Thue-Morse system of difference equations

Francisco Balibrea

balibrea@um.es
Departamento de Matemáticas
Universidad de Murcia (Spain)

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The Thue-Morse Sequence

The Thue-Morse sequence

$$\mathbf{t} = (t_n)_{n \geq 0} = 0110100110010...$$

is an ubiquitous mathematical object.
It comes up in algebra, number theory, combinatorics, topology and other areas.
It has different but equivalent definitions.

The Thue-Morse Sequence

Define a sequence of strings of 0's and 1's as follows:

$$X_0 = 0$$

$$X_{n+1} = X_n \bar{X}_n$$

where $\bar{X}$ means change all the 0's in X into 1's and vice-versa.

For example, we find

$$X_0 = 0$$

$$X_1 = 01$$

$$X_2 = 0110$$

$$X_3 = 01101001$$

$$X_4 = 0110100110010110$$

$$\dots$$

The Thue-Morse Sequence

Then

$$\lim_{n\to\infty} X_n = \mathbf{t}$$

The Thue-Morse Sequence

Another definition

A **morphism** is a map h on strings that satisfies the identity $h(XY) = h(X)h(Y)$ for all strings X, Y.
Define the Thue-Morse morphism $\mu(0) = 01$, $\mu(1) = 10$. Then

$$\mu(0) = 01$$
$$\mu^2(0) = \mu(\mu(0)) = 0110$$
$$\mu^3(0) = 01101001$$
$$\mu^4(0) = 0110100110010110$$
$$\ldots$$

Then can be proved by induction on n that $\mu^n(0) = X_n$ and $\mu^n(1) = \bar{X}_n$

The Thue-Morse Sequence.- Repetitions in Strings

A **square** is a string of the form XX. Examples in English are

mama

murmur

hotshots

A finite string of consecutive symbols is called a word. A word is **squarefree** if it contains no subword (a word contained) that is a square. Note that the English word **squarefree** is not squarefree, but **square** is.

The Thue-Morse Sequence.- Repetitions in Strings

A **cube** is a string of the form XXX. Examples in English include

hahaha

shshsh

A word is **cubefree** if it contains no subword that is a cube. A **fourthpower** is a string of the form $XXXX$. The only example I know in English is

tratratratra

which is an extinct lemur from Madagascar

The Thue-Morse Sequence.- Repetitions in Strings

An **overlap** is a string of the form *aXaXa* where *a* is a single letter and *X* ia string. Examples in English include

alfalfa

entente

A word is **overlap** $-$ **free** if it contains no word that is an overlap
One example of overlap-free is the Thue-Morse infinite word **t** (it is necessary to prove it and was made by Morse in the twenties)

The Thue-Morse sequence. Another definition

Given a positive integer number n whose binary decomposition is

$$n = \sum_{0 \leq i \leq k} a_i 2^i$$

we define the sum of digits, function $s_2(n)$ to be the sum of the a_i's. Then it can be proved that

$$t_n = s_2(n) (\mathrm{mod} 2)$$

The Thue-Morse sequence

Given the shift space of two symbols $((\Sigma^2, d), \sigma))$, the Thue-Morse sequence $\mathbf{t}$ is a uniform recurrent point.The proof can be done seeing that the words in the sequence are syndetic.

The Thue-Morse system of difference equations

In a paper of 1992, Y.Avishai and D.Berend, considered a one dimensional array of N- δ-function potentials

$$V_n(x) = v \sum_{i=1}^{n} \delta(x - x_i)$$

where $v > 0$
and $\mathbf{x} = (x_n)_{n \geq 1}$
is an infinite real sequence whose difference sequence $y_n = x_{n+1} - x_n$ which assumes two possible positive values d_1 and d_2. When we take $\mathbf{x} = \mathbf{t}$, then

The Thue-Morse system of difference equations

$$y_n = x_{n+1} - x_n = d_1 \text{ or } d_2$$

depending on if $\psi_n = 0$ or $\psi = 1$ where $\psi_n = [1 + (-1)^{s(n)}]/2$

Then they study the reflection proccess of a plane wave through a one dimensional array of n-δ-functions located in the Thue-Morse chain with the former distances d_1 and d_2, arriving after application of the Schrödinger equation and some transformations to the following system of difference equations where x_n and y_n for every n are the trace of some matrices associated to the numerical scheme. Finally it is decided if the array behaves as an electrical insulator or conductor.

