Topology of the parameter plane of Newton's method on Bring-Jerrard polynomials

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Abstract
We study the topology of the hyperbolic components of the parameter plane for the Newton's method applied to n-th degree Bring-Jerrard polynomials, given by \( P_n(z) = z^n - c \), \( c \in \mathbb{C} \).

For \( n = 5 \), using the Tscheinhaus-Bring-Jerrard nonlinear transformations, this family controls, at least theoretically, the roots of all quintic polynomials.

1. Introduction
The main goal of this work is to study some topological properties of the parameter plane of Newton's method applied to the family

\[ P_n(z) = z^n - c \]

where \( n \geq 2 \) (to simplify the notation we will assume that \( n \) is fixed); so, we erase the dependence on \( n \) unless we need to refer to it explicitly.

The interest to consider this family is that the general quintic equation \( P(z) = 0 \) can be transformed (through a strictly nonlinear change of variables) to one of the form \( P_n(z) = z^n - c \), \( c \in \mathbb{C} \).

Letting \( n \) as a parameter \( n \) (1) allows us to have a better understanding of the problems we are dealing with.

Easily, the expression of the Newton's map applied to (1) can be written as:

\[ N(z) = \frac{z^n - c}{n(z^{n-1})} \]

The following quite general topological properties of the basins of attraction and hyperbolic components of the Julia set are well known (see [1]), for instance, where they studied Newton's method for a general polynomial, and Shishikura [2].

Proposition 2. The following statements hold:
(a) \( \mathcal{A}(\alpha) \) is unbounded.
(b) The number of accesses to infinity of \( \mathcal{A}(\alpha) \) is either 2 or n − 1.
(c) \( \mathcal{J}(N) \) is connected, so, any connected component of the Fatou set is simply connected.

The classical Böttcher Theorem provides a tool related to the behavior of holomorphic maps near a superattracting fixed point \( \alpha \), which we apply to make a detailed description of the superattracting basin of each simple root \( \alpha_j \) for \( j = 0 \ldots n − 1 \) of \( N \).

Theorem 3. Suppose that \( f \) is a holomorphic map, defined in some neighborhood \( U \) of \( \alpha \), having a superattracting fixed point at \( \alpha \), i.e.,

\[ f(z) = \alpha^n + a_{n-1}z^{n-1} + \ldots \quad \text{where} \quad n \geq 2 \quad \text{and} \quad a_{n-1} \neq 0 \]

Then, there exists a local conformal change of coordinate \( w = \varphi(z) \), called Böttcher coordinate at \( \alpha \) (or Böttcher map), such that \( \varphi^{-1} w = z^{n} \) through some neighborhood of \( \varphi(0) = 0 \).

3.3. Symmetries in the parameter plane of \( \mathcal{N} \)

The hyperbolic components in the parameter plane correspond to open subsets of \( \mathcal{N} \) in which the unique free critical point \( z = 0 \) either is a superattracting root of the family \( F \) (those were denoted \( c_{\alpha} \)), such on one hand, it indicates the end of the capture component, and on the other hand, it means a fixed point of degree \( n \) of the Newton's map. As a consequence, any other capture component should be bounded (see the previous figures), which is turn implies that for a fixed \( n \), all non-captured hyperbolic components correspond to \( n − 1 \) generalized Mandelbrot sets.

The following lemma removes from our parameter plane those \( \epsilon \)-values for which the roots of \( P_n \) are not simple and so the Newton's method is not a rational map of degree \( n \).

Lemma 4. Fix \( \epsilon > n \). The Newton's map \( N \) is a degree \( n \) rational map if and only if

\[ \epsilon < \frac{n^2}{n} = \frac{n}{n-1} \]

With the next result we prove that we can focus on a sector in the parameter plane due to the following symmetries (see figures).

Lemma 5. Let \( n \geq 3 \). The following symmetries in the parameter plane hold:
(a) The roots \( \alpha_j = \frac{a_j}{a_{n-1}} \) are conjugate through the anti-holomorphic map \( h(z) = z^n \).

(b) The maps \( N(z) \) and \( N(\omega z) \), for \( \omega = e^{2\pi i/n} \) are conjugate through the anti-holomorphic map \( h(z) = z^n \).

The parameter plane for \( n = 3 \) and a zoom.

4. Topology of the hyperbolic components
We first study the capture components \( \mathcal{C}_j \), for \( 0 \leq j \leq n − 1 \).

The first result determines that one of the roots, \( \alpha_0 \), is playing a different role, since for all \( z \) outside a certain ball around the origin the free critical point \( z = 0 \) lies in its immediate basin of attraction. This is due to the fact that the free critical point is \( z = 0 \) for all \( n \geq 3 \) and for all \( n \) in the parameter space. As a consequence, any other capture component should be bounded (see the previous figures), which is turn implies that for a fixed \( n \), all non-captured hyperbolic components correspond to \( n − 1 \) generalized Mandelbrot sets.

References


