Inverse limits on Riemann Spheres

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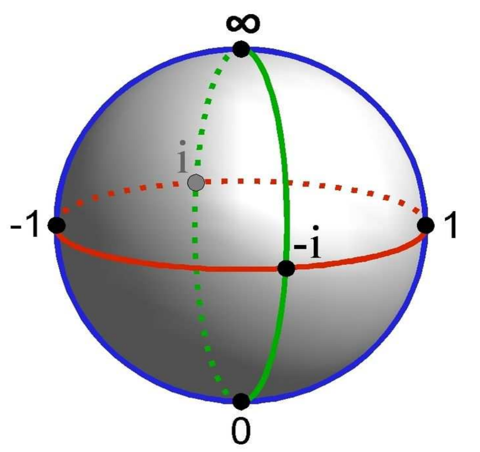
Introduction

The two important objects that we use are:
(i) The Riemann Sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
(ii) The functions $P_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ where $P_c(z) = z^2 + c, c \in \mathbb{C}$.

These two objects are very studied in the complex dynamics context. We present the diferent inverse limits induced by take diferent values of c.

Plain

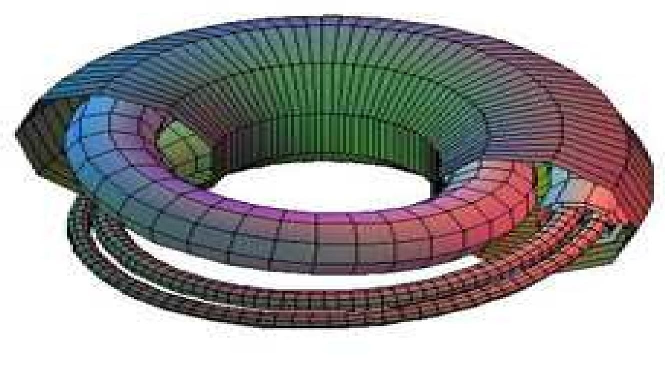
1. Topological Results
2. Cases P_0 and P_{-2}
3. Dynamic Results
4. The clasification theorem



Riemann Sphere

1 Topologycal Results

First we says that a continuum is a metric, compact and connected space. The continua that we use in this talk are *the arc* $\mathbb{I}$, *the closed curve* $\mathbb{S}^1$, *the knaster continuum* $\mathbb{K}$, *the Riemann Sphere* $\hat{\mathbb{C}}$, *the dyadic solenoid* $\sum_2$.
Now we talk about *inverse limits*. All the functions that we consider are surjective continuous functions and all spaces are continua.
An *inverse system* $\{X_n, f_n^{n+1}\}_{n=1}^{\infty}$ is a double sequence of spaces X_n and functions $f_n^{n+1} : X_{n+1} \to X_n$. The functions f_n^{n+1} are called *bonding functions*. Sometimes we write


Dyadic Solenoid

$$X_1 \xleftarrow{f_1^2} X_2 \xleftarrow{f_2^3} \cdots X_{n1} \xleftarrow{f_{n-1}^{n}} X_n \xleftarrow{f_n^{n+1}} X_{n+1} \xleftarrow{f_{n+1}^{n+2}} \cdots$$

Given two inverse systems, $\{X_n, f_n^{n+1}\}_{n=1}^{\infty}$ and $\{Y_n, g_n^{n+1}\}_{n=1}^{\infty}$ we says that are *topologically conjugate* if for all $n \in \mathbb{N}$ exists a homeomorphism $h_n : X_n \to Y_n$, for all $n \in \mathbb{N}$ such that $g_n^{n+1} \circ h_{n+1} = h_n \circ f_n^{n+1}$.
Given an inverse system, their *inverse limit* is

$$X_\infty = \varprojlim\{X_n, f_n^{n+1}\} = \left\{ \{x_i\}_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_n^{n+1}(x_{n+1}) = x_n \right\}$$

- The inverse limit of continuum is a continuum.
- If two inverse systems are topologically conjugate, then they inverse limits are homeomorphic.

Very Important Result Let $\{X_n, f_n^{n+1}\}$ be an inverse sequence of metric spaces whose inverse limit is X_∞. If A is a closed subset of X_∞, then the double sequence $\left\{\pi_n(A), f_n^{n+1}|_{\pi_{n+1}(A)}\right\}$ is an inverse sequence with surjective bonding maps and

$$\varprojlim \left\{\pi_n(A), f_n^{n+1}|_{\pi_{n+1}(A)}\right\} = A = \left[\prod_{n=1}^{\infty} \pi_n(A)\right] \cap X_\infty.$$

References

[1] Robert L. Devaney. The complex dynamics of quadratics polynomials. *Proceedings of Symposia in Applied Mathematics*, 49(1):1–29, 1994.

