Topology of the parameter plane of Newton's method on Bring-Jerrard polynomials

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Abstract

We study the topology of the hyperbolic components of the parameter plane for the Newton's method applied to n-th degree Bring-Jerrard polynomials given by $P_n(z) = z^n - cz + 1,\ c \in \mathbb{C}$.

For $n = 5$, using the Tschirnhaus-Bring-Jerrard nonlinear transformations, this family controls, at least theoretically, the roots of all quintic polynomials.

1. Introduction

The main goal of this work is to study some topological properties of the parameter plane of Newton's method applied to the family

$$P_{n,c} := P_c(z) = z^n - cz + 1, \tag{1}$$

where $n \geq 3$ (to simplify the notation we will assume that n is fixed; so, we erase the dependence on n unless we need to refer to it explicitly). The interest to consider this family is that the general quintic equation $P_5(z) = 0$ can be *transformed* (through a strictly nonlinear change of variables) to one of the form $P_{5,c} := z^5 - cz + 1 = 0,\ c \in \mathbb{C}$ [10]. Letting n as a parameter in (1) allows us to have a better understanding of the problems we are dealing with.

Easily, the expression of the Newton's map applied to (1) can be written as:

$$N_c(z) = z - \frac{P_c(z)}{P_c'(z)} = z - \frac{z^n - cz + 1}{nz^{n-1} - c} = \frac{(n-1)z^n - 1}{nz^{n-1} - c}. \tag{2}$$

So, the critical points of N_c correspond to the zeroes of P_c, which we denote by $\alpha_j,\ j = 0, \ldots, n-1$ and $z = 0$ which is the unique *free* critical point of N_c of multiplicity $n-2$. We notice that since all critical points except $z = 0$ coincide with the zeroes of P_c they are superattracting fixed points; so, their dynamics is fixed for all $c \in \mathbb{C}$. Note that for certain values of n and c, this rational map is not irreducible.

For each root $\alpha_j(c) := \alpha_j,\ j = 0, \ldots, n-1$ we define its *basin of attraction*, $\mathcal{A}_c(\alpha_j)$, as the set of points in the complex plane which tend to α_j under the Newton's map iteration. That is

$$\mathcal{A}_c(\alpha_j) = \{z \in \mathbb{C},\ N_c^k(z) \to \alpha_j \text{ as } k \to \infty\}.$$

In general $\mathcal{A}_c(\alpha_j)$ may have infinitely many connected components but only one of them, denoted by $\mathcal{A}_c^*(\alpha_j)$ and called *immediate basin of attraction of* α_j, contains the point $z = \alpha_j$.

2. Dynamical plane: Distribution of the roots and attracting basins

In this section we prove some estimates for the relative distribution of the roots $\alpha_j,\ j = 0, \ldots n-1$ of the polynomials in family (1), assuming they are all different roots.

Fix $c \in \mathbb{C}$ and denote by $D(z_0, r)$ the disc centered at $z = z_0$ of radius $r > 0$. Let $w_j := w_j(c),\ j = 0, \ldots, n-1$ be the n different solutions of $z(z^{n-1} - c) = 0$. In particular, we set $w_0 = 0$. Next lemma shows that if $|c|$ is large enough we have $\alpha_j \in D(w_j, 1),\ j = 0, \ldots, n-1$. In particular if $|c|$ is large enough we set α_0 to be the root of the corresponding polynomial such that $\alpha_0 \in D(0,1)$ and $\alpha_j,\ j = 1, \ldots n-1$ to be the root of the corresponding polynomial such that $\alpha_j \in D(w_j, 1)$. That is to say, the root α_0 is always inside a disc of radius 1 centered at 0 and the other roots $\alpha_j,\ j = 1, \ldots, n-1$ are inside discs centered at w_j. As we see in the following, α_0 behaves as $\frac{1}{c}$ for c large enough.

