On a Family of Rational Perturbations of the Doubling Map

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The Blaschke family

We are interested in the Blaschke products of the form

$$B_a(z) = z^3 \frac{z-a}{1-\overline{a}z}$$

where $a \in \mathbb{C}$. They leave $\mathbb{S}^1$ invariant and, when $|a| \to \infty$, converge to $e^{4\pi i \mathrm{Arg}(a)} z^2$ uniformly on compact sets of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The main properties of the B_a are the following:

- If $|a| > 1$, the circle map $B_a|_{\mathbb{S}^1}$ has degree 2. If moreover $|a| \geq 2$, $B_a|_{\mathbb{S}^1}$ is a degree 2 cover.
- They are symmetric w.r.t. $\mathbb{S}^1$, i.e. $B_a(\tau(z)) = \tau(B_a(z))$, where $\tau(z) = 1/\overline{z}$.
- $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3.
- They have two "free" critical points $c_\pm = \frac{a}{3|a|^2}\left(2 + |a|^2 \pm \sqrt{(|a|^2-4)(|a|^2-1)}\right)$.
- B_a and $B_{\xi a}$ are conjugate, where ξ is a third root of unity.

Connectivity of the Julia Set

We study the connectivity of the Julia set $\mathcal{J}(B_a)$ depending on the position of c_+ with respect to the immediate basin of attraction of infinity $A^*(\infty)$.

Theorem. *Given a Blaschke product B_a, the following statements hold:*

- *If $|a| \leq 1$, then $\mathcal{J}(B_a) = \mathbb{S}^1$.*
- *If $|a| > 1$, then $A(\infty)$ and $A(0)$ are simply connected if and only if $c_+ \notin A^*(\infty)$.*
- *If $|a| \geq 2$, then every Fatou component U such that $U \cap A(\infty) = \emptyset$ and $U \cap A(0) = \emptyset$ is simply connected.*

Consequently, if $|a| \geq 2$, then $\mathcal{J}(B_a)$ is connected if and only if $c_+ \notin A^*(\infty)$.

There may exist parameters a, $1 < |a| < 2$, for which B_a has disconnected Julia set.

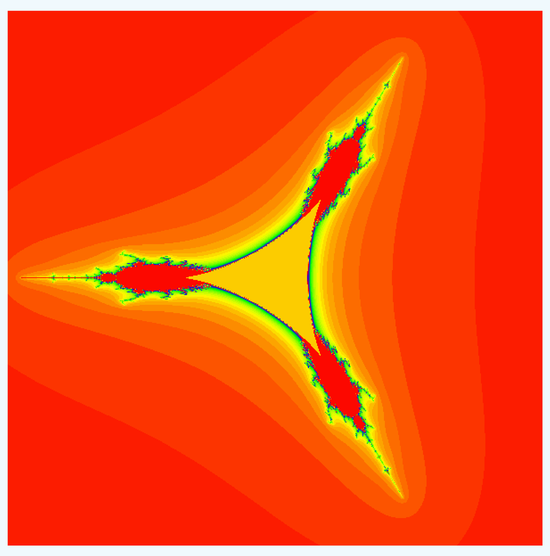
Hyperbolic dynamics

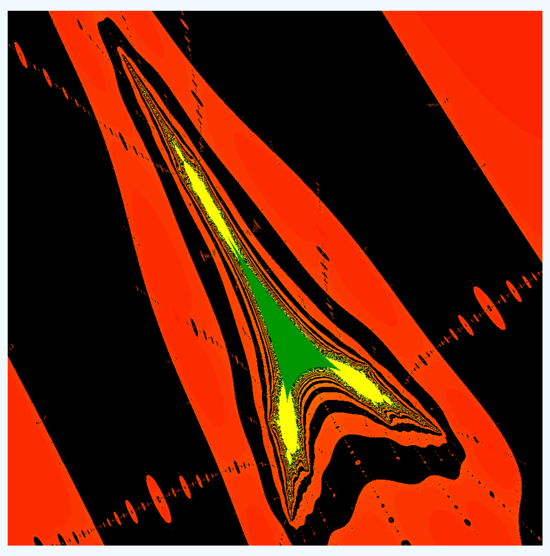
Due to symmetry, either a free critical orbit is captured by the basin of 0 and the other one by the basin of ∞ or they are both bounded in $\mathbb{C}^*$ ($a \in \mathcal{B}$). Using the notation of bicritical maps, if $a \in \mathcal{B}$ and B_a is hyperbolic, we say that the parameter a is **adjacent** if the critical points belong to the same Fatou component, **bitransitive** if the critical points belong to different components of the same immediate basin of attraction, **capture** if one critical point belongs to a preperiodic Fatou component (only possible for $1 < |a| < 2$) or **disjoint** if the critical points belong to different attracting basins.

