, T^+) \to (-x, -y, T^+, T^-)$$

Therefore, it is enough to analyze dynamics in the half plane $T^+ \leq T^-$

2) Map **G** is reversible under involution $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that is $R\mathbf{G} = \mathbf{G}^{-1}R$

Consequently, if $\mathbf{G}(x_0, y_0) = (x_1, y_1)$, then $\mathbf{G}^{-1}(y_0, x_0) = (y_1, x_1)$

Also, if $\Gamma = (\theta_1 \ldots \theta_q)$ is a q-periodic orbit of the circle map g, then

- $(\theta_1' \ldots \theta_q')$ is a q-periodic orbit of g^{-1}, where $\theta_i' = \pi/2 - \theta_i$
- $\Gamma' = (\theta_q' \ldots \theta_1')$ is a q-periodic orbit of g with opposite stability that Γ

To every value θ_i in Γ (analogously to θ_i' in Γ') we can associate the ray

$$\Pi^-_{\tan(\theta_i)} \text{ if } \pi/2 < \theta_i < 3\pi/2 \quad \text{or} \quad \Pi^+_{\tan(\theta_i)} \text{ if } -\pi/2 \leq \theta_i \leq \pi/2$$

If Γ and Γ' are the stable and the unstable orbits in g respectively,
then the dynamics in **G** is unbounded tending to the rays associated to Γ,
except for the orbits in the rays associated to Γ' that converge to the origin

THE CPWL MAP **G**. THE CASE $T^{-} \geq 2$

If $T^{-} \geq 2$, then $\Pi^{-}_{\lambda_1^-}$ and $\Pi^{-}_{\lambda_2^-}$ are invariant rays for **G**, and θ_1^- and θ_2^- are fixed points for the circle map g, where

$$\lambda_1^- = \frac{T^- - \sqrt{\left(T^-\right)^2 - 4}}{2}, \quad \lambda_2^- = \frac{T^- + \sqrt{\left(T^-\right)^2 - 4}}{2}, \quad \theta_k^- = \pi + \tan^{-1}\left(\lambda_k^-\right), \quad k = 1,2$$

For map g, θ_1^- is an attractor and θ_2^- is a repellor

Assume $T^+ < 2$, thus an orbit starting in Σ^+ will eventually go into Σ^-

- If $T^- > 2,\ T^+ < 2,$ then the orbits of **G** are unbounded tending to $\Pi^{-}_{\lambda_1^-}$, except those in $\Pi^{-}_{\lambda_2^-}$ that converge to the origin

- If $T^- = 2,\ T^+ < 2,$ then $\lambda_1^- = \lambda_2^-, \theta_1^- = \theta_2^-$ and we have a fold bifurcation for the circle map g. The ray $\Pi^{-}_{\lambda_1^-}$ is plenty of equilibria and any other orbit is unbounded

- If $T^- > 2,\ T^+ > 2,$ the orbits of **G** will tend either to the ray $\Pi^{-}_{\lambda_1^-}$ or to the ray $\Pi^{+}_{\lambda_1^+}$, which can be analogously defined, depending on the initial point

Due to the symmetry, the case $T^{+} \geq 2$ can be reduced to the case $T^{-} \geq 2$

THE CPWL MAP **G**. THE CASE $T^{-}T^{+} \geq 4$, $T^{\pm} < 0$

We analyze the properties of the composition of the linear maps $A(T^{-}) \circ A(T^{+})$

The corresponding slope map $\left(h^{-} \circ h^{+}\right)(\lambda) = \dfrac{T^{+} - \lambda}{T^{-}T^{+} - T^{-}\lambda - 1}$

has the fixed points $\mu_1^{+} = \dfrac{\Delta^{-}}{T^{-}}$ and $\mu_2^{+} = \dfrac{\Delta^{+}}{T^{-}}$, with μ_1^{+} stable and μ_2^{+} unstable

and being $\Delta^{\pm} = \dfrac{1}{2}\left(T^{-}T^{+} \pm \sqrt{\left(T^{-}T^{+}\right)^{2} - 4T^{-}T^{+}}\right)$

The fixed points of the composition $\left(h^{+} \circ h^{-}\right)$ are $\mu_1^{-} = \dfrac{\Delta^{-}}{T^{+}}$ and $\mu_2^{-} = \dfrac{\Delta^{+}}{T^{+}}$

The corresponding angles are $\theta_{1,2}^{+} = \tan^{-1}\left(\mu_{1,2}^{+}\right)$, $\theta_{1,2}^{-} = \pi + \tan^{-1}\left(\mu_{1,2}^{-}\right)$, so that $\left(\theta_1^{+}, \theta_1^{-}\right)$ and $\left(\theta_2^{+}, \theta_2^{-}\right)$ are 2-periodic orbits (repellor and attractor) of map g

