DARBOUX THEORY OF INTEGRABILITY ON THE CLIFFORD n-DIMENSIONAL TORUS

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ABSTRACT. For the polynomial vector fields on a Clifford n-dimensional torus, we develop a Darboux theory of integrability. Moreover, we study the optimal maximal number of invariant meridians in terms of the degree of the polynomial vector field.

1. Introduction and statement of the main results

Nonlinear ordinary differential equations are vastly used to model processes in many fields. First integrals are important in particular because they help to obtain the phase portrait of the system and to reduce the dimension of the system by its number of independent first integrals. For all this, the corresponding methods are very important.

The existence of first integrals for non Hamiltonian vector fields can be studied for example using Noether symmetries [3], the Darboux theory of integrability [9], Lie symmetries [25], the Painlevé analysis [2], the use of Lax pairs [14], and the direct method [11] and [12]. There are also many extensions to $\mathbb{R}^n$. In particular, the Darboux theory of integrability can be applied to polynomial vector fields using a sufficient number of invariant algebraic hypersurfaces. It was extended succesfully to $\mathbb{R}^2$ [4, 5, 6, 7, 9, 13, 15, 23, 26, 27, 28, 29, 30, 31, 32] and to $\mathbb{R}^n$ [16, 17, 18, 20, 21, 22, 24].

In this paper we first develop a Darboux theory of integrability on the n-dimensional Clifford torus $\mathbb{T}$ and, second, we study the maximal number of invariant meridians of polynomial vector fields on this torus.

We recall that the Clifford n-dimensional torus $\mathbb{T}$ is the n-dimensional torus whose circles have all equal radius. A torus of this type embeds into $\mathbb{R}^{2n}$ by the parametrization

$$x_i = \cos\theta_i, \quad y_i = \sin\theta_i, \quad i = 1, \dots, n.$$

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Following partly [20], we recall a few necessary definitions. Given a C^1 map $G\colon \mathbb{R}^\ell \to \mathbb{R}$, a hypersurface

$$S = \{(x_1, \dots, x_\ell) \in \mathbb{R}^\ell \colon G(x_1, \dots, x_\ell) = 0\}$$

is *regular* if $\nabla G \neq 0$ on S. A hypersurface S is *algebraic* of *degree* d if G is an irreducible polynomial of degree d.

A polynomial vector field $X = (P_1, \dots, P_\ell)$ on a regular hypersurface S is a polynomial vector field satisfying $X \cdot \nabla G = 0$ on S. An algebraic hypersurface $\{f = 0\} \cap S \subset \mathbb{R}^\ell$ is said to be *invariant* under a polynomial vector field X if:

(a) for some $k \in \mathbb{C}[x_1, \dots, x_\ell]$ (the *cofactor* of $f = 0$ on S) we have

$$Xf = \sum_{i=1}^{\ell} P_i \frac{\partial f}{\partial x_i} = kf \quad \text{on } S; \tag{1}$$

(b) the hypersurfaces $f = 0$ and S are transverse.

Note that X is tangent to $\{f = 0\} \cap S$. Hence, the intersection is composed by orbits of X.

Assume that X has degree m. We say that $F = F(x_1, \dots, x_\ell) = \exp(g/h)$ is an *exponential factor* of X on the regular hypersurface S if $g, h \in \mathbb{C}[x_1, \dots, x_\ell]$ and $XF = LF$ on S for some $L \in \mathbb{C}_{m-1}[x_1, \dots, x_\ell]$ (the set of polynomials in $\mathbb{C}[x_1, \dots, x_\ell]$ of degree at most $m-1$). Given a regular algebraic hypersurface $S = \{G = 0\}$ in $\mathbb{R}^\ell$, two polynomials $f, g \in \mathbb{C}_m[x_1, \dots, x_\ell]$ are said to be *related* (and we write $f \sim g$), if $f/g = \text{constant}$ or $f - g = hG$ for some polynomial h. One can easily verify that $\sim$ is an equivalence relation.

The dimension $d(m)$ of $\mathbb{C}_m[x_1, \dots, x_\ell]/\sim$ is called the dimension of $\mathbb{C}_m[x_1, \dots, x_\ell]$ on S. It is proved in [20, Proposition 1] that

$$d(m) = \binom{\ell + m}{\ell} - \binom{\ell + m - d}{\ell}, \tag{2}$$

where d is the degree of the algebraic hypersurface S.