For $|a| \geq 2$, bitransitive and adjacent parameters are of special interest since they may lead to dynamics related with antipolynomials and tongues.

Relation with antipolynomials

Copies of the Tricorn, the bifurcation locus of the antipolynomials $p_c(z) = \overline{z}^2 + c$ (left figure), seem to appear embedded in the **swapping regions** of parameters for which the critical points enter and exit the unit disk (right figure).





Following Milnor [1], we use the Theory of polynomial-like mappings [2] to explain this phenomenon.

Definition. *A triple $(f; U, V)$ is called a **polynomial-like** mapping of degree d if U and V are bounded simply connected subsets of the plane, $\overline{U} \subset V$ and $f : U \to V$ is holomorphic and proper of degree d.*

Theorem. *Let a_0 be a swapping parameter with an attracting or parabolic cycle of period $p > 1$. Then, there are two open sets U and V and a minimal $p_0 > 1$ dividing p such that $(B_a^{p_0}; U, V)$ is a polynomial-like map. Moreover,*

- *If a_0 is bitransitive, $\left(B_{a_0}^{p_0}; U, V\right)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = \left(z^2 + \overline{c}\right)^2 + c$.*
- *If a_0 is disjoint, $\left(B_{a_0}^{p_0}; U, V\right)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = \left(z^2 + \overline{c}\right)^2 + c$ or of the form $z^2 + c$.*

The Tongues

We follow Misiurewicz and Rodrigues [3] to introduce the concept of tongue for the Blaschke family.

Let a s.t. $|a| \geq 2$. Since $B_a|_{\mathbb{S}^1}$ is a degree 2 covering, $B_a|_{\mathbb{S}^1}$ is semiconjugate to the doubling map $\theta \to 2\theta (\text{mod } 1)$ by a unique continuous map H_a which sends periodic points to periodic points of the same period.

Definition. *We say that a, $|a| \geq 2$, is of **type** τ if $B_a|_{\mathbb{S}^1}$ has an attracting cycle and $H_a(x_0) = \tau$, where x_0 is the marked point of the attracting cycle. We define the **tongue** of type τ as $T_\tau = \{a| \;\; 2 \leq |a|, a \text{ has type } \tau\}$.*

Parameters in orange outside the annulus of radius 1 and 2 (blue) correspond to the fixed tongue T_0.

Connectivity of the Tongues

Following Misiurewicz and Rodrigues [3] Dezotti [4], we prove the following result.

Theorem. *Given any periodic point τ of the doubling map, T_τ is not empty and consists of three connected and simply connected components, each containing a unique parameter a_0 such that B_{a_0} has a superattracting cycle in $\mathbb{S}^1$. Moreover $|a_0| = 2$.*
*The boundary of a connected component of a tongue consists of the union of two curves which depend injectively on $|a|$ and intersect on the **tip** a_τ of the tongue.*

<u>Idea of the proof:</u> First we perform a qc surgery connecting any $a \in T_\tau$ with a parameter $a_0 \in T_\tau$ having a superattracting cycle. The conclusion holds by seeing that there exist only 3 parameters in T_τ having superattracting cycles.

Bifurcations along curves

Bifurcations along curves are known to appear for antipolynomials $P_c(z) = \overline{z}^d + c$ (see [5, 6]). The following result shows that it does happen in a similar way in a neighbourhood of the tip of every tongue.

Theorem. *Given any tongue T_τ, there exists a neighbourhood U of the tip of the tongue in which only the following can occur:*

- *$a \in T_\tau$ and $B_{a|\mathbb{S}^1}$ has an attracting cycle.*
- *$a \in \partial T_\tau$ and $B_{a|\mathbb{S}^1}$ has a parabolic cycle.*
- *$a \notin \overline{T_\tau}$ and B_a has two different attracting cycles not lying in $\mathbb{S}^1$.*

<u>Idea of the proof:</u> Use the holomorphic index of the periodic cycles obtained when perturbing a parameter on the tip of any tongue.

Extending the Tongues

Definition. *An extended tongue ET_τ is defined to be the set of parameters for which the attracting cycle of T_τ can be continued.*