Thue-Morse quasicrystal

Figure: 1D Thue-Morse quasicrystal

The Thue-Morse system of difference equations

The previous problem leads to the following system of difference equations

$$x_{n+1} = x_n(4 - x_n - y_n)$$

$$y_{n+1} = x_n y_n$$

where x_n, y_n are the trace of two matrices associate to the process. The system can be seen as a two dimensional dynamical system given by the pair $(\mathbb{R}^2, T)$ where

$$T(x, y) = (x(4 - x - y), xy)$$

The Thue-Morse system, first facts

The most interesting part of the dynamics is concentrated in the interior of the triangle Δ obtained connecting the three points $(0,0),(4,0),(0,4)$. The line Γ connecting $(4,0)$ and $(0,4)$ is given by $x+y=4$. Δ is invariant by H, that is, $H(\Delta)=\Delta$. The segment $\Gamma_1=\{(x,0):x(4-x)\}$ is also invariant ($H(\Gamma_1)=\Gamma_1$). If $\Gamma_2=\{(0,y):0\leq y\leq 4\}$, then $H(\Gamma)=\Gamma_2$ and $H(\Gamma_2)=\{(0,0)\}$.

The Thue-Morse system, first facts

It is easy to test that all points belonging to Δ have no preimages outside it, which means that Δ and its complements with respect to $\mathbb{R}^2$ are invariant.

The Sharkovskĭi's problem

In 1993 in a conference in Oberwolfach, Sharkovskĭi proposed to study the two dimensional dynamical system

$$H(x, y) = ((y - 2)^2, xy)$$

In particular, answer to the following questions:

1. Are the periodic points dense in Δ?
2. Is $H|\Delta$ transitive?
3. Is Γ an attractor of Δ in Milnor's sense?
4. Does there exist a point P such that $\omega_H(P)$ be unbounded but holding $\omega_H(P) \cap \Gamma \neq \emptyset$

References

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Periodic points

By elementary algebraic computations it is esay to see that inside Δ there is only one fixed point, $(1,2)$. At the boundary of Δ we have two fixed points, $(0,0)$ and $(3,0)$. Outside the triangle we have no fixed point. There are no two-periodic points, neither three periodic points.

By the algebraic method of resultant, we obtain that the interior point

$$(1-\sqrt{2}/2, 1+\sqrt{2}/2)$$

is periodic of period 4 and its orbit is the unique having such a period. By numerical methods we can prove the existence of a unique interior periodic points of period 5 and by numerical calculus that explicitely

$$(1, (3+\sqrt{5})/2))$$

is periodic of period six, but it is not unique.

Periodic points

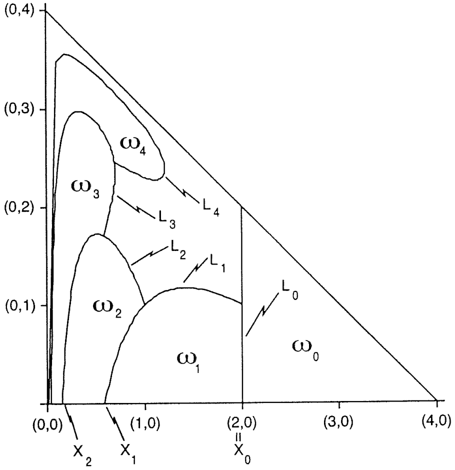
Using a particular symbolic dynamics, P. Malicky has shown that for $n \geq 4$ there is an interior point of H in Δ of period n. It remains open if such points are unique or not.