Since $\mu_{1,2}^{\pm} > T^{\pm}$, **G** transforms ray $\Pi_{\mu_{1,2}^{+}}^{+}$ into ray $\Pi_{\mu_{1,2}^{-}}^{-}$ and vice verse. The orbits of **G** are unbounded and agglomerate around $\Pi_{\mu_1^{\pm}}^{\pm}$, except in $\Pi_{\mu_2^{\pm}}^{\pm}$ that converge to origin

DYNAMICS CLASSIFICATION FOR MAPS g AND $\mathbf{G}$

The remaining parameter region is $\{(T^-, T^+), T^-T^+ < 4, T^- < 2, T^+ < 2\}$
Here, there is an intricate pattern of sub regions with different dynamics

Let us classify the dynamics found in this parameter region by considering the number of periodic orbits for the circle map g, having a period $q \geq 3$

1) Every orbit is q-periodic. Here, $\mathbf{G}^q = I$, and we say $\mathbf{G}$ is a finite order map

2) There is only a pair of q-periodic orbits. Then, due to the reversibility, they are symmetric each other and they have opposite stability. The origin is a saddle point for map $\mathbf{G}$ and *we call this dynamics as ray type* because unbounded orbits of $\mathbf{G}$ agglomerate around the rays associated to the points of the stable orbit of g

3) There is only one q-periodic orbit, which due to reversibility is R-symmetric
Then, if q is odd a point of the orbit must be either $\pi/4$ or $5\pi/4$, and if q is even two points of the orbit are either $\pi/4$ and $5\pi/4$ or none of them.
If $T^- = T^+ = T$, this case is only possible if $T = 2$ with $q = 1$ or if $T = -2$ with $q = 2$

4) There are no periodic orbits. Then the rotation number is irrational

FIBONACCI POLYNOMIALS FOR $A^n(T)$

Proposition : If $A(T)=\begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix}$, then $A^n(T)=\begin{pmatrix} \psi_n(T) & -\psi_{n-1}(T) \\ \psi_{n-1}(T) & -\psi_{n-2}(T) \end{pmatrix}$,

where

$$\psi_n(T)=T\psi_{n-1}(T)-\psi_{n-2}(T) \quad \text{and} \quad \psi_n(T)=\sum_{k=0}^{floor\left(n/2\right)}(-1)^k\begin{pmatrix} n-k \\ k \end{pmatrix}T^{n-2k}$$

Proof: From Cayley Hamilton theorem, we have $A^n = TA^{n-1} - A^{n-2}$ for $n \geq 2$, then we apply induction with $A^0 = I$ and $A^1 = A(T)$

The generalized Fibonacci polynomial sequence is defined recursively as

$$u_0(x,y)=0, \qquad u_1(x,y)=1$$

$$u_n(x,y)=xu_{n-1}(x,y)+yu_{n-2}(x,y)$$

By induction can be proved that

$$u_n(x,y)=\sum_{k=0}^{floor\left(n/2\right)}(y)^k\begin{pmatrix} n-k-1 \\ k \end{pmatrix}T^{n-2k-1}, \quad \text{thus} \quad \psi_n(T)=u_{n+1}(T,-1)$$

LINES WITH CONTINUOUS ROTATION NUMBER

Proposition : Along the line in the parameter plane

$$CR^- = \left\{ \left(T^-, T^+\right):\ -2 < T^- < 2,\ T^+ = 2\cos\left(\frac{\pi}{p^+}\right) \text{ for } p^+ \in \mathrm{Z} \geq 2 \right\}$$

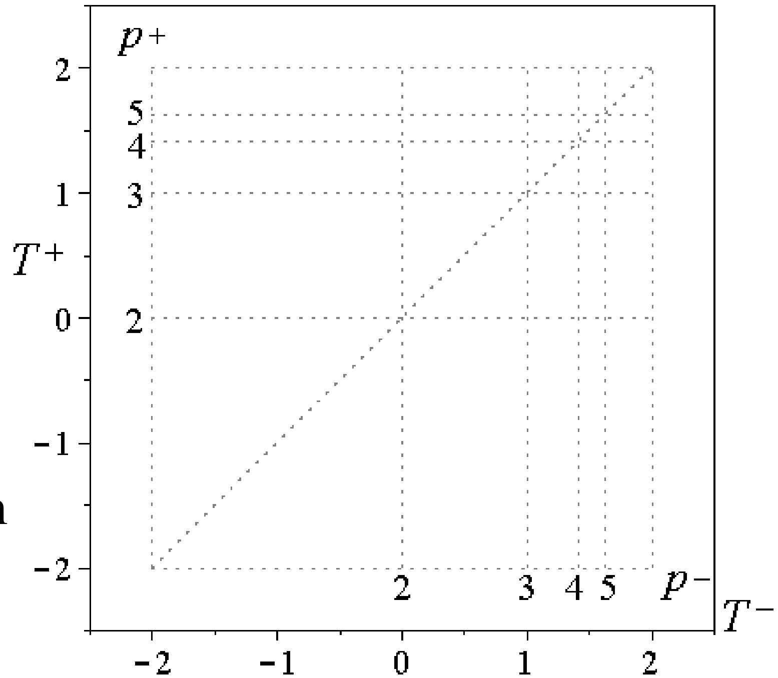
the rotation number is $\rho = \dfrac{2\rho^-}{1+2p^+\rho^-}$, where $\rho^- = \dfrac{\cos^{-1}\left(T^-/2\right)}{2\pi}$