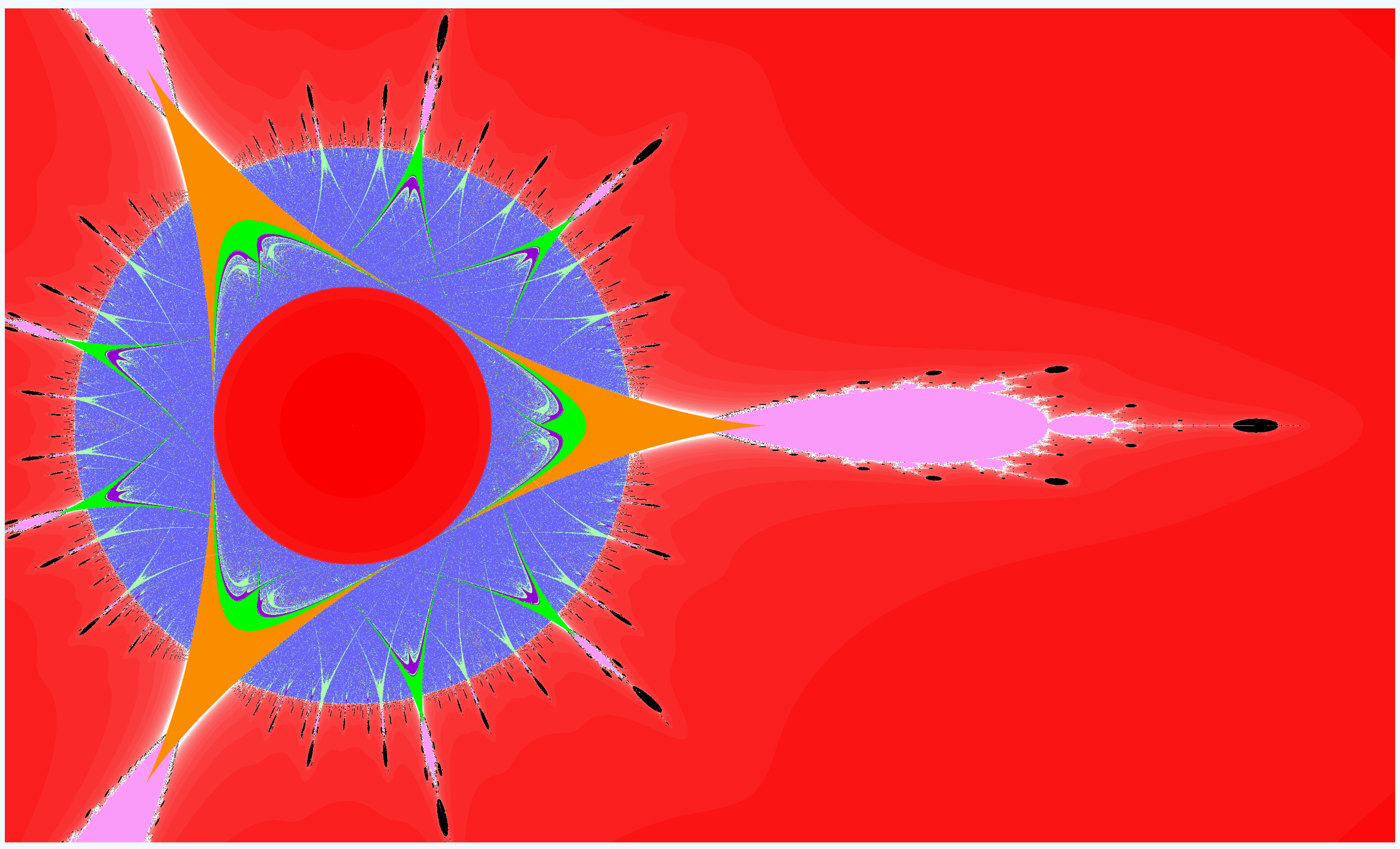
The big figure shows, in color orange, the extended fixed tongue ET_0. Notice that, since there are two different critical points moving independently, two different tongues may intersect each other.

<u>Work in progress:</u> We are studying how these tongues extend through the annulus $1 < |a| < 2$.

Parameter plane

The Blaschke products B_a satisfy the following, depending on the modulus of the parameter.

- If $|a| \leq 1$, there are no other attractors than 0 and ∞.
- If $1 < |a| < 2$, there are two different critical points $c_+, c_- \in \mathbb{S}^1$.
- If $|a| > 2$, B_a has a critical point $c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and a critical point $c_- \in \mathbb{D}^*$. For $|a| = 2$ those points collapse in $c \in \mathbb{S}^1$.



Parameter plane of the Blaschke family. The parameters correspond to $-0.5 < \mathrm{Re(a)} < 7.5$ and $-3 < \mathrm{Im(a)} < 3$. We have plotted in orange the parameters for which there is an attracting fixed point in $\mathbb{S}^1$. Strong green corresponds to parameters having a period 2 attracting cycle in the unit circle, whereas violet corresponds to period 4 cycles; red for $c_+ \in A(\infty)$, black for $c_+ \in A(0)$, pallid green if $O^+(c_+)$ accumulates on a periodic orbit in $\mathbb{S}^1$, pink if $O^+(c_+)$ accumulates in a periodic orbit not in $\mathbb{S}^1$ and blue in any other case.

References

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[6] J. H. Hubbard and D. Schleicher, Multicorns are not Path Connected *arXiv:1209.1753 (2012)*