TANGENTIAL TRAPEZOID CENTRAL CONFIGURATIONS

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Abstract. A tangential trapezoid, also called a circumscribed trapezoid, is a trapezoid whose four sides are all tangent to a circle within the trapezoid: the in-circle or inscribed circle. In this paper we classify all planar four-body central configurations, where the four bodies are at the vertices of a tangential trapezoid.

1. Introduction and statement of the results

The classical n-body problem concerns the study of the dynamics of n particles interacting among themselves by their mutual attraction according to Newtonian gravity.

Let $x_i \in \mathbb{R}^d (i = 1, \ldots, n)$ denotes the position vector of the i-body, $m_i \in \mathbb{R}^+ (i = 1, \ldots, n)$ denotes the mass of the i-body. $\mathbb{R}^d$ is the Euclidean space ($d = 2$ or 3). By Newton's law of motion and Newton's gravitational law the equations of the motion of the n-body problem are governed by

$$\ddot{x}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j(x_i - x_j)}{r_{ij}^3}, \quad 1 \leq i \leq n.$$

Where $r_{ij} = |x_i - x_j|$ is the mutual Euclidean distance between the i-body and the j-body. Here we take the gravitational constant $G = 1$.

The vector $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ is called the *configuration* of the system. Define $\delta(x)$, the dimension of a configuration x, i.e. the dimension of the smallest affine space of $\mathbb{R}^d$ containing all of the points x_i. Configurations with $\delta(x) = 1, 2, 3$ are called collinear, planar, spacial, respectively.

When $n = 2$ the n-body problem has been completely solved. However for the n-body problem for $n \geq 3$ the complete solution remains open.

Let

$$M = m_1 + \cdots + m_n, \quad c = \frac{m_1 x_1 + \cdots + m_n x_n}{M},$$

be the total mass and the center of masses of the n bodies, respectively.

A *configuration* x is called a *central configuration* if the acceleration vectors of the n bodies are proportional to their positions with respect to the center of masses

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with the same constant of proportionality, i.e.

$$\sum_{j=1, j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3} = \lambda(x_i - c), \quad 1 \leq i \leq n, \tag{1}$$

where λ is the constant of proportionality.

A *central configuration* is invariant under isometries and homotheties with respect to the center of masses. So we say that two central configurations $x = (x_1, \ldots, x_n)$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ are *equivalent* if we can pass from one to the other through a dilation or a rotation with respect to the center of the mass. This relation is of equivalence. Therefore when studying central configurations, we only count the classes of central configurations defined by the above equivalence relation.

Central configurations play an important role in celestial mechanics. First, the knowledge of central configurations allows us to obtain homographic solutions of the n-body problem (see [35]). We recall that a homographic solution of the n-body problem is a solution such that the configuration remains constant up to rotation and scaling. Second, there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy h and angular momentum c (see [37, 46]). Third, if the n bodies are going to a simultaneous collision or to a total parabolic escape to infinity, then the configuration of n bodies is asymptotic to a central configuration(see [20, 25, 43, 47]).

There is an extensive literature on the study of central configurations, see Euler [21], Lagrange [28], Albouy [1, 2], Albouy and Chenciner [3], Albouy and Fu [4], Albouy and Kaloshin [6], Davis et al. [18], Hampton and Moeckel [26], Moeckel [36], Llibre [22, 23, 30], Long [33], *É*rdi and Czirj*á*k [20], Moulton [38], Palmore [39], Schmidt [44], Smale [46], Xia [48, 49], Xie [50], . . .

In this paper we are interested in the planar 4- body problem. Sim*ó* [45] studied numerically the class of central configurations for the 4-body problem with arbitrary masses . In 2006 the finiteness of central configurations for the 4-body problem has been proved by Hampton and Moeckel [26] with the assistance of a computer. Later on Albouy and Kalsoshin [6] proved this result without using the computer.

For $m_1 = m_2 = m_3 = m_4$ Llibre found all the planar central configurations by studying the intersections of two curves and assuming that the central configurations have a straight line of symmetry, see [30]. Later on Albouy proved the existence of such symmetry and provide a more analytical proof for the central configurations with equal masses.