Proof : Since $A^{p^+}\left(T^+\right) = -I$, for every value $\mathbf{x}_0$ with a corresponding angle $-\pi/2 \leq \theta_0 \leq 0$, $\mathbf{G}^{p^+}\left(x_0\right) = A^{p^+}\left(T^+\right)x_0 = -x_0$. Then, there exists an integer k such that $\mathbf{G}^k\left(-x_0\right) = x_1$, which corresponding angle satisfies $-\pi/2 \leq \theta_1 \leq 0$ In fact, this is equivalent to the linear map $A\left(T^-\right)$ by excluding the p^+ right iterations and then, the expression of the rotation number is straightforward

POINTS WITH FINITE ORDER DYNAMICS

We define the set of points in the parameter plane $FO = \left(T^-, T^+\right)$:

$$\begin{cases} T^- = T_{p^-} = 2\cos\left(\dfrac{\pi}{p^-}\right), \ p^- \in \mathrm{Z} \geq 2 \\ T^+ = T_{p^+} = 2\cos\left(\dfrac{\pi}{p^+}\right), \ p^+ \in \mathrm{Z} \geq 2 \end{cases}$$



In fact, points in *FO* are the intersection of CR^- lines with analogous CR^+ lines

For points in *FO*, $A^{p^-}\left(T^-\right) = A^{p^+}\left(T^+\right) = -I$,

so **G** is of finite order with period $p^- + p^+$

Next, we study the dynamics of **G** with parameter values around *FO* points

RAY DYNAMICS WITH PATTERN (p^-, p^+)

We say that a periodic orbit of g has the pattern $\left(p^-, p^+\right)$ if it is constitued by p^- consecutive points with $\pi/2 < \theta < 3\pi/2$, followed by p^+ points $-\pi/2 < \theta < \pi/2$

Thus the dynamics of $\mathbf{G}$ is represented by $\Lambda_{p^-,p^+}\left(T^-, T^+\right) = A^{p^-}\left(T^-\right)A^{p^+}\left(T^+\right) =$

$$\begin{pmatrix} \psi_{p^+}\left(T^+\right)\psi_{p^-}\left(T^-\right) - \psi_{p^+-1}\left(T^+\right)\psi_{p^--1}\left(T^-\right) & -\psi_{p^+-1}\left(T^+\right)\psi_{p^-}\left(T^-\right) + \psi_{p^+-2}\left(T^+\right)\psi_{p^--1}\left(T^-\right) \\ \psi_{p^+}\left(T^+\right)\psi_{p^--1}\left(T^-\right) - \psi_{p^+-1}\left(T^+\right)\psi_{p^--2}\left(T^-\right) & -\psi_{p^+-1}\left(T^+\right)\psi_{p^--1}\left(T^-\right) + \psi_{p^+-2}\left(T^+\right)\psi_{p^--2}\left(T^-\right) \end{pmatrix}$$

If $T^- = T_{p^-}$ and $T^+ = T_{p^+}$, then $\Lambda_{p^-,p^+}\left(T_{p^-}, T_{p^+}\right) = I$, that is $\mathbf{G}$ is of finite order and so *every orbit is periodic with pattern* $\left(p^-, p^+\right)$ *and two of the orbits are* R*-symmetric*

We have $\det\left(\Lambda\right) = 1$ and let us define $\Theta_{p^-,p^+}\left(T^-, T^+\right) = \mathrm{trace}\left(\Lambda_{p^-,p^+}\left(T^-, T^+\right)\right) - 2$

We will see that the pattern $\left(p^-, p^+\right)$ is admissible for $\mathbf{G}$ in some regions containing $\left(T_{p^-}, T_{p^+}\right)$, so we have a *ray type dynamics* in the quoted regions excluding $\left(T_{p^-}, T_{p^+}\right)$, being the boundary determined by the condition $\Theta_{p^-,p^+}\left(T^-, T^+\right) = 0$

