On the dynamics of an affine system with non-reducible linear behaviour

Marc Jorba, Àngel Jorba
Universitat de Barcelona

Universitat de Barcelona

Description of the problem

Consider the following quasi-periodically forced discrete dynamical system:

$$\begin{cases} \bar{x} = f(x,\theta), \\ \bar{\theta} = \theta + \omega. \end{cases}$$

Where $x \in U$, $\theta \in \mathbb{T}$, $f : U \times \mathbb{T} \mapsto U$ is of class C^{r+1}, $r \geq 0$, and $\omega \in (0, 2\pi) \setminus 2\pi\mathbb{Q}$.
Suppose there exists an invariant curve, a function $x \in C^r(\mathbb{T}, U)$, such that

$$x(\theta + \omega) = f(x(\theta), \theta).$$

The linear dynamics around the **invariant curve** is described by:

$$\begin{cases} \bar{x} = A(\theta)x, \\ \bar{\theta} = \theta + \omega. \end{cases} \qquad (1)$$

We have named $A(\theta) := D_x f(x(\theta))$. Any system such as (1) is called **quasi-periodic linear skew product**, or a quasi-periodic C^r cocycle.

Reducibility

Definition

System (1) is said to be **C^r-reducible** if there exists a C^r change of variables $x = C(\theta)y$ such that transforms the former system into:

$$\begin{cases} \bar{y} = By, \\ \bar{\theta} = \theta + \omega. \end{cases}$$

Where the matrix $B = C^{-1}(\theta + \omega)A(\theta)C(\theta)$ does not depend on θ.

Essential non-reducibility

Reducibility can have a topological obstruction. In that case we speak about **essential non-reducibility**:

Definition

Let $A \in C^r(\mathbb{T}, \mathrm{GL}_2\,\mathbb{R})$, $r \geq 0$. Fix a vector $v \in \mathbb{R}^2 \setminus \{0\}$ and consider the curve v_A at $\mathbb{R}^2 \setminus \{0\}$ given by $v_A(\theta) = A(\theta)v$. We define the winding number of A, **wind A**, as the winding number of v_A around the origin of $\mathbb{R}^2$.

- The winding number of a cocycle does not depend on the choice of the vector v.
- The winding number is invariant under conjugation. Since a constant matrix B verifies **wind** $B = 0$, if A is a cocycle such that **wind** $A \neq 0$, then A is essentially non-reducible.
- A linear map coming from a Poincaré section of a quasi-periodic linear ODE has winding number zero.

Affine systems

Consider the following affine system on the plane:

$$\begin{cases} \bar{x} = \mu A(\theta)x + b(\theta), \\ \bar{\theta} = \theta + \omega, \end{cases}$$

with $A \in C^r(\mathbb{T}, \mathrm{GL}_2\,\mathbb{R})$ and $b \in C^r(\mathbb{T}, \mathbb{R}^2) =: E$, $r \geq 0$, endowed with the standard C^r norm. Let $\|A\| = \sup_{\|x\|=1} \|Ax\|$ and $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.

- If $\rho(A) < 1$ an attracting invariant curve appears as a fixed point of the operator:

$$\mathcal{T}(x(\theta)) = \mu A(\theta - \omega)x(\theta - \omega) + b(\theta - \omega).$$

Goal of this work

- We show that non-reducibility has **dynamical manifestation**.
- We study the destruction of the family of attracting invariant curves of the model:

$$\begin{cases} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \mu \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \\ \bar{\theta} = \theta + \omega. \end{cases} \qquad (2)$$

- Here, μ is a **positive** parameter.
- The linear behaviour of the model is essentially non-reducible.
- There exists a smooth invariant curve z_μ for each $\mu \neq 1$.
- The curve z_μ is attracting when $\mu < 1$.