When one of the 4 masses is sufficiently small, Pedersen [40], Barros and Leandro [12, 13] found the classes of central configurations of the 4 body, also see Arenstorf [11], Fernandes et al [23] and Gannaway [24].

The central configurations with three equal masses was studied by Bernat at al. They classified the non-collinear kite central configurations. For more details, see [14], also see [29].

In 2010 Pi*ñ*a and Lonngi [42] found new properties of the symmetric and non-symmetric central configurations for the 4-body problem.

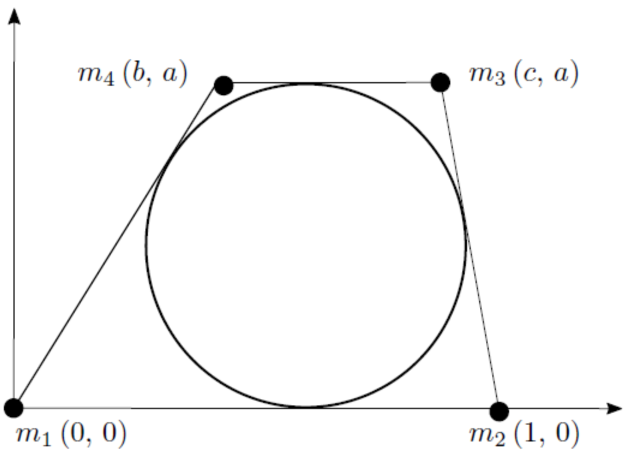

Figure 1. A tangential trapezoid.

MacMillan and Bartky [34] proved that there is a unique isosceles trapezoid central configuration for the 4-body problem when two pairs of equal masses are located at the adjacent vertices of a trapezoid. Long and Sun [33] studied the convex central configurations with three equal masses and they proved that the central configurations must posses a symmetry. Pérez-Chavela and Santoprete [41] generalized further and proved that central configurations must posses such symmetry when two equal masses are located at the opposite vertices of a quadrilateral and at most only one of the remaining masses is larger than the equal masses. Later on Albouy [5] obtain the symmetry of convex central configuration with two equal masses at the opposite vertices.

For $m_1 = m_2 \neq m_3 = m_4$ Álvarez and Llibre [7] characterized the convex and concave central configurations with an axis of symmetry.

Using these previous results on the symmetries Corbera and Llibre [15] gave a complete description of the families of central configurations with two pairs of equal masses and two equal masses sufficiently small.

Cors and Robert [16] studied the case when 4 masses are located at the vertices of a cyclic quadrilateral, see also [10].

Recently Álvarez and Llibre [8, 9] classified the Hjelmslev and the equilic quadrilateral central configurations.

The trapezoid central configurations have been studied in [16], here we want to see which of these trapezoid central configurations are tangential.

A *tangential trapezoid,* also called a circumscribed trapezoid, is a trapezoid whose four sides are all tangent to a circle within the trapezoid: the in-circle or inscribed circle.

Without loss of generality we take $m_1 = 1$ and assume that the positions of four bodies at the vertices of a trapezoid are

$$x_1 = (0, 0), \quad x_2 = (1, 0), \quad x_3 = (c, a), \quad x_4 = (b, a), \tag{2}$$

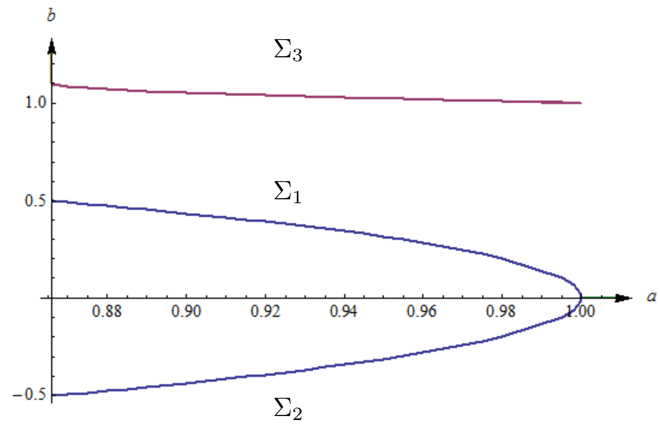

Figure 2. Tangential trapezoid central configurations Σ_1, Σ_2, Σ_3.

where $a > 0$.