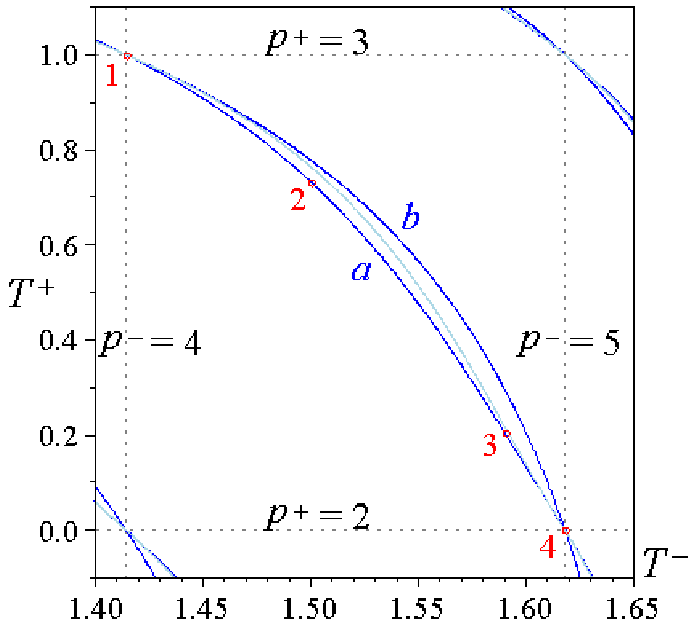
EXAMPLE: 7-PERIODIC ORBIT FOR MAP g

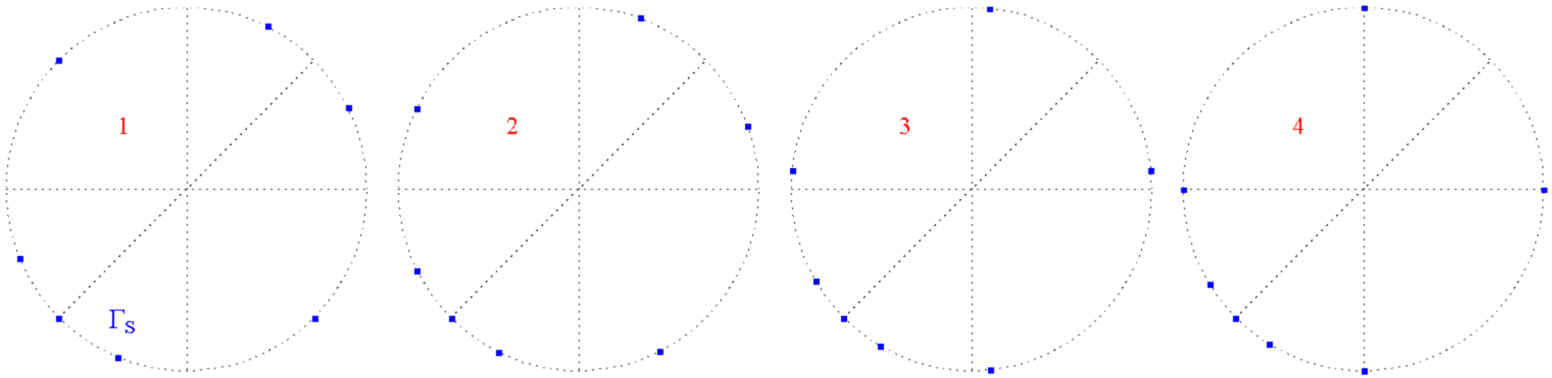
Let us consider the *FO* point 1, $p^- = 4, p^+ = 3$

Let Γ_S be one of the two *R*-symmetric periodic orbits of the map g, which is drawn on the left

The line a is defined by $\Theta_{4,3}\left(T^-, T^+\right) = 0$

Along this line, the *R*-symmetric periodic orbit of map g evolves as shown below