Lemma 1. *The following statements hold:*

(a) For all c in the parameter space, the roots $\alpha_0, \ldots, \alpha_{n-1}$ of (1) belong to the set

$$\mathcal{D} = \bigcup_{j=0}^{n-1} D(w_j, 1)$$

(b) Let $c \in$ such that

$$|c| > \max\{2^{n-1}, \frac{1}{\sin^{n-1}\left(\frac{\pi}{n-1}\right)}\}.$$

Then, $D(w_j, 1) \cap D(w_k, 1) = \emptyset,\ j \neq k$. Moreover, each $D(w_j, 1)$ contains one and only one of the roots of (1).

(c) If c is large enough, there exists $M := M(n) > 0$ such that

$$|\alpha_0 - N_c(0)| < M|c|^{-(n+1)}.$$

In particular, for a fixed n, as c goes to infinity the *small* root of (1) tends to $1/c$ (exponentially) faster than c approaches infinity. Statement (c) of Lemma 1 is equivalent to say that for c outside a certain disc in the parameter plane, the free critical point $z = 0$ always belongs to the same immediate basin of attraction, the one of $\alpha_0 \sim 1/c$.

The following quite general topological properties of the basins of attraction and hyperbolic components of the Julia set are well known (see [6], for instance, where they studied Newton's method for a general polynomial, and Shishikura [9]).

Proposition 2. *The following statements hold:*

(a) $\mathcal{A}_c^*(\alpha_j)$ is unbounded.

(b) The number of accesses to infinity of $\mathcal{A}_{n,c}^*(\alpha_j)$ is either 1 or $n-1$.

(c) $\mathcal{J}(N_c)$ is connected. So, any connected component of the Fatou set is simply connected.

The classical Böttcher Theorem provides a tool related to the behavior of holomorphic maps near a superattracting fixed point [1], which we apply to make a detailed description of the superattracting basin of each simple root α_j for $j = 0, \ldots, n-1$ of N_c.

Theorem 3. *Suppose that f is an holomorphic map, defined in some neighborhood U of 0, having a superattracting fixed point at 0, i.e.,*

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots \text{ where } m \geq 2 \text{ and } a_m \neq 0.$$

Then, there exists a local conformal change of coordinate $w = \varphi(z)$, called Böttcher coordinate at 0 (or Böttcher map), such that $\varphi \circ f \circ \varphi^{-1}$ is the map $w \to w^m$ throughout some neighborhood of $\varphi(0) = 0$. Furthermore, φ is unique up to multiplication by an $(m-1)$-st root of unity.

3. Symmetries in the parameter plane of N_c

The hyperbolic components in the parameter plane correspond to open subsets of $\mathbb{C}$ in which the unique *free* critical point $z = 0$ either eventually maps to one of the immediate basins of attraction corresponding to one of the roots of P_c (those were denoted by $\mathcal{C}_j^k$ where j explains the *catcher* root and k the minimum number of iterates for which $z = 0$ reaches $\mathcal{A}_c^*(\alpha_j)$), or it has its own hyperbolic dynamics associated to an attracting periodic point of period strictly greater than one (black components in the following figures). We use the following notation:

$$\begin{aligned}
\mathcal{H} &= \{c \in \mathbb{C},\ 0 \text{ is attracted by an attracting cycle of period } p \geq 2\}.\\
\mathcal{B} &= \{c \in \mathbb{C},\ \text{the Julia set } \mathcal{J}(N_c) \text{ does not move continuously}\\
&\qquad \text{(in the Hausdorff topology) over any neighborhood of } c\}.
\end{aligned}$$

The following lemma removes from our parameter plane those c-values for which the roots of P_c are not simple and so the Newton's method is not a rational map of degree n.

