# Approximating invariant objects using wavelet expansions


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joint work with

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## The problem statement

*Using wavelets, we give an analytical approximation of a case of a Strange Non–Chaotic Attractor (SNA) given by the solution of a (non) – linear system of equations.*

## Dynamical motivation

Consider a skew product on the cylinder $\begin{pmatrix}\theta_{k+1}\\ x_{k+1}\end{pmatrix} = \begin{pmatrix}R_\omega(\theta_k)\\ F_{\sigma,\varepsilon}(\theta_k, x_k)\end{pmatrix}$, where $R_\omega$ is an irrational rotation and $F_{\sigma,\varepsilon} : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{R}$. There are several works (see [3] and references therein) where it is shown that, under mild conditions, there exists an upper semi continuous function $\varphi$such that verifies the *Invariance Equation* and if we restrict ourselves to the Keller setting such $\varphi$ is a SNA if $\sigma > 1$ and $\varepsilon = 0$, where the strangeness of the attractor refers to complicated geometry (see Figure 1). On the other side, provided that $F_{\sigma,\varepsilon} \in C^{n+1}(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R})$ then the linear operator between the functional spaces given by

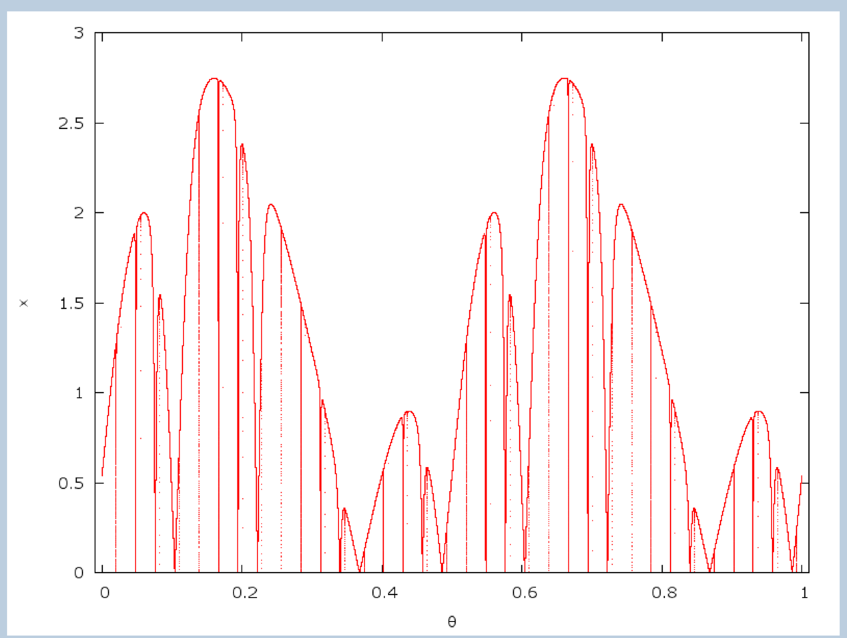

Figure 1: *Notice the abrupt changes in the graph of the attractor ($\sigma = 1.5$ and $\varepsilon = 0$).*

$$\begin{aligned} \mathbf{T} : C^n(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}) &\longrightarrow C^n(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}) \\ \varphi &\longmapsto F_{\sigma,\varepsilon}(\theta, \varphi(\theta)) - \varphi(R_\omega(\theta)) \end{aligned} \tag{1}$$

is a differentiable operator and also, notice, that if there exists $\varphi$ such that $\mathbf{T}\varphi(R_\omega(\theta)) = 0$ then $\varphi$ is an invariant curve: verifies the Invariance Equation.

## Wavelet tools

Let $\psi(x) \in \mathscr{L}^2(\mathbb{R})$ be a function whose integer translates and dilations by powers of two: $2^{-j/2}\psi\left(\frac{x-2^j n}{2^j}\right)$, is an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$. Such a function is called *mother wavelet* (see [4]). Since our framework is $\mathbb{S}^1$, we use the common trick of the periodization of a $\mathbb{R}$-function by setting $\psi^{\text{PER}}_{j,n}(x)$ as follows:

$$\psi^{\text{PER}}_{j,n}(x) = \sum_{\ell\in\mathbb{Z}} \psi_{j,n}(x+\ell) = 2^{-j/2} \sum_{\ell\in\mathbb{Z}} \psi\left(\frac{(x+\ell)-2^j n}{2^j}\right).$$

**Theorem.** *[(see [2])] An orthonormal basis of $\mathscr{L}^2(\mathbb{S}^1)$ is given by the sytem $\{1, \psi^{\text{PER}}_{j,n}$ with $j \le 0$ and $n = 0, 1, \dots, 2^{-j} - 1\}$.*

Four our purposes we will be focused on the *Daubechies wavelets* which are a family of orthogonal wavelets characterized by a maximal number of vanishing moments, $p$, for some given support, $[1-p, p]$: $\int_{1-p}^{p} x^k \psi(x)\, dx = 0$ for $0 \le k < p$.