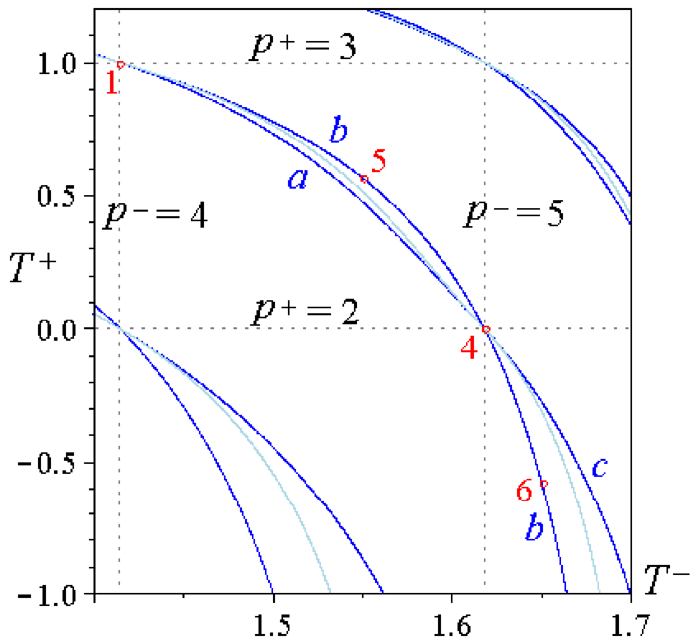
POCKETS WITH 7-PERIODIC ORBITS FOR g

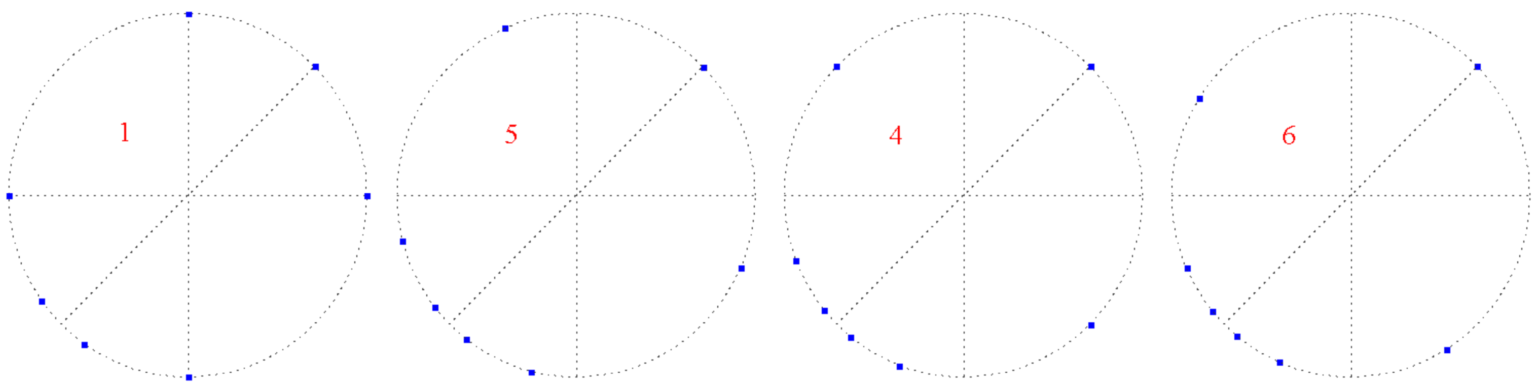
Let us consider the *FO* point 4, $p^- = 5$, $p^+ = 2$

We saw the line *a* ends at the point 4, where the *R*-symmetric orbit has the transition from the pattern (4, 3) to the ones (5, 2) and (6, 1)

The pattern (6, 1) continues along the line *c*

The *R*-symmetric orbit with the pattern (5, 2) along the line *b* is plotted below. Note this is the other *R*-symmetric orbit at point 4