[2] Sergio Macias. *Topics on Continua*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Chapman Hall/CRC, Taylor Francis Group, Boca Raton, FL USA, 2005. Advanced Topics.

2 Cases P_0 and P_{-2}

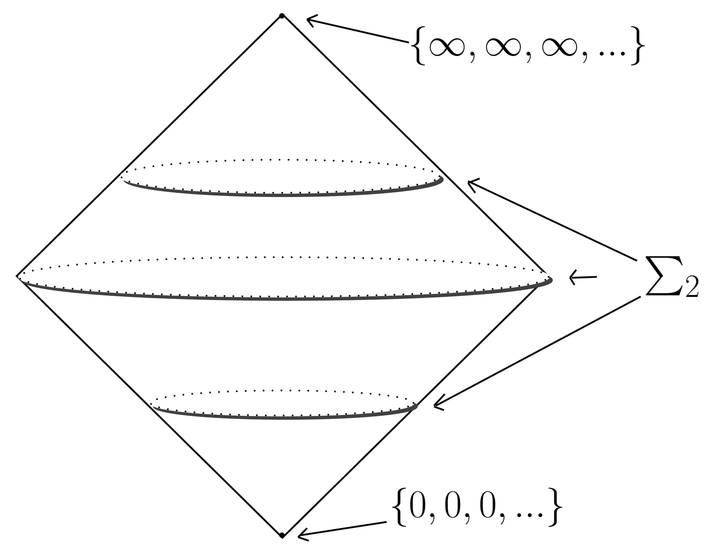
We show $\varprojlim\{\hat{\mathbb{C}}, P_0\}$ is the *solenoid suspension*.
$P_0(0) = 0$ and $P_0(\infty) = \infty$, then $\{0,0,0,0,0,0,\ldots,\}, \{\infty,\infty,\infty,\infty,\ldots,\} \in \varprojlim\{\hat{\mathbb{C}}, z^2\}$.
Set $\mathbb{S}^{(r)} = \{z \in \mathbb{C} : |z| = r\}$. For all $r \in \mathbb{R}$, $P_0\left(\mathbb{S}^{(\sqrt{r})}\right) = \mathbb{S}^{(r)}$, this induces the next inverse systems between closed curves

$$\mathbb{S}^{(r)} \xleftarrow{z^2} \mathbb{S}^{(\sqrt{r})} \xleftarrow{z^2} \cdots \xleftarrow{z^2} \mathbb{S}^{\left(\sqrt[2(n-1)]{r}\right)} \xleftarrow{z^2} \mathbb{S}^{\left(\sqrt[2n]{r}\right)} \xleftarrow{z^2} \mathbb{S}^{\left(\sqrt[2(n+1)]{r}\right)} \xleftarrow{z^2} \cdots$$

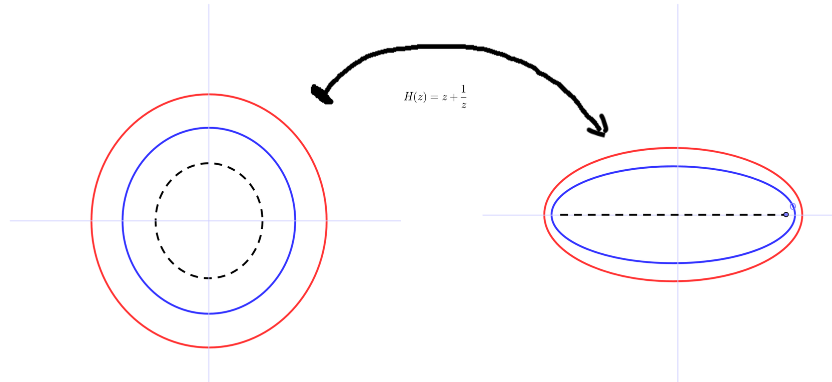
We denote this inverse sistem as $\left\{\mathbb{S}^{\left(\sqrt[2n]{r}\right)}, z^2\right\}_{n=1}^{\infty}$.
Observe that is topologically conjugate to $\{\mathbb{S}^1, z^2\}_{n=1}^{\infty}$ by $h_n : \mathbb{S}^{\left(\sqrt[2n]{r}\right)} \to \mathbb{S}^1$ defined by

$$h_n((\theta, \sqrt[2n]{r})) = (\theta, 1) \text{ for all } n \in \mathbb{N}.$$

We know that $\varprojlim\{\mathbb{S}^1, z^2\}$ is homeomorphic to the dyadic solenoid [2].
Conclusion For all $r \in \mathbb{R}$ the cercle with radiuos r induces a dyadic solenoid in the inverse limit, and the points ∞ and 0 a point. Therefore the inverse limit is homeomorphic to the solenoid suspension.

