Dynamics of integrable birational maps preserving genus 0 foliations.

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Introduction and some results

A planar *rational* map $F:\mathcal{U}\to\mathcal{U}$, where $\mathcal{U}\subseteq\mathbb{K}^2$ is an open set and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, is called *birational* if it has a rational inverse F^{-1}. Also, we will say that a map F is *integrable* if there exists a non-constant function $V:\mathcal{U}\to\mathbb{K}$ such that $V(F(x,y))=V(x,y)$ which is called a *first integral*. In this work, we will consider only integrable maps that have *rational first integrals* $V(x,y)=V_1(x,y)/V_2(x,y)$. In other words, the map preserves the foliation of $\mathcal{U}$ given by the algebraic curves

$$C_h=\{V_1(x,y)-hV_2(x,y)=0\}_{h\in Im(V)}.$$

It is known that any birational map F preserving a foliation of *nonsingular* curves of generic genus greater or equal than 2 is a globally periodic map, as a consequence of Hurwitz (1893) and Montgomery (1937) theorems. If the genus is generically 1, either F or F^2 are conjugate to a linear action.

Our goal is to derive a methodology to study the genus 0 case.

The Cayley-Riemann Theorem ensures that genus 0 curves are rationally parametrizable. Furthermore, this parametrization can be chosen to be *proper*, i.e. a birational, which is unique modulus Möbius transformations, [2]. Hence we obtain:

Proposition

Let F be a planar birational map preserving a foliation of algebraic curves $\{C_h\}$ of genus 0, and $\{P_h\}$ a family of proper parametrizations of $\{C_h\}$. Then, each $F_{|C_h}$ is conjugated to a Möbius map M_h, where $M_h=P_h^{-1}\circ F\circ P_h$.

This is because each P_h is birational, so M_h is a one-dimensional birational map.

Since every rational real curve can be properly parametrized over the reals [2], then *real* birational maps preserving a foliation of real algebraic curves of genus 0 can be represented by *real* Möbius transformations.

Corollary

Let F be a *real* planar birational map preserving a foliation of algebraic curves $\{C_h\}$ of genus 0, and $\{P_h\}$ a family of proper *real* parametrizations of $\{C_h\}$. Then each $F_{|C_h}$ is conjugated to a Möbius map M_h, with real coefficients.

As the dynamics of the Möbius maps is well-known, we are able to give a description of the global dynamics of our maps, and in particular, the *explicit* expression of the rotation number function associated to given invariant curves in an open set foliated by closed ones. The explicitness of the rotation number function allow us to fully characterize the set of periods of these maps.

We also have proved that our maps have an associated *Lie symmetry*. Specifically,

Theorem

Any birational map F preserving a foliation given by algebraic curves of genus 0, $\{C_h\}$ where $h=V(x,y)$, has a Lie symmetry. Furthermore, if $\{P_h(t)=(P_{1,h}(t),P_{2,h}(t))\}$ is a family of proper parameterizations of $\{C_h\}$, then there is a Lie symmetry of F given by the vector field in $\mathbb{K}^2$

$$X(p)=P'_{1,h}(t)(P_h^{-1}(p))Y(P_h^{-1}(p))\frac{\partial}{\partial x}+P'_{2,h}(t)(P_h^{-1}(p))Y(P_h^{-1}(p))\frac{\partial}{\partial y}$$

where $p=(x,y)$, and $Y(t)=-b(h)+(d(h)-a(h))t+c(h)t^2$, is the Lie symmetry associated to the Möbius map

$$M_h(t)=\frac{a(h)t+b(h)}{c(h)t+d(h)},$$

given by $M_h(t)=P_h^{-1}\circ F\circ P_h(t)$.