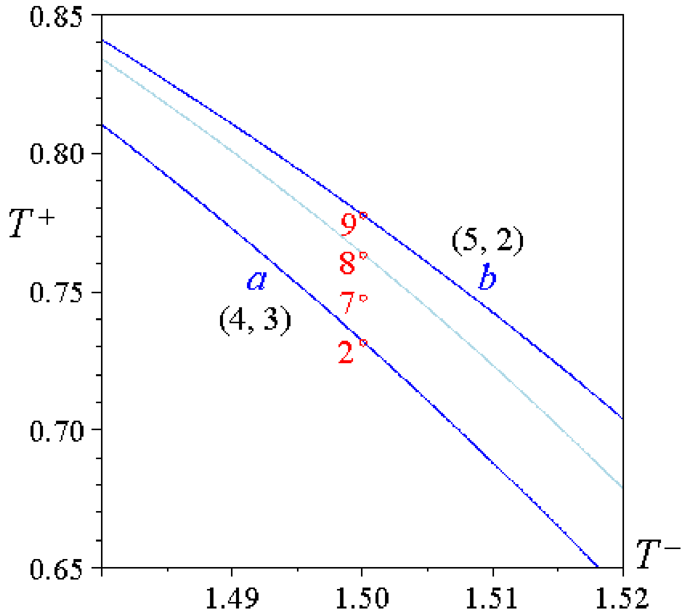


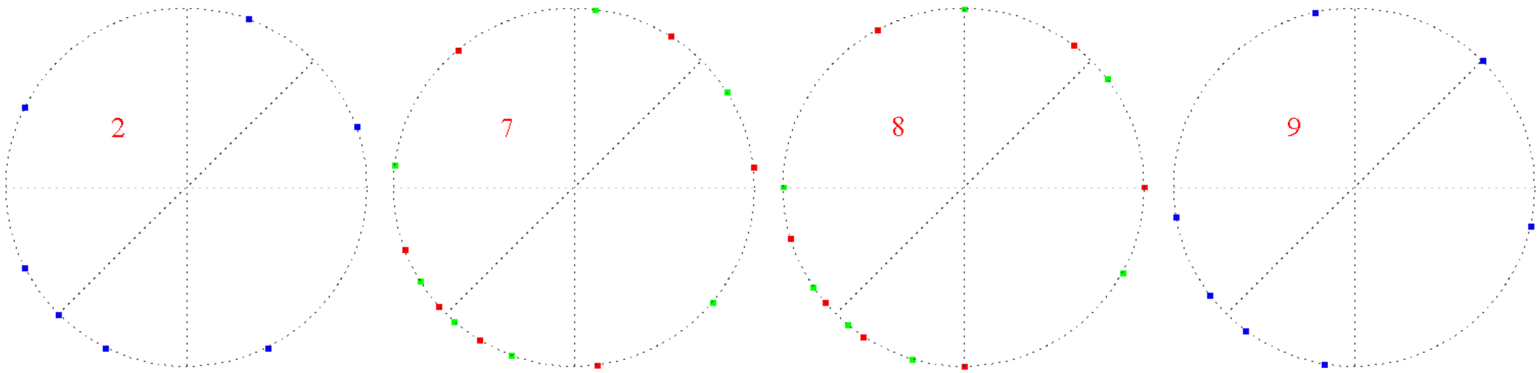


INSIDE THE POCKET

Inside the parameter region with ray dynamics, we have two periodic orbits for map *g* with opposite stability arising from each one of the *R*-symmetric periodic orbits at the boundaries (see the diagrams plotted below)

There exists a line splitting the ray dynamics region in two parts. Each one has the pattern of the corresponding boundary. At this transition line (see point 8), the pattern changes and can be computed from $\left(\Lambda_{4,3}\left(T^{-},T^{+}\right)\right)_{12}=0$





PROPERTIES OF Θ AT *FO* POINTS

Proposition : If $T_n = 2\cos\left(\frac{\pi}{n}\right)$, then $\psi_n\left(T_n\right) = -1, \psi_{n-1}\left(T_n\right) = 0, \psi_{n-2}\left(T_n\right) = 1$

Proof :

Recall that $A(T) = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix}$, $A^n\left(T\right) = \begin{pmatrix} \psi_n\left(T\right) & -\psi_{n-1}\left(T\right) \\ \psi_{n-1}\left(T\right) & -\psi_{n-2}\left(T\right) \end{pmatrix}$ and $A^n\left(T_n\right) = -I$

Proposition: The derivative of the function $\psi_n\left(T\right)$ is

$$\frac{\mathrm{d}\psi_n\left(T\right)}{\mathrm{d}T} = \frac{nT\psi_n\left(T\right) - 2\left(n+1\right)\psi_{n-1}\left(T\right)}{T^2 - 4}$$

Corollary : From $\psi_{n-k}\left(T_n\right)$ and the derivative $\mathrm{d}\psi_n\left(T\right)/\mathrm{d}T$,

by induction we can evaluate $\mathrm{d}^j\psi_{n-k}\left(T\right)/\mathrm{d}^jT$ at T_n for $k = 0...2$

These results allow us to compute the gradient and the hessian of the function $\Theta_{p^- -k, p^+ +k}\left(T^-, T^+\right)$, for $k = 0, \pm 1$, at the point $\left(T_{p^-}, T_{p^+}\right)$

PROPERTIES OF GRADIENT AND HESSIAN OF Θ

- $\Theta_{p^-,p^+}\left(T_{p^-},T_{p^+}\right) = \mathrm{grad}\left(\Theta_{p^-,p^+}\left(T_{p^-},T_{p^+}\right)\right) = 0$
- $\Theta_{p^- -k,p^+ +k}\left(T_{p^-},T_{p^+}\right) = 0$ and $\mathrm{grad}\left(\Theta_{p^- -k,p^+ +k}\left(T_{p^-},T_{p^+}\right)\right) \neq 0$ for $k = \pm 1$
- $\det\left(\mathrm{hessian}\left(\Theta_{p^-,p^+}\left(T_{p^-},T_{p^+}\right)\right)\right) = -\left(\dfrac{p^- p^+\left(\cos\left(\dfrac{\pi}{p^-}\right)-\cos\left(\dfrac{\pi}{p^+}\right)\right)}{2\sin\left(\dfrac{\pi}{p^-}\right)^2\sin\left(\dfrac{\pi}{p^+}\right)^2}\right)^2 \leq 0$

If $p^- \neq p^+$, then Θ_{p^-,p^+} is saddle shapped around $\left(T_{p^-},T_{p^+}\right)$

Thus there exist at least two lines through $\left(T_{p^-},T_{p^+}\right)$ satisfying $\Theta_{p^-,p^+}\left(T^-,T^+\right) = 0$

The pattern $\left(p^-,p^+\right)$ is admissible for the dynamics along one of these lines, which is the boundary of the ray dynamics region in the parameter plane with that pattern.