Theorem

Consider the system (2). Assume the rotation number ω to be of constant type. Then, when $\mu \to 1$:

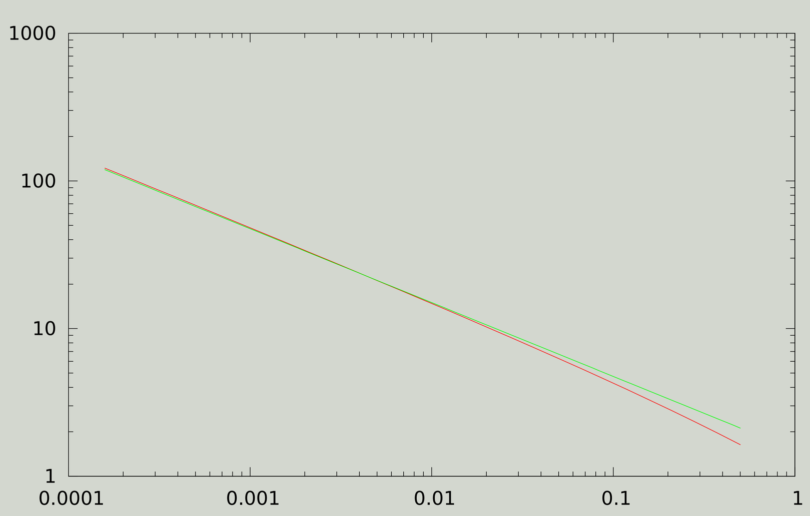
1. *The invariant curve z_μ undergoes a **fractalization** process, i.e.*
$$\frac{\|z'_\mu\|_\infty}{\|z_\mu\|_\infty} = \mathcal{O}(1-\mu)^{-1}.$$
2. *The winding number of z_μ around any point of $\mathbb{R}^2$ verifies the following:*
$$\text{wind } z_\mu = \mathcal{O}(1-\mu)^{-1}.$$

Corollary

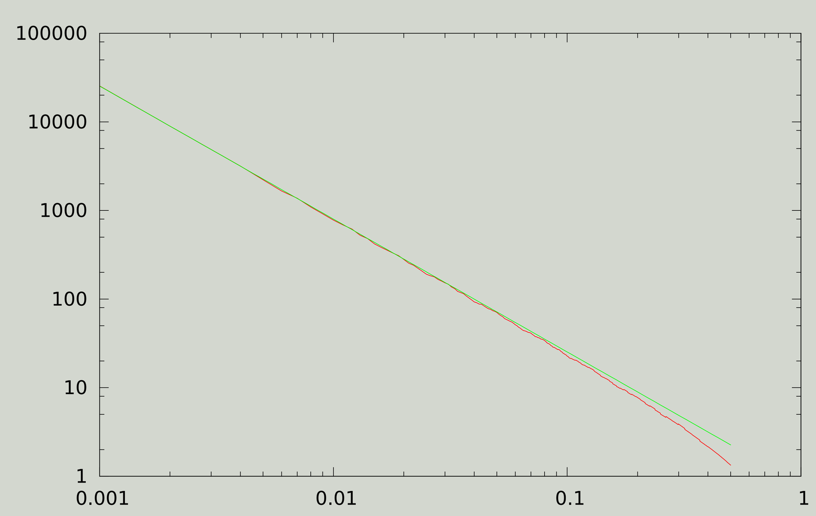
Let z_μ be the invariant curve of (2). It holds

$$\bigcup_{\mu \in (\mu_0, 1)} \text{graph } z_\mu = \mathbb{R}^2 \quad \text{for any } \mu_0 \in (0,1).$$

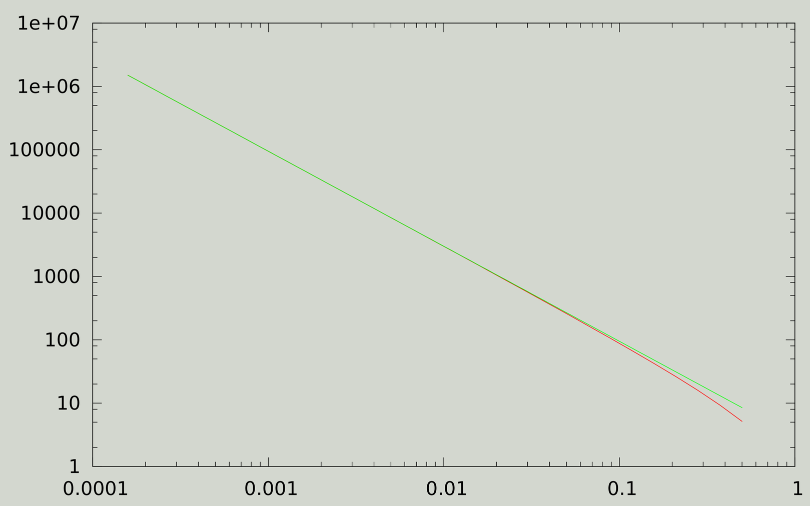
Numerical experiments: exploring the breakdown scenario