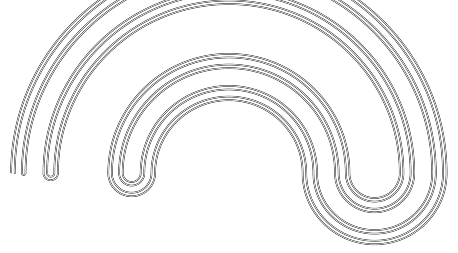


Now we show that $\varprojlim\{\hat{\mathbb{C}}, z^2 - 2\}$ is not the solenoid suspension but is a space "very similar" to it.

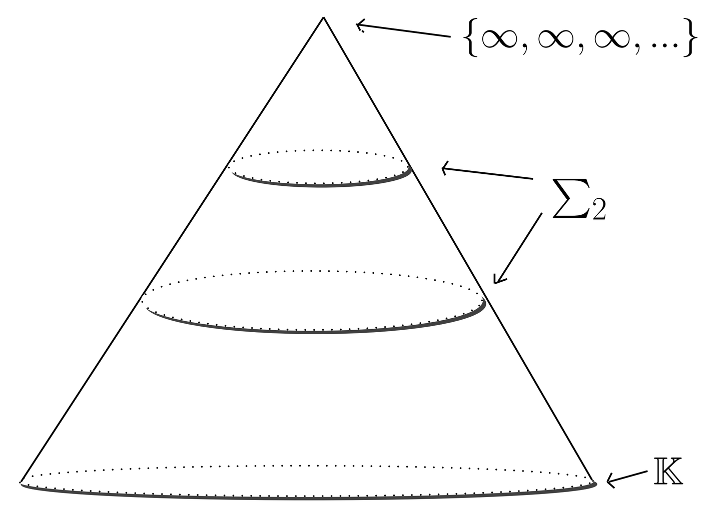


$\{\infty,\infty,\infty,\infty,\ldots,\} \in \varprojlim\{\hat{\mathbb{C}}, z^2 - 2\}$.
Let $H(z) = z + \frac{1}{z}$ defined in the set $\{z \in \mathbb{C} : |z| \geq 1\}$.
Remark H conjugates P_0 on $\{z : |z| > 1\}$ to P_{-2} in all $\mathbb{C} \setminus [-2, 2]$ in fact $H \circ P_0 = P_{-2} \circ H$, and $\mathbb{S}^1$, is mapped in 2-to-1 fashion onto the interval $[-2, 2]$.

Fix $r > 1$, observe that H conjugate the inverse system $\left\{\mathbb{S}^{\left(\sqrt[2n]{r}\right)}, z^2\right\}_{n=1}^{\infty}$ with $\varprojlim \left\{H\left(\mathbb{S}^{\left(\sqrt[2(n-1)]{r}\right)}\right), z^2\right\}$. In consequence we have $H\left(\mathbb{S}^{\left(\sqrt[2(n-1)]{r}\right)}\right)$ induces a dyadic solenoid in the $\varprojlim\{\hat{\mathbb{C}}, z^2 - 2\}$ for all $r > 1$. Now, the inverse limit of the function restricted to $[-2, 2] \times \{0\}$. is homeomorphic to the Knaster continuum $\mathbb{K}$.



Conclusion The inverse limit is homeomorphic to the suspension solenoid with the point $\{0, 0, 0, \ldots\}$ exploited to the Knaster continuum.



Remark in case P_0 the set $\mathbb{S}^1$ plays an important role, and in addition is an invariant set of $\hat{\mathbb{C}}$ by P_0. Also the the interval $[-2, 2]$ is an *invariant* set of $\hat{\mathbb{C}}$ by P_{-2}.
This observation leads us to introduce some concepts of the complex dynamics.

3 Preliminary Dynamic Results

Do you find the next results in [1].
Given $P_c = z^2 + c$ we says that the *Filled Julia Set* is

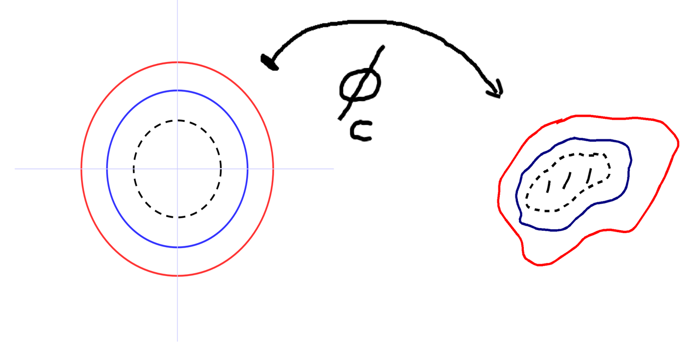
$$\mathcal{K}_c = \{z \in \mathbb{C} : P_c^n(z) \nrightarrow \infty \text{ when } n \to \infty\}$$