This line exits until the two neighbor points $\left(T_{p^- \mp 1},T_{p^+ \pm 1}\right)$

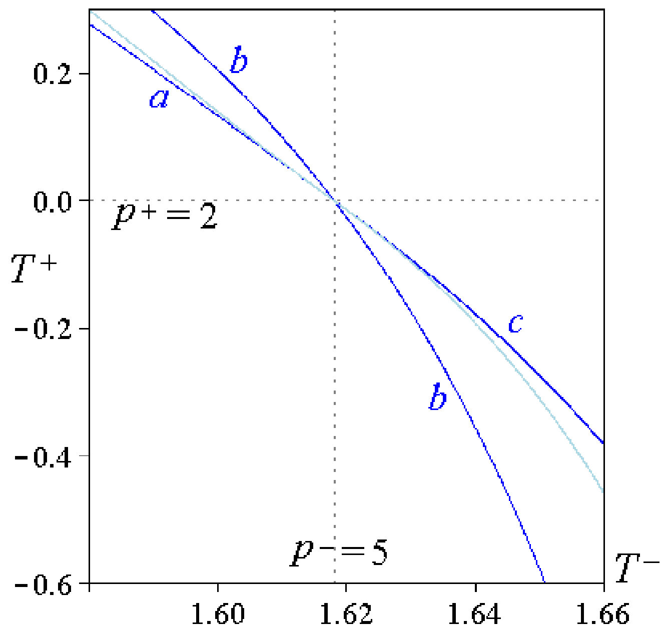
SUBREGION TONGUES AT *FO* POINTS

Proposition : The boundaries of the subregions with ray dynamics and patterns $\left(p^- \mp 1, p^+ \pm 1\right)$ collapse at the point $\left(T_{p^-}, T_{p^+}\right)$, so they are tongue shapped

Proof : For $k = \pm 1$, we have $\operatorname{grad}\left(\Theta_{p^- - k, p^+ + k}\left(T_{p^-}, T_{p^+}\right)\right) \neq 0$, then we compute the slope of the boundary lines for ray dynamics with patterns $\left(p^- - k, p^+ + k\right)$ at the point $\left(T_{p^-}, T_{p^+}\right)$ and obtain that both boundary lines have the common slope

$$\frac{\mathrm{d}T^+}{\mathrm{d}T^-}\left(T_{p^-}, T_{p^+}\right) = -\frac{p^-}{p^+}\left(\frac{\sin\left(\dfrac{\pi}{p^+}\right)}{\sin\left(\dfrac{\pi}{p^-}\right)}\right)^2$$

Moreover, the transition lines where the pattern $\left(p^-, p^+\right)$ changes to one of the patterns $\left(p^- \mp 1, p^+ \pm 1\right)$ also has the previous common slope



SUBREGION TONGUES AT DIAGONAL POINTS

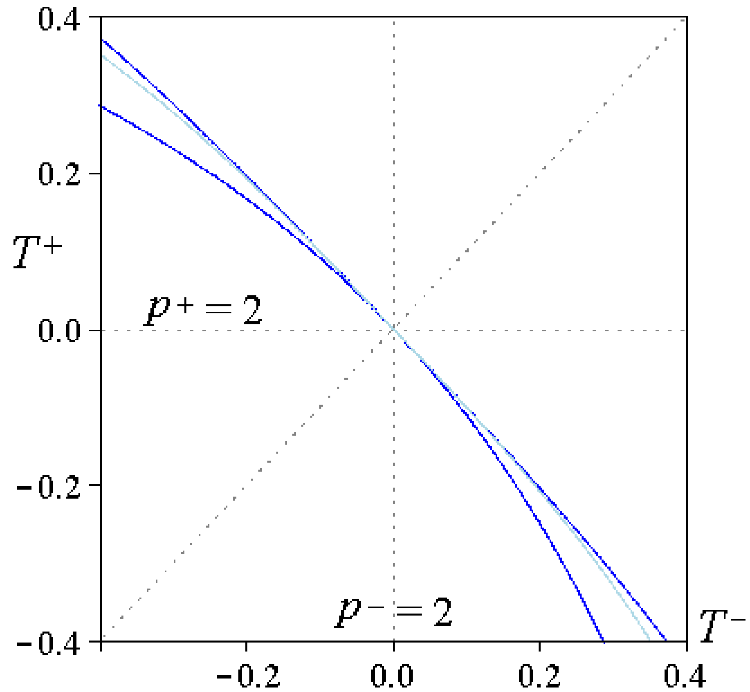
We have $\Theta_{p,p}\left(T_p,T_p\right)=\mathrm{grad}\left(\Theta_{p,p}\left(T_p,T_p\right)\right)=\det\left(\mathrm{hessian}\left(\Theta_{p,p}\left(T_p,T_p\right)\right)\right)=0$

It can be shown that $\Theta_{p,p}\left(T_p+t,T_p+t\right)<0$ and $\Theta_{p,p}\left(T_p-t,T_p+t\right)>0$

for $|t|$ sufficiently small, then $\Theta_{p,p}$ is saddle shapped around $\left(T_p,T_p\right)$

Considering the parameter symmetry, we obtain that the slopes of both boundary lines are $\mathrm{d}T^+/\mathrm{d}T^- = -1$ at $\left(T_p,T_p\right)$