Lemma 4. *Fix $n \geq 3$. The Newton's map N_c is a degree n rational map if and only if*

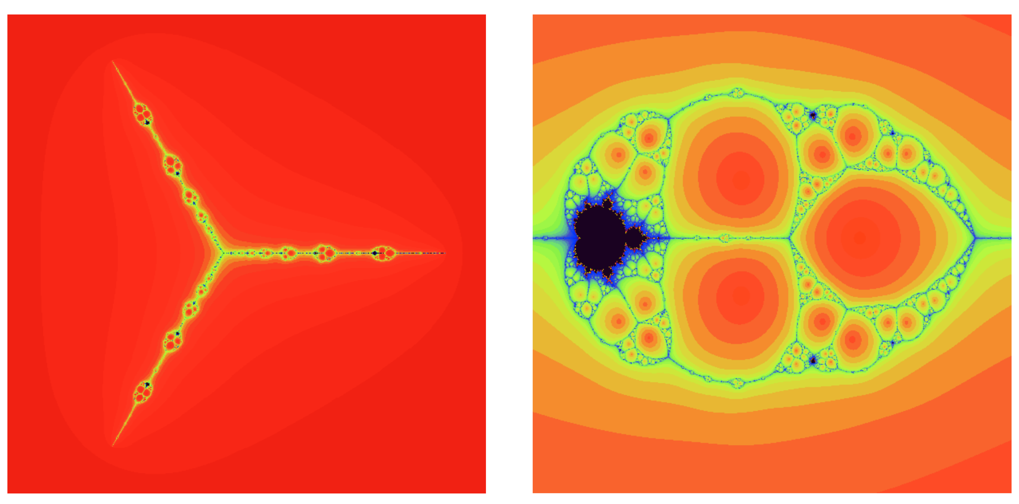
$$c \neq c_k^* := \frac{n}{(n-1)^{\frac{n-1}{n}}} e^{2k\pi i/n},\ k = 0, \ldots, n-1.$$

With the next result we prove that we can focus on a *sector* in the parameter plane due to the following symmetries (see figures).

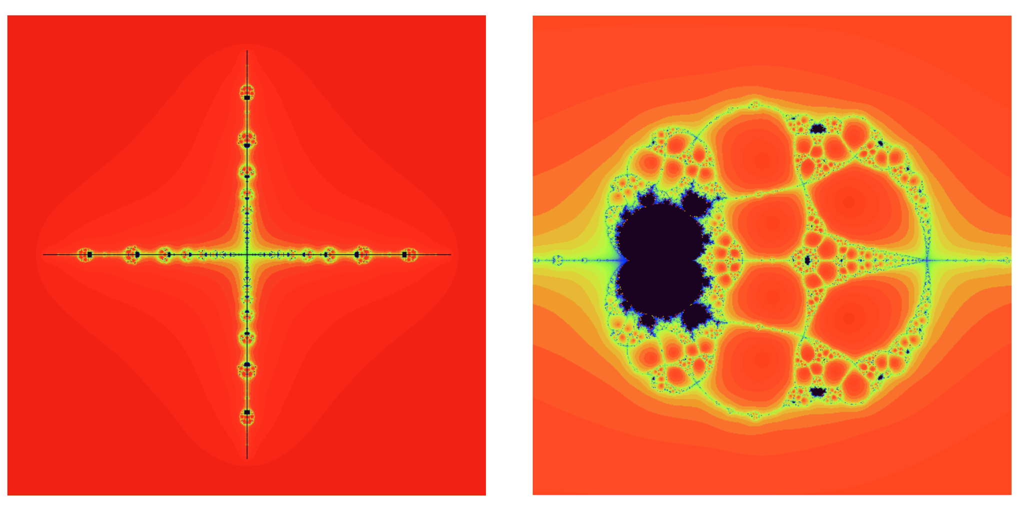
Lemma 5. *Let $n \geq 3$. The following symmetries in the c-parameter plane hold:*

(a) The maps $N_c(z)$ and $N_{\hat{c}}(z)$ with $\hat{c} = e^{\frac{2\pi i}{n}} c$, are conjugate through the holomorphic map $h(z) = e^{\frac{2\pi i}{n}} z$.