The key point of Malicky's proof is to prove that given a saddle periodic point P in Γ_2, there exists in the interior of Δ a periodic point having the same itinerary. It is interesting to have a criterium to prove the existence of saddle points in Γ_2. In fact, let $P = [4\sin^2(k\pi/(2^n + (-)1), 0]$, where $n > 0, k$ are integers numbers. If

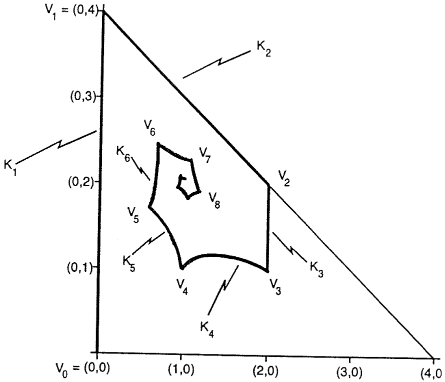
$$1 \leq k \leq \frac{\sqrt{2}(2^n + (-)1)}{\pi 2^{\sqrt{2n+1/4}}}$$

then P is a saddle fixed point of H^n.

Decomposition of the interior of Δ



An invariant curve approaching the fixed point



Periodic points

It remains open if outside Δ there are or not periodic points. We claim that there are some periodic points, a non-wandering set of points and the rest tending to infinite when $n \to \infty$. The set

$$\Delta_1 = \{(x, y) : x, y \geq 0, \;\; x + y \geq 4\}$$

has no wandering points and the set

$$\Delta_2 = \{(x, y) : x, y \geq 0, \;\; x + y \leq 4\}$$

is also T-invariant and can contain non-wandering points. Using the former arguments the answer to problems 3 and 4 from Sharkovsk$\breve{i}$i is negative.

Non-wandering points

According to numerical experiments it seems that most points go to infinite when $n \to \infty$

Representation of saddle points

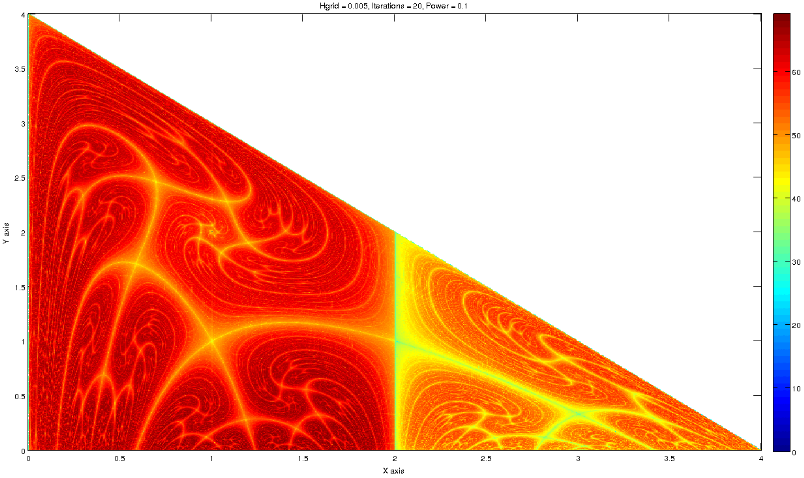


Figure: Triangle

Fibonacci systems of equations

Y.Avishai and D.Berend in *Transmission through a one-dimensional Fibonacci sequence of δ-function potentials*, repeating what was done in the case of a Thue-Morse case, but using the substitutions rules

$$0 \to 01$$

$$1 \to 0$$

we obtain the so called Fibonacci sequence

$$(00101001001....)$$

with it we construct the Fibonacci quasicrystal and then the trace maps associated to the resolution of a partial differential equation, can be solutions of the system of difference equations which unfolding is

$$F(x, y, z) = (y, z, yz - z)$$

it is open to study the dynamics of such map.

One day in Galicia

Toast with wine

Figure: Two good friends!