Lemma 1. *If the configuration of* 4 *masses is a tangential trapezoid with the vertices* x_1, x_2, x_3 *and* x_4*, as it is shown in Figure* 1 *, then*

$$c = \frac{(1-b)(a^2+2(b+\sqrt{a^2+b^2}))}{4-a^2-4b}. \tag{3}$$

Lemma 1 is proved in section 3.

We characterize all planar 4-body problem central configurations, where the four bodies are at the vertices of a tangential trapezoid.

Theorem 2. *We take positive masses for the* 4*-body problem with* $m_1 = 1$*. Then we have a tangential trapezoid central configuration given by* (2) *for each value of* $(a,\, b)$ *in the arc* Σ_1, Σ_2, Σ_3*, see Figure* 2*.*

(a) *The arc* Σ_1 *and* Σ_2 *is symmetric with respect to the* a*-axis. The arc* Σ_1 *goes from the point* $(1,\, 0)$ *to the point* $(\frac{\sqrt{3}}{2},\, \frac{1}{2})$*, it is open at* $(\frac{\sqrt{3}}{2},\, \frac{1}{2})$ *and closed at* $(1,\, 0)$*. On* Σ_1 *we have* $m_1 = m_3 = 1$, $m_2 = m_4 > 1$*, and* $m_2 = m_4$ *increases from* 1 *to* ∞*. The arc* Σ_2 *goes from the point* $(1,\, 0)$ *to the point* $(\frac{\sqrt{3}}{2},\, -\frac{1}{2})$*, it is open at* $(\frac{\sqrt{3}}{2},\, -\frac{1}{2})$ *and closed at* $(1,\, 0)$*. On* Σ_2 *we have* $m_1 = m_3 = 1$, $m_2 = m_4 < 1$*, and* $m_2 = m_4$ *decreases from* 1 *to* 0*.*

(b) *The value* $(a, b) = (1,\, 0)$ *corresponds to the tangential trapezoid given by the square with four equal masses at its vertices.*

(c) *The arc* Σ_3 *goes from the point* $(1,\, 1)$ *to the point* $(\frac{\sqrt{3}}{2},\, \frac{11}{10})$*, it is closed at* $(1,\, 1)$ *and open at* $(\frac{\sqrt{3}}{2},\, \frac{11}{10})$*. On* Σ_3 *we have* $m_2 > 1 > m_3 > m_4$*, and* m_2 *increases from* 1 *to* 1.24789..*,* m_3 *decreases from* 1 *to* 0.29184..*,* m_4 *decreases from* 1 *to* 0*.*

(d) *The value* $(a, b) = (1,\, 1) \in \Sigma_3$ *corresponds to the tangential trapezoid given by the square with four equal masses at its vertices.*

Theorem 2 is proved in section 3.

2. Dziobek's equations

In this section we present the equations of the central configurations provided by Dziobek for the 4-body problem.

Let $x = (x_1, x_2, x_3, x_4) \in (\mathbb{R}^2)^4$. We associated with x the matrix:

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

X_k denotes the matrix obtained deleting from the matrix X its k-th column and its last row. Then let $D_k = (-1)^{k+1} \det(X_k)$ for $k = 1, \ldots, 4$.

The equations for the central configurations (1) of the 4-body problem were written by Dziobek [17] (see also equations (8) and (16) of Moeckel [35] or [24]) as the following 12 equations with 12 unknowns.

$$\begin{aligned} e_{i,j} = c_1 + c_2 \frac{D_i D_j}{m_i m_j} - \frac{1}{r_{ij}^3} &= 0, \\ t_i - t_j &= 0, \end{aligned} \tag{4}$$

for $1 \leq i < j \leq 4$, with

$$t_i = \sum_{j=1, j\neq i}^{4} D_j r_{ij}^2.$$

The unknowns of equation (4) are the mutual distances r_{ij}, the variables D_i, and the constants $c_k (k = 1, 2)$.