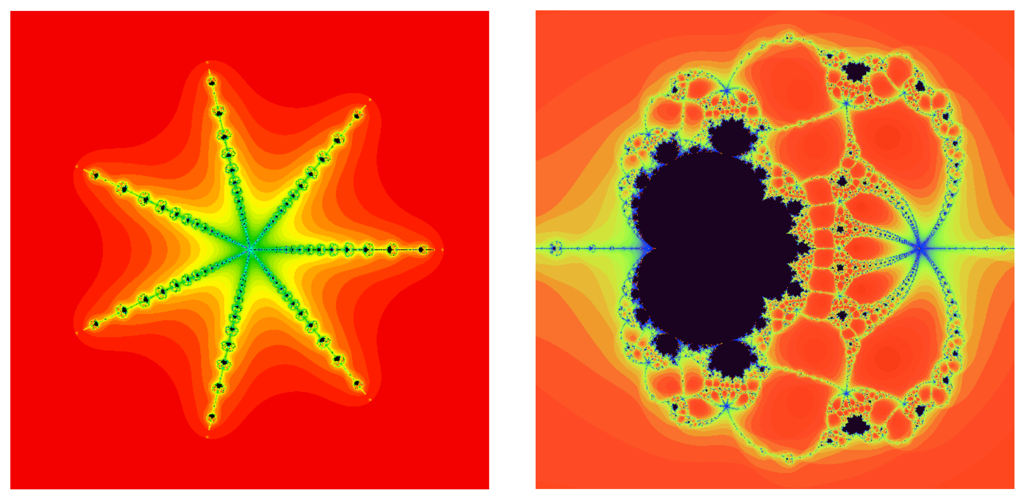
(b) The maps $N_c(z)$ and $N_{\bar{c}}(z)$ are conjugate through the anti–holomorphic map $h(z) = \bar{z}$.



The parameter plane for $n = 3$ and a zoom.



The parameter plane for $n = 4$ and a zoom.



The parameter plane for $n = 7$ and a zoom.

4. Topology of the hyperbolic components

We first study the capture components $\mathcal{C}_j^0$, for $0 \leq j \leq n-1$.
The first result determines that one of the roots, α_0, is playing a differentiated role, since for all c outside a certain ball around the origin the free critical point $z = 0$ lies in its immediate basin of attraction. This is due to the fact that the free critical point is $z = 0$ for all $n \geq 3$ and for all c in the parameter space. As a consequence, any other capture component should be bounded (see the previous figures), which in turn implies that $\mathcal{C}_j^0,\ j = 1, \ldots, n-1$ are empty.

Proposition 6. *Fix $n \in \mathbb{N}$.*

(a) $\mathcal{C}_0^0$ is unbounded. In fact we have $\mathcal{C}_0^0 \supset \{c \in \mathbb{C}, |c| > 4\}$.

(b) $\mathcal{C}_0^0$ is connected and simply connected.

(c) $\mathcal{C}_j^0 = \emptyset$ for all $j \geq 1$.

Next, we investigate the rest of the capture components $\mathcal{C}_j^0$, for $0 \leq j \leq n-1, k \geq 1$, showing that every connected component is simply connected.

Proposition 7. *Fix $n \in \mathbb{N}$.*

(a) $\mathcal{C}_j^1 = \emptyset$ for all $j = 0, \ldots, n-1$.

(b) If $\mathcal{C}_j^k \neq \emptyset$, its connected components are simply connected.

Some topological results about those stable subsets of the parameter plane are:

Theorem 8. *The following statements hold.*

(a) $\mathcal{C}_0^0$ is connected, simply connected and unbounded.

(b) $\mathcal{C}_j^0,\ 1 \leq j \leq n-1$ are empty.

(c) $\mathcal{C}_j^1,\ 0 \leq j \leq n-1$ are empty.

(d) $\mathcal{C}_j^k,\ 0 \leq j \leq n-1$ and $k \geq 2$ are simply connected as long as they are nonempty.

Apart from the captured components we also observe the presence of Generalized Mandelbrot sets $\mathcal{M}_k$ (the bifurcation locus of the polynomial families $z^k + c,\ c \in$). As an application of a result of C. McMullen [8], we can show that for a fixed n, all non-captured hyperbolic components correspond to $n-1$ Generalized Mandelbrot sets.

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