**Proposition 1.** *Let $\psi(x)$ be a $\mathbb{R}$-Daubechies wavelet with $p > 1$ vanishing moments. Then, given $j \le 0$ and $\theta \in \mathbb{S}^1$ $\psi^{\text{PER}}_{j,n}(\theta) = \sum_{\ell\in\Lambda_\theta}\sum_{n\in\aleph_\theta} \psi_{j,n}(\theta)$, where $\Lambda_\theta \subset \left[\lceil \frac{1-p}{2^{-j}} - \theta\rceil, \lfloor \frac{p-1}{2^{-j}} - \theta\rfloor\right]$ and, being $t = \lfloor 2^{-j}\theta\rfloor$ and $\alpha = \{2^{-j}\theta\}$, $\aleph_\theta \subset [\max(0, 2^{-j}\ell + t + \lceil\alpha\rceil - p), \min(2^{-j} - 1, 2^{-j}\ell + t + p - 1)]$.* ■

As an easy corollary, we have that for an integer $J$ "*big enough*" then the following equality holds $\psi^{\text{PER}}_{j,n}(\theta) = \psi_{j,n}(x)$ if $j > J$.

**Observation 2.** *The Daubechies wavelets, in contrast to the trigonometric polynomials, do not have a closed expression. Therefore, it is necessary to perform a strategy to, given a point $\tilde\theta \in \mathbb{S}^1$, evaluate $\psi^{\text{PER}}_{j,n}(\tilde\theta)$. To do this, in the $\mathbb{R}$-case there are some methods and we have modified, using Proposition 1, one of them: the Daubechies – Lagarias algorithm (see [1, 5]) in order to use it on $\mathbb{S}^1$ .* ■

With the above Observation and Proposition 1 in mind we can prove the following

**Proposition 3.** *Let $\tilde\theta$ be an arbitrary point of $\mathbb{S}^1$ and $\psi(x)$ be a $\mathbb{R}$-Daubechies wavelet with $p > 1$ vanishing moments. Then, for the vector $u$ and the matrcies $T_0$ and $T_1$,*

$$\psi^{\text{PER}}(\tilde\theta) = \sum_{\ell\in\Lambda_\theta} \lim_{k\to\infty} u(\tilde\theta + \ell)' v(\tilde\theta + \ell, k) = \sum_{\ell\in\Lambda_\theta} \lim_{k\to\infty} u(\tilde\theta + \ell)' \left(\frac{1}{2p-1} 1' \prod_{i\in \text{dyad}(\{2\tilde\theta+\ell\},k)} T_i\right),$$

*where $u(\cdot)$, $T_0$ and $T_1$ are fully determined by $\psi$ and $1'$ is a row vector of ones.* ■

## Forging a nonlinear system of equations

Using $\mathbf{T}$, from Equation (1), and a equidistributed partition of $\mathbb{S}^1$ with $N = 2^J$ points, namely $\theta_i = i/N$, we want to express the invariant curve, $\varphi$, as a finite expansion like $\varphi \sim a_0 + \sum_{j=-J}^{0}\sum_{n=0}^{2^{-j}-1} \left\langle \varphi, \psi^{\text{PER}}_{j,n}\right\rangle \psi^{\text{PER}}_{j,n} = a_0 + \sum_{l\in\Lambda} d^{\text{PER}}_l \psi^{\text{PER}}_l$, where $l = 2^j + n$ (therefore $j = \lfloor \log_2(l)\rfloor$ and $n = l - 2^j$), performing a (non) – linear system of equations where the unknowns are $a_0$ and the coefficients $d^{\text{PER}}_l$ by setting:

$$\mathbf{T}\varphi(R_\omega(\theta_i)) = 0 \Leftrightarrow \overbrace{a_0 + \sum_{l\in\Lambda} d^{\text{PER}}_l \psi^{\text{PER}}_l(R_\omega(\theta_i)) - F_{\sigma,\varepsilon}\left(\theta_i, a_0 + \sum_{l\in\Lambda} d^{\text{PER}}_l \psi^{\text{PER}}_l(\theta_i)\right)}^{\mathcal{F}\text{ of Newton's method}}.$$

In order to solve the above System, as usual, we will use Newton's method: $\mathbf{J}\mathcal{F}(x_n)(\mathrm{X}) = -\mathcal{F}(x_n)$ for the unknown $\mathrm{X} = x_{n+1} - x_n$ and given an initial seed $x_0$. In our particular framework, $\mathbf{J}\mathcal{F}$ has some interesting properties. Indeed,