Also the *Julia Set* is $\mathcal{J}_c = \mathrm{Bd}(\mathcal{K}_c)$
Theorem $\mathcal{K}_c$ is connected if and only if $0 \in \mathcal{K}_c$.
If $\mathcal{K}_c$ is not connected then it is totally disconnected, in fact is a Cantor space (Compact, perfect totally disconnected space).
A compact subset K of the complex plane is called full if its complement is connected. For example, the closed unit disk is full, while the unit circle is not.
Important Theorem If $\mathcal{K}_c$ es connected then $\hat{\mathbb{C}} \setminus \mathcal{K}_c$ is an open simply connected and exists a Riemann mapping

$$\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathcal{K}_c$$

such that $\phi_c(\infty) = \infty$ and

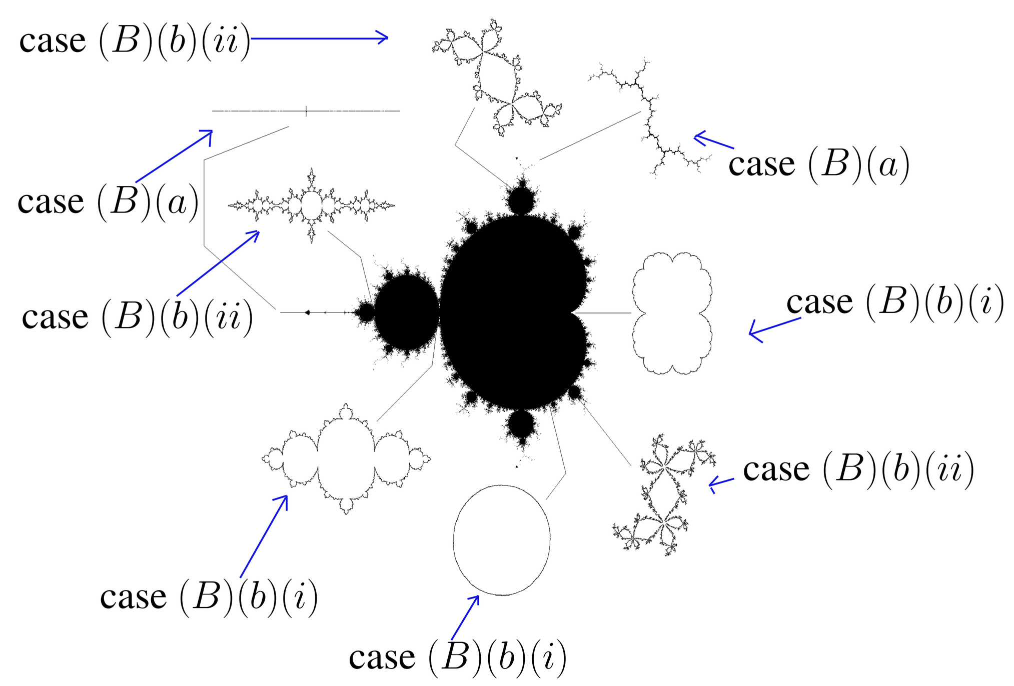
$$P_c \circ \phi_c = \phi_c \circ P_0.$$



The implications of the above theorems can be summarized in the following clasification theorem

4 The clasification Theorem

Theorem Given $P_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the next is true:
(A) If $c \notin \mathcal{K}_c$ then $\mathcal{K}_c$ is totally disconnected and $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is homeomorphic to ...?
(B) If $c \in \mathcal{K}_c$ then $\mathcal{K}_c$ is a continuum and we have diferent cases for $\mathcal{K}_c$.
(a) $\mathrm{Int}(\mathcal{K}_c) = \emptyset$ then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is a solenoid suspension with the point $\{0, 0, 0, \ldots\}$ "exploited" to a indecomposable continuum $M_{\mathcal{K}_c}$ tipe-$\mathcal{K}_c$.
(b) $\mathrm{Int}(\mathcal{K}_c) \neq \emptyset$ then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ have two posibilities
(i) $\mathrm{Int}(\mathcal{K}_c)$ have only one coneccted components, and the options are:
$\mathcal{J}_c$ arcwise connected then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is an solenoid suspension.
$\mathcal{J}_c$ is not arcwise connected then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is an solenoid suspension with $\mathbb{S}^1$ exploited to one continuum like-$\mathbb{S}^1$.
(ii) $\mathrm{Int}(\mathcal{K}_c)$ have numerable coneccted components,
$\mathcal{J}_c$ arcwise connected then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is an solenoid suspension with...?
$\mathcal{J}_c$ is not arcwise connected then $\varprojlim\{\hat{\mathbb{C}}, z^2 + c\}$ is an solenoid suspension with...?



Now we work to find the inverse limits that we obtain with the case (A), and $(B)(b)(ii)$. We intend to extend this study to $P_{(n,c)}(z) = z^n + c$ with $n \in \mathbb{N}$ and $c \in \mathbb{C}$.