Now take $f, g \in \mathbb{C}_m[x_1, \dots, x_\ell]$ and let $S = \{G_1 = 0\} \cap \dots \cap \{G_q = 0\}$ be the intersection of q regular algebraic hypersurfaces in $\mathbb{R}^\ell$ of degree d_i for $i = 1, \dots, q$. Similarly, we say that f and g are *related* (and again we write $f \sim g$), if either $f/g = \text{constant}$ or $f - h = \sum_{i=1}^{q} h_i G_i$ for some polynomials h_i. Then $\sim$ is an equivalence relation in $\mathbb{C}_m[x_1, \dots, x_\ell]$. We denote the quotient space $\mathbb{C}_m[x_1, \dots, x_\ell]/\sim$ by $d(m)$ and called it the dimension of $\mathbb{C}_m[x_1, \ldots, x_\ell]$ on S. It follows from (2) that

$$d(m) = \binom{\ell+m}{\ell} - \sum_{i=1}^{q} \binom{\ell+m-d_i}{\ell}.$$

Given an open set $U \in \mathbb{R}^\ell$, a function $H(x_1, \ldots, x_\ell, t)\colon \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}$ is said to be an *invariant of* X *on* $S \cap U$ if $H(x_1(t), \ldots, x_\ell(t), t)=$ constant for all t such that $(x_1(t), \ldots, x_\ell(t))$ belongs to $S \cap U$. When it is independent of t we call it a *first integral* and when it is a rational function we call it *rational first integral.*

Now we present the extension of the Darboux theory of integrability to polynomial vector fields on $\mathbb{T}$.

In the case of $\mathbb{T} = (\mathbb{S}^1)^n$, i.e. the Clifford n-dimensional torus, we have that $d_i = 2$ for $i = 1, \ldots, n$ and so

$$\begin{aligned} d(m) &= \binom{2n+m}{2n} - \sum_{i=1}^{n} \binom{2n+m-2}{2n} \\ &= \frac{(2n+m-2)!}{(2n)!m!}\big((2n+m)(2n+m-1) - nm(m-1)\big). \end{aligned}$$

Note that for the Clifford torus $\mathbb{T}$ it is necessary that $m \geq 2$. Moreover, $m = 2$ then $d(2) = 2n^2 + 2n + 1$.

We recall the following result (see [16, 20]).

Theorem 1. *Assume that* $X = (P_1, \ldots, P_n)$ *is a polynomial vector field on* $\mathbb{T}$ *of degree* $m = (m_1, \ldots, m_n)$, *i.e.* $\deg P_i = m_i$, *having* p *invariant algebraic hypersurfaces* $\{f_i = 0\} \cap \mathbb{T}$ *with cofactors* K_i *for* $i = 1, \ldots, p$ *and* q *exponential factors* $F_1, \ldots, F_q$ *with* $F_j = \exp(g_j/h_j)$ *with cofactors* L_j *for* $j = 1, \ldots, q$. *Then the following statements hold.*

(a) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0 \quad \text{on } \mathbb{T},$$

if and only if the real (multi-valued) function of Darboux type $f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$ *substituting* $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbb{R}$ *is a first integral of the vector field* X *on* $\mathbb{T}$.

(b) *If* $p+q \geq d(m)+1$ *then there exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ *on* $\mathbb{T}$.

(c) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\sigma \quad on\ \mathbb{T}$$

for some $\sigma \in \mathbb{R} \setminus \{0\}$ *if and only if the real (multi-valued) function of Darboux type* $f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{\sigma t}$ *substituting* $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbb{R}$ *is an invariant of the vector field* X *on* $\mathbb{T}$.

(d) *The vector field* X *on* $\mathbb{T}$ *has a rational first integral if and only if* $p+q \geq d(m)+n$. *Moreover all the trajectories are contained in invariant algebraic hypersurfaces.*

See [20] for the proof of statements (a), (b) and (c) and see [16] for the proof of statement (d).

We shall use extactic polynomials [10] (see also [1]) for obtaining invariant algebraic hypersurfaces. For a finitely generated subspace W of $\mathbb{C}[x_1, \ldots, x_d]$ with basis $\{v_1, \ldots, v_l\}$, the *extactic polynomial* of a polynomial vector field X associated to W is defined by

$$\mathcal{E}_W(X) = \mathcal{E}_{\{v_1,\ldots,v_l\}}(X) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \vdots & \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix}$$

(although it does not depend on the choice of the basis). We shall use the following result from [8].

Proposition 2. *Let* W *be a finitely generated vector subspace of dimension* $\dim W > 1$ *of* $\mathbb{C}[x_1, \ldots, x_d]$. *Then every algebraic invariant hypersurface* $f = 0$ *of a polynomial vector field* X *in* $\mathbb{C}^d$, *with* $f \in W$, *is a factor of the polynomial* $\mathcal{E}_W(X)$.