The first six Dziobek's equation (4) are

$$\begin{aligned} m_1 m_2 (r_{12}^{-3} - c_1) &= c_2 D_1 D_2, \quad & m_3 m_4 (r_{34}^{-3} - c_1) &= c_2 D_3 D_4, \\ m_1 m_3 (r_{13}^{-3} - c_1) &= c_2 D_1 D_3, \quad & m_2 m_4 (r_{24}^{-3} - c_1) &= c_2 D_2 D_4, \\ m_1 m_4 (r_{14}^{-3} - c_1) &= c_2 D_1 D_4, \quad & m_2 m_3 (r_{23}^{-3} - c_1) &= c_2 D_2 D_3, \end{aligned} \tag{5}$$

where

$$D_1 = \begin{vmatrix} 1 & c & b \\ 0 & a & a \\ 1 & 1 & 1 \end{vmatrix} = a(c-b), \qquad D_2 = \begin{vmatrix} b & c & 0 \\ a & a & 0 \\ 1 & 1 & 1 \end{vmatrix} = a(b-c),$$

$$D_3 = \begin{vmatrix} 0 & 1 & b \\ 0 & 0 & a \\ 1 & 1 & 1 \end{vmatrix} = a, \qquad D_4 = \begin{vmatrix} c & 1 & 0 \\ a & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -a.$$

Multiplying equations (5) in order that each one at the right has the expression $c_2^2 D_1 D_2 D_3 D_4$, and since the masses must be positive we obtain the Dziobek relation:

$$(r_{12}^{-3} - c_1)(r_{34}^{-3} - c_1) = (r_{13}^{-3} - c_1)(r_{24}^{-3} - c_1) = (r_{14}^{-3} - c_1)(r_{23}^{-3} - c_1). \tag{6}$$

The relation holds for every planar central configuration of the 4-body problem.

We can solve c_1 from the Dziobek relation and we have

$$
c_1 = \frac{r_{12}^{-3} r_{34}^{-3} - r_{13}^{-3} r_{24}^{-3}}{r_{12}^{-3} + r_{34}^{-3} - r_{13}^{-3} - r_{24}^{-3}}
= \frac{r_{13}^{-3} r_{24}^{-3} - r_{14}^{-3} r_{23}^{-3}}{r_{13}^{-3} + r_{24}^{-3} - r_{14}^{-3} - r_{23}^{-3}}
= \frac{r_{14}^{-3} r_{23}^{-3} - r_{12}^{-3} r_{34}^{-3}}{r_{14}^{-3} + r_{23}^{-3} - r_{12}^{-3} - r_{34}^{-3}}. \tag{7}
$$

Defining

$$
\begin{aligned}
s_1 &= r_{12}^{-3} + r_{34}^{-3}, & p_1 &= r_{12}^{-3} r_{34}^{-3},\\
s_2 &= r_{13}^{-3} + r_{24}^{-3}, & p_2 &= r_{13}^{-3} r_{24}^{-3},\\
s_3 &= r_{14}^{-3} + r_{23}^{-3}, & p_3 &= r_{14}^{-3} r_{23}^{-3},
\end{aligned}
$$

then equation (7) becomes

$$
c_1 = \frac{p_1 - p_2}{s_1 - s_2} = \frac{p_2 - p_3}{s_2 - s_3} = \frac{p_3 - p_1}{s_3 - s_1}, \tag{8}
$$

which imply that the point (s_1, p_1), (s_2, p_2), (s_3, p_3) are on the same line with slope c_1, i.e.

$$
\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = 0.
$$

So we can write Dziobek equation as the following nice factorization

$$
D = (r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) = 0. \tag{9}
$$

The equation $D = 0$ must be satisfied for every planar central configuration of the 4-body problem.

3. Proofs of Lemma 1 and Theorem 2

For proving Lemma 1 we shall use Pitot's Theorem which states that *in a tangential quadrilateral the two sums of lengths of opposite sides are the same*, for a proof see for instance [27].