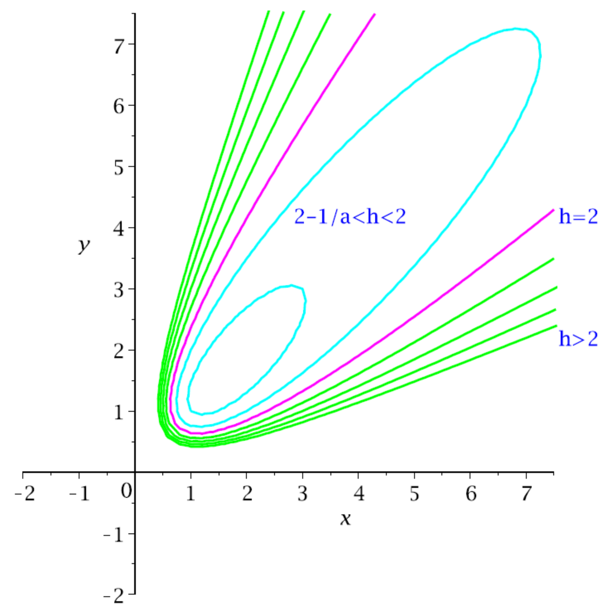
Example: The Bastien and Rogalski map

We consider the planar birational map defined in $\mathbb{R}^{2,+}$.

$$F_a(x,y)=\left(y,\frac{a-y+y^2}{x}\right)\text{ with } a>1/4,$$

This map was studied in [1]. It preserves the foliation given by the algebraic curves of genus 0:

$$C_h=\{x^2+y^2-x-y+a-hxy=0\}.$$



Using an appropriate algorithm we find the *proper real* parametrization of C_h given by $P_h(t)=(P_{1h}(t),P_{2h}(t))$ where

$$P_{1h}(t)=\frac{2\delta t-ah^2-h\delta-h-2+4a}{2(-t^2+ht-1)}+a,$$

$$P_{2h}(t)=\frac{(-ah+\delta-1)\,t^2+(4\,a-2)\,t-ah-\delta-1}{2(-t^2+ht-1)},$$

and $\delta=\sqrt{(ha+1-2a)(ha+1+2a)}$. Its inverse is

$$P_h^{-1}(x,y)=\frac{-2\,\delta\,x+(ah^2+h\delta-4\,a+h+2)\,y-ah+2\,a+\delta-1}{(ah^2-h\delta-4\,a+h+2)\,x+2\,\delta\,y+ah-2\,a-\delta+1}.$$

Then, as show us the previous results, the composition $M_h=P_h^{-1}\circ F\circ P_h$ gives

$$M_h(t)=\frac{(h+1)t-1}{t+1},$$

a Möbius map with real coefficients, and the Lie symmetry

$$X(x,y)=\frac{x^2-y^2+a-x}{y}\frac{\partial}{\partial x}+\frac{x^2-y^2-a+y}{x}\frac{\partial}{\partial y}.$$

Both of these results allow us to give a global description of the dynamics:

Proposition

Set $a>1/4$. Let $\tilde{C}_h$ and $\tilde{F}$ denote the extensions of C_h and F to $\mathbb{R}P^2$. Then, we get:

- For $h>2$, C_h is a hyperbola, $F_{|C_h}$ is conjugated to a translation, and there are two fixed points of $\tilde{F}_{|\tilde{C}_h}$ at infinity, an attractor and a repeller.
- For $h=2$, C_h is a parabola, $F_{|C_h}$ is conjugated to a translation, and there is a non stable attractor of $\tilde{F}_{|\tilde{C}_h}$ at infinity.
- For $2-1/a<h<2$, C_h is an ellipse, $F_{|C_h}$ is conjugated to a rotation with explicit rotation number

$$\theta(h)=\arg\left(\frac{h-i\sqrt{4-h^2}}{2}\right)\mod 2\pi.$$

Using this rotation number we reobtain the results in [1]:

Proposition

Set $a>1/4$, $h_c=2-1/a$, and $\rho_a=\frac{1}{2\pi}\arg\left(\frac{2a-1-i\sqrt{4a-1}}{2a}\right)$. Then, we have:

- For any fixed $a>1/4$ and $p\geq \mathrm{E}\left(1/(1-\rho_a)\right)+1$ there exists $h_p\in(h_c,2)$ such that C_{h_p} is filled of p-periodic orbits.
- For all $p\in\mathbb{N}$, $p\geqslant 3$ there exists $a>1/4$ and $h_p\in(h_c,2)$ such that C_{h_p} is filled of p-periodic orbits.

References

[1] G. Bastien, M. Rogalski. *On some algebraic difference equations $u_{n+2}u_n=\psi(u_{n+1})$ in $\mathbb{R}^+$, related to families of conics or cubics: generalization of the Lyness' sequences*, J. Math. Anal. Appl. 300 (2004), 303–333.

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