For all $(a_i, b_i) \in \mathbb{R}^2$ such that $a_i^2 + b_i^2 = 1$, a *meridian* of $\mathbb{T}$ is defined by

$$\Big\{(x_1, x_2, a_1, b_1, \ldots, a_{n-1}, b_{n-1}) : x_1^2 + x_2^2 = 1\Big\}. \tag{3}$$

The following result gives the maximal number of invariant meridians that a polynomial vector field X on $\mathbb{T}$ can have as function of its degree.

Theorem 3. *Let* X *be a polynomial vector field on the Clifford* n*-dimensional torus* $\mathbb{T}$ *of degree* $m = (m_1, \ldots, m_{2n})$ *with* $m_1 \geq m_2 > 0$ *and* $m_{2n-1} \geq m_{2n} > 0$. *Assume that* X *has finitely many invariant* *meridians. Then their number is at most* $2(m_2 - 1)$ *taking into account their multiplicities.*

Theorem 3 is proved in section 2. The case when $n = 2$ was treated in [19].

2. Proof of Theorem 3

Before proving Theorem 3 we state and prove an auxiliary result.

Proposition 4. *The polynomial differential systems* $X = (P_1, \ldots, P_{2n})$ *having an invariant Clifford* n*-dimensional torus* $\mathbb{T}$ *are*

$$P_{2j+1} = A_j(x_{2j+1}^2 + x_{2j+2}^2 - 1) - 2C_j x_{2j+2},$$
$$P_{2j+2} = B_j(x_{2j+1}^2 + x_{2j+2}^2 - 1) + 2C_j x_{2j+1},$$

for $j = 0, 1, \ldots, n-1$*, where* A_j, B_j, C_j *are arbitrary polynomials in* $(x_1, \ldots, x_{2n})$*.*

Proof. Fix $j \in \{1, \ldots, n-1\}$ and take $f^{(j)} = x_{2j+1}^2 + x_{2j+2}^2 - 1$. Since there are no points at which $f^{(j)}, f^{(j)}_{x_{2j+1}}, f^{(j)}_{x_{2j+2}}$ vanish simultaneously, from Hilbert's nullstellensatz we obtain that there exist polynomials $E^{(j)}, F^{(j)}, G^{(j)}$ such that

$$E^{(j)} f^{(j)}_{x_{2j+1}} + F^{(j)} f^{(j)}_{x_{2j+2}} + G^{(j)} f^{(j)} = 1. \tag{4}$$

If X is a polynomial vector field on $\mathbb{T}$, then $f^{(j)} = 0$ is an invariant hypersurface of X with cofactor $K^{(j)}$. As $f^{(j)}$ satisfies equation (1) we get from (1) and (4) that

$$K^{(j)} = (K^{(j)}E^{(j)} + G^{(j)}P_{2j+1}) f^{(j)}_{x_{2j+1}} + (K^{(j)}F^{(j)} + G^{(j)}P_{2j}) f^{(j)}_{x_{2j+2}}.$$

Substituting $K^{(j)}$ into (1) we get

$$\begin{aligned} &\big(P_{2j+1} - (K^{(j)}E^{(j)} + G^{(j)}P_{2j+1}) f^{(j)}\big) f^{(j)}_{x_{2j+1}} \\ &= -\big(P_{2j+2} - (K^{(j)}F^{(j)} + G^{(j)}P^{(j)}_{2j}) f^{(j)}\big) f^{(j)}_{x_{2j+2}}. \end{aligned}$$

Since $(f^{(j)}_{x_{2j+1}}, f^{(j)}_{x_{2j+2}}) = 1$, there exists a polynomial $D^{(j)}$ such that

$$P_{2j+1} - (K^{(j)}E^{(j)} + G^{(j)}P_{2j+1}) f^{(j)} = -D^{(j)} f^{(j)}_{x_{2j+2}} = -D^{(j)} x_{2j+2}$$

and

$$P_{2j+2} - (K^{(j)}F^{(j)} + G^{(j)}P_{2j+2}) f^{(j)} = D^{(j)} f^{(j)}_{x_{2j+1}} = D^{(j)} x_{2j+1}.$$

This proves the theorem for P_{2j+1} and P_{2j} taking $A_j = K^{(j)}E^{(j)} + G^{(j)}P_{2j+1}$ $B_j = K^{(j)}F^{(j)} + G^{(j)}P_{2j+2}$ and $C_j = D^{(j)}$. Since this procedure can be done for any j the proof of the proposition is complete. $\square$

We finally establish our main result.