**Lemma 4.** *The $(i, l)$-th entry of the Jacobian matrix of $\mathbf{J}\mathcal{F}$, for the case (1) and using a wavelet expression of $\varphi$, is given by:*

$$\begin{cases} 1 - \dfrac{\partial F_{\sigma,\varepsilon}\left(\theta_i, a_0 + \sum_{l\in\Lambda} d^{\text{PER}}_l \psi^{\text{PER}}_l(\theta_i)\right)}{\partial x} & \text{if } l = 0, \\ \psi^{\text{PER}}_l(R_\omega(\theta_i)) - \dfrac{\partial F_{\sigma,\varepsilon}\left(\theta_i, a_0 + \sum_{l\in\Lambda} d^{\text{PER}}_l \psi^{\text{PER}}_l(\theta_i)\right)}{\partial x} \psi^{\text{PER}}_l(\theta_i) & \text{otherwise.} \end{cases}$$

*Moreover, if $\psi(x)$ is a $\mathbb{R}$-Daubechies wavelet then $\mathbf{J}\mathcal{F}$ is a* "highly structured" *sparse matrix.* ■

To get the initial guess $x_0$ is enough to use the *Trapezoidal rule*. Indeed,

**Lemma 5.** *Let $\Psi^{\text{PER}}_N$ be the $N\times N$ matrix whose columns are $\psi^{\text{PER}}_l(\theta_i)$ and consider the* Transfer Operator, *$\mathfrak{T}(\varphi)(\theta) = F_{\sigma,\varepsilon}(\theta, \varphi(\theta))$. Then $x_0 = \Psi^{\text{PER}}_N \tilde\varphi$ is an exponentially close "good seed" where, being $k > k_0 \in \mathbb{N}$, $\tilde\varphi = \mathfrak{T}(\varphi_{k-1}) = \mathfrak{T}^k(c)$ and $\varphi_0 = c$ is sufficiently large positive constant function.* ■

## A (*small*) linear system of equations

Fix $N \in \mathbb{N}$ such that is "*big enough*" and consider $\left(\Psi_{R_\omega N}{}^{\text{PER}} - \Delta_N \Psi^{\text{PER}}_N\right)\mathrm{X} = -\mathcal{F}(x_n)$ to be the linear system of equations given – created by Proposition 3 and Lemmae 4 & 5. The system, large and sparse, is solved using a Krylov method: Generalized Minimal Residual Method (GMRES). Such method is an iterative method that seek the solution on a linear subspace generated by the powers of the system matrix against the residual vector $r = \mathbf{J}\mathcal{F}\big|_{x_n} - \mathcal{F}(x_n)$.

For a fixed $\sigma > 1$ and $\varepsilon \ge 0$ and different levels of tolerance in the Newton's method we have carried out, for a particular instance of the Keller setting ($F_{\sigma,\varepsilon}(\theta, x) = 2\sigma(\varepsilon + |\cos(2\pi\theta)|)\tanh(x)\big|_{\mathbb{R}^+}$), some succesful experiments in terms of expended time and the $N$'s coice ($2^J$ with $J = 10, 11, \dots, 15$) with GMRES.

## One step beyond: switch *small* by *huge*, but... why?

It is known that the wavelet coefficients can characterize the lack of regularity of a parameter dependent function (see [2]) and, also, the behaviour of the Newton's method can be used in the same way. On the other side, since in our case $\mathbf{J}\mathcal{F}$ is a sparse matrix the "*large matrix problems*" appear noticeably after (respect a dense matrix). In view of that, in order to have more information the increase of $N$ seems to be necessary but, the standard techniques to speed up the convergence of the GMRES method seems to be not very useful in our framework.

Recall that the system to solve is $\left(\Psi_{R_\omega N}{}^{\text{PER}} - \Delta_N \Psi^{\text{PER}}_N\right)\mathrm{X} = -\mathcal{F}(x_n)$ and in a future work it will be preconditioned using two matrices. The first one is the analogous of the Fast Fourier Transform in the wavelets framework: the Discrete Wavelet Transform DWT (see[4]). The second one is recalling that $\Psi^{\text{PER}}_N$ must be an orthogonal matrix. Therefore, $(\Psi_{R_\omega N}{}^{\text{PER}} \Psi^{\text{PER}\top}_N - \Delta_N)\mathrm{Y} = -\mathcal{F}(x_n)$, with $\mathrm{Y} = \Psi^{\text{PER}}_N \mathrm{X}$ must be the reformulation of the system to solve using the GMRES method.

## References


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