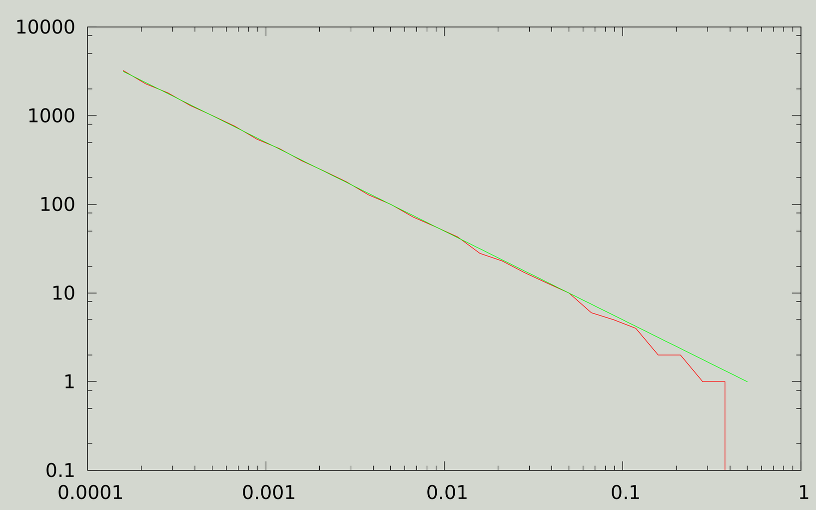
Red: Plot of $\|z_\mu\|_\infty$. **Green:** Plot of the function $2(1-\mu)^{-1/2}$.



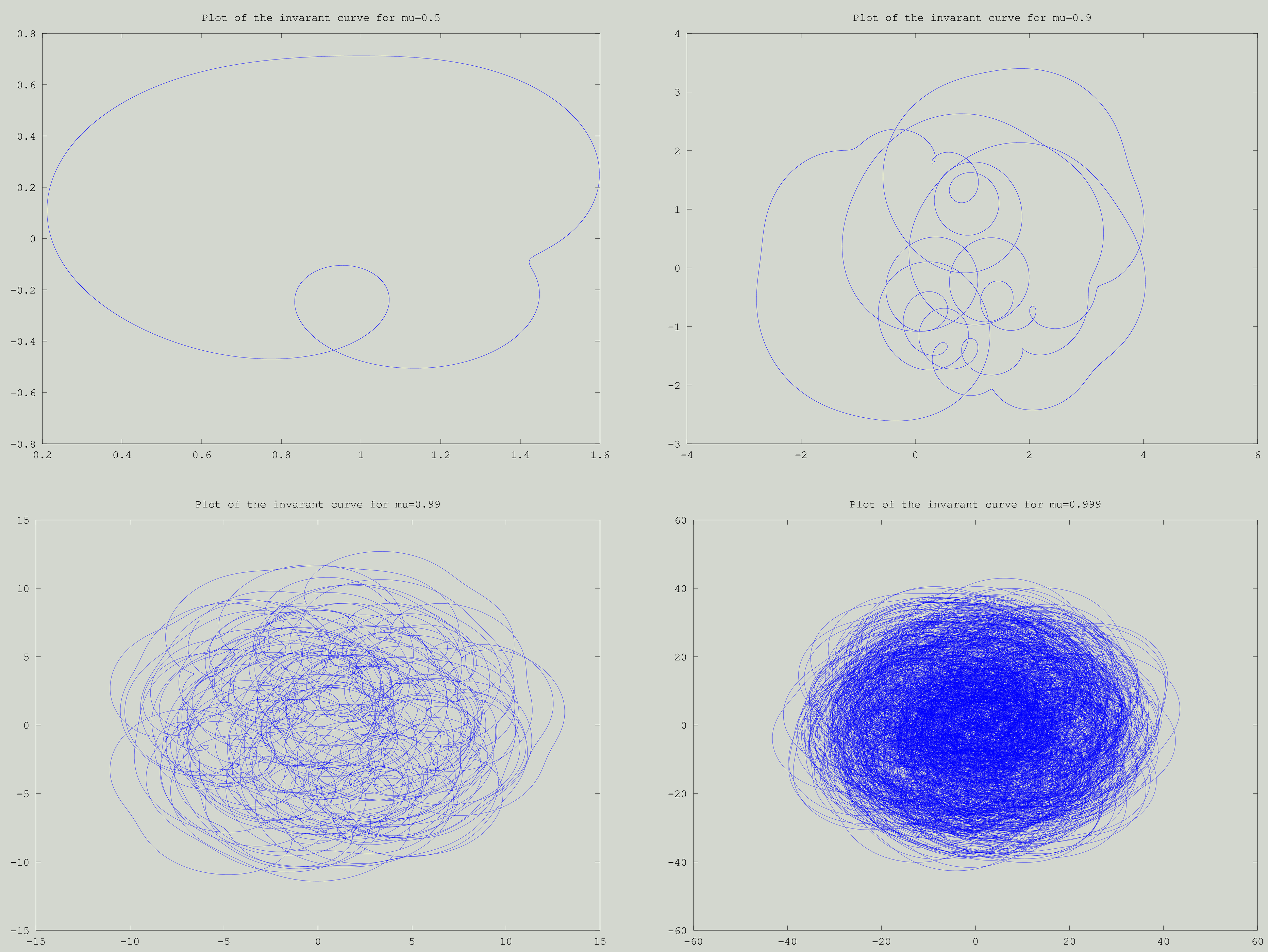
Red: Plot of $\|z'_\mu\|_\infty$. **Green:** Plot of the function $\frac{4}{5}(1-\mu)^{-3/2}$.