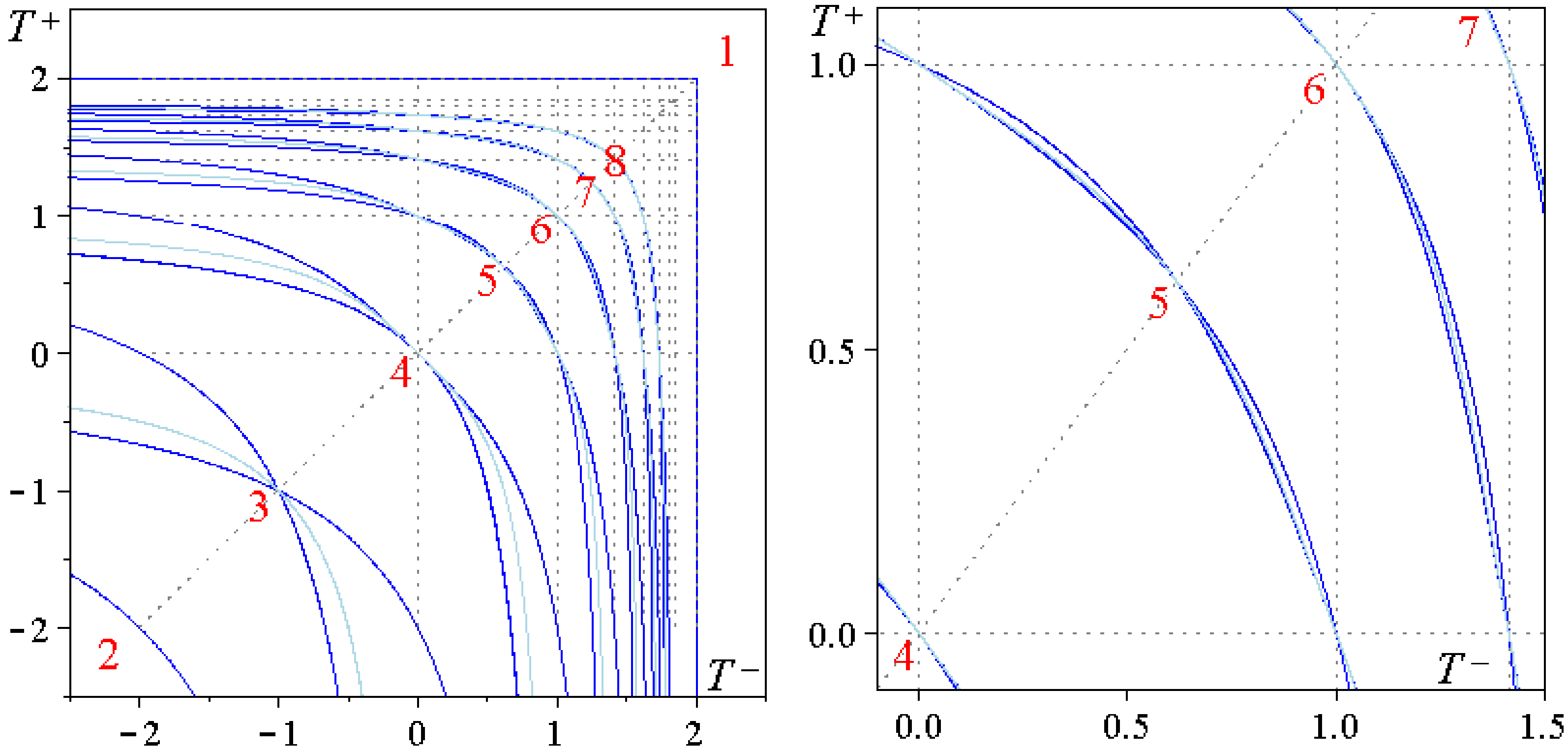
Thus, we conclude that the three patterns $\left(p-k,p+k\right)$ for $k=0,\pm 1$ are admissible in a region which is tongue shaped and symmetric with respect to the diagonal



SUMMARY OF RAY DYNAMICS REGIONS

Some bifurcation lines in the parameter plane have been plotted on the left panel and a zoom is showed on the right panel

The number of pockets with period q for map g increases with q



CONCLUSIONS

A circle map g is defined for the area preserving CPWL map $\mathbf{G}$, so that the periodic dynamics of g gives account for the ray dynamics of $\mathbf{G}$

Lines in the parameter plane with a continuous rotation number $CR^{\pm}$ exist for the map $\mathbf{G}$, apart from the linear case $\mathbf{G}(T, T)$, for $-2 < T < 2$

Around the finite order points FO (defined by the intersection of the CR^{+} and CR^{-} lines) there exist tongue shaped regions with ray dynamics. For each periodicity, several pockets linked through those FO points exist

Conjecture: Associated to some of the finite order points at the diagonal of the parameter plane, there exists a parametric ray dynamics region having a pocket structure as the one obtained from the crossing of the $CR^{\pm}$ lines. These regions follow the well known Farey sequence

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