Proof of Theorem 3. A meridian of the Clifford n-dimensional torus $\mathbb{T}$ is obtained intersecting $\mathbb{T}$ with the hyperplanes $x_1 = a$ and $x_2 = b$ taking $a^2 + b^2 = 1$ (see (3)). Therefore the hyperplanes $x_1 - a = 0$ and $x_2 - b = 0$ must be invariant under the polynomial vector field X. In view of Proposition 2, the polynomial $x_1 - a$ must divide the extactic polynolmial

$$\begin{aligned}\mathcal{E}_{\{1,x_1\}} &= \det\begin{pmatrix} 1 & x_1 \\ X(1) & X(x_1)\end{pmatrix} = \det\begin{pmatrix} 1 & x_1 \\ 0 & P_1\end{pmatrix} \\ &= P_1(x_1, x_2, \ldots, x_{2n-1}, x_{2n}),\end{aligned}$$

and so $x_1 - a$ must divide the polynomial $P_1(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$. In a similar way we have that $x_2 - b$ must divide the extactic polynolmial

$$\begin{aligned}\mathcal{E}_{\{1,x_2\}} &= \det\begin{pmatrix} 1 & x_2 \\ X(1) & X(x_2)\end{pmatrix} = \det\begin{pmatrix} 1 & x_2 \\ 0 & P_2\end{pmatrix} \\ &= P_2(x_1, x_2, \ldots, x_{2n-1}, x_{2n}),\end{aligned}$$

and so $x_2 - b$ must divide the polynomial $P_2(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$.

Since the degree of P_2 is m_2, it follows that the polynomials of the form $x_2 - b$ can divide the polynomial P_2 at most m_2 times. If this is the case then

$$P_2 = \kappa \prod_{j=1}^{m_2} (x_2 - b_j)$$

with $\kappa \in \mathbb{R} \setminus \{0\}$ and $|b_j| \leq 1$ (so that we have a meridian).

It follows from Proposition 4 that

$$P_2 = B_1(x_1^2 + x_2^2 - 1) + 2C_1 x_1.$$

Therefore we have that

$$B_1(x_1^2 + x_2^2 - 1) = -2C_1 x_1 + \kappa_4 \prod_{j=1}^{m_2} (x_2 - b_j).$$

Hence, from $x_1 = 0$ it follows that $x_1^2 - 1$ must divide $\prod_{j=1}^{m_2} (x_2 - b_j)$. Then the two planes $x_2 = \pm 1$ can only produce meridians with $x_1 = 0$. So the two planes $x_2 = \pm 1$ can produce at most two meridians. The other $m_2 - 2$ planes $x_2 = b_j \neq 1$ can produce each one at most two meridians. We conclude that the maximum number of meridians is $2(m_2 - 2) + 2 = 2(m_2 - 1)$. $\square$

Now we provide an example realizing the upper bound for the meridians provided in Theorem 3 (thus showing that the upper bound is optimal). Consider the vector field X on the Clifford n-dimensional torus $\mathbb{T}$ given by

$$X=\sum_{j=0}^{n-1}(x_{2j+1}^2+x_{2j+2}^2-1)\frac{\partial}{\partial x_{2j+1}}+x_{2j+1}x_{2j+2}\frac{\partial}{\partial x_{2j}},$$

thus of degree $(2,\ldots,2)$. We prove that the upper bound $2(m_1-1)=2$ for the number of meridians provided in Theorem 3 is attained. Since

$$X(x_{2j+1}^2+x_{2j+2}^2-1)=2x_{2j+1}(x_{2j+1}^2+x_{2j+2}^2-1)$$

for $j=0,\ldots,n-1$ it follows that X defines a vector field on $\mathbb{T}$.

Note that

$$(x_1,x_2,a_1,b_1,\ldots,a_{n-2},b_{n-2})=(1,0,a_1,b_1,\ldots,a_{n-2},b_{n-2}),$$
$$(x_1,x_2,a_1,b_1,\ldots,a_{n-2},b_{n-2})=(-1,0,a_1,b_1,\ldots,a_{n-2},b_{n-2}),$$

for any $a_j,b_j\in\mathbb{R}$ satisfying $a_j^2+b_j^2=1$ are two meridians for X.

3. Conclusions

This paper is devoted to the Darboux theory of integrability for the polynomial vector fields on the Clifford torus. In Theorem 1 we summarize what is known for the Clifford torus. The theory is based on the study of the invariant algebraic hypersurfaces of polynomial vector fields.

One of the best tools for obtaining invariant algebraic hypersurfaces for a polynomial vector field is the extactic polynomial (see Proposition 2 for the precise relation between both of them). While in Theorem 1 the extactic polynomial is not present, in Theorem 3 proven in our paper we use it for studying the maximal number of invariant meridians that a polynomial vector field on the Clifford torus can exhibit as a function of its degree.

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