Red: Length of z_μ. **Green:** Plot of the function $3(1-\mu)^{-3/2}$.



Red: Plot of **wind** z_μ. **Green:** Plot of the function $\frac{1}{2}(1-\mu)^{-1}$.



Scenario for the breakdown of the curve.

Sketch the proof

We take advantage on the identification between $\mathbb{R}^2$ and the complex plane.

$$\begin{cases} \bar{z} = \mu e^{i\theta} z + c, \\ \bar{\theta} = \theta + \omega. \end{cases}$$

It is possible to compute explicitly the Fourier expansion for the invariant curve:

$$z_\mu(\theta) = c \sum_{k=0}^{\infty} \mu^k e^{-i\frac{k(k+1)}{2}\omega} e^{ik\theta}, \qquad (3)$$

which is convergent whenever $\mu < 1$. The series was studied by Hardy and Littlewood in 1914 , from their work it is easy to see that, if ω is of **constant type**:

$$\|z_\mu\|_\infty = \mathcal{O}(1-\mu)^{-1/2}.$$
$$\|z'_\mu\|_\infty = \mathcal{O}(1-\mu)^{-3/2}.$$

By the principle argument, the winding number of (3) is the number of zeros in the unit disk of the function

$$f(z) = c \sum_{k=0}^{\infty} e^{-i\frac{k(k+1)}{2}\omega} z^k.$$

The winding number, hence, can be defined for a full measure set of μ's and

$$\text{wind } z_\mu = \mathcal{O}(1-\mu)^{-1}.$$

On the reducible case

Assume we deal with a reduced affine system with one dimensional complex coordinates:

$$\begin{cases} \bar{\zeta} = \mu\zeta + b(\theta), \\ \bar{\theta} = \theta + \omega. \end{cases}$$

Assume:

- The rotation number ω is **Diophantine**.
- The independent term b is smooth.

We look for a curve $\zeta_\mu \in C^r(\mathbb{T}, \mathbb{C})$ which satisfies the condition of invariant curve. It is possible to find explicitly the Fourier coefficients.

$$\zeta_k^\mu = \frac{b_k}{e^{ik\omega} - \mu}, \quad k \in \mathbb{Z}.$$

The smoothness of b implies a suitable decay on the values $|b_k|$, then:

- If b is average free, the decay on the values $|b_k|$, imply the smoothness of ζ_μ.
- If b is average different from zero, the invariant curve diverges when $\mu \to 1$ with velocity $\mathcal{O}(1-\mu)^{-1}$ and with bounded derivative.
- **No fractalization occurs** in the reducible case.

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