Proof of Lemma 1. By Pitot's and Pythagoras Theorems and Figure 1 we get that

$$
1 + c - b = \sqrt{a^2 + b^2} + \sqrt{(1-c)^2 + a^2}.
$$

Isolating c from this equality we obtain the conclusion of the lemma. □

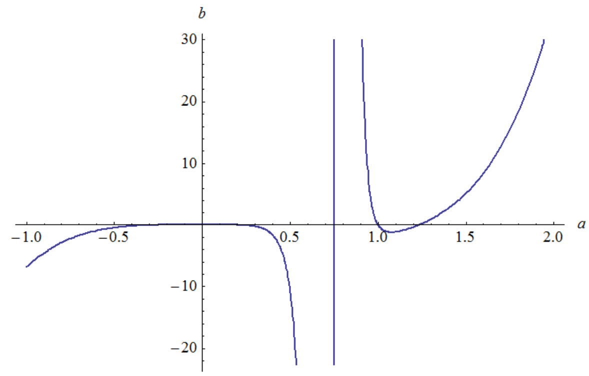

Figure 3. The graphic of Dziobek$=0$ for $a=1$.

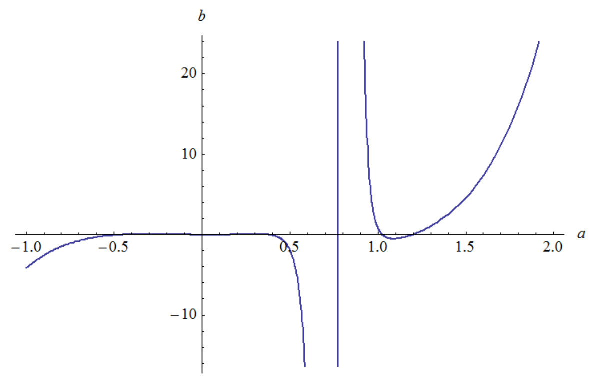

Figure 4. The graphic of Dziobek$=0$ for $a=0.95$.

Proof of Theorem 2. From (2) we have

$$\begin{aligned}
&r_{12}=1,\quad r_{14}=\sqrt{a^2+b^2},\quad r_{24}=\sqrt{a^2+(1-b)^2},\\
&r_{13}=\sqrt{a^2+\frac{(1-b)^2(a^2+2(b+\sqrt{a^2+b^2}))^2}{(a^2+4b-4)^2}},\\
(1)\qquad &r_{23}=\sqrt{a^2+\frac{(a^2(b-2)+2(b-1)(b-2+\sqrt{a^2+b^2}))^2}{(a^2+4b-4)^2}},\\
&r_{34}=\sqrt{\frac{(a^2-(b-1)(\sqrt{a^2+b^2}-b))^2}{(a^2+4b-4)^2}}.
\end{aligned}$$

Substituting these expressions together with the values of D_k for $k=1,2,3,4$ into the last six equations of (4), we found that they are identically zero for a tangential trapezoid configuration.

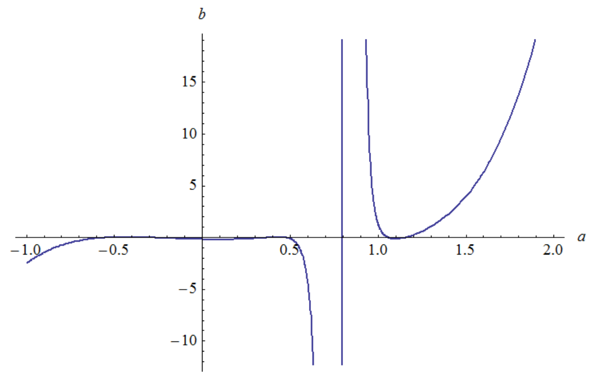

Figure 5. The graphic of Dziobek$=0$ for $a=0.9$.

Dividing conveniently two different equations of (5) we obtain

$$\tag{11} \begin{aligned} \frac{m_2(r_{23}^{-3}-c_1)}{m_1(r_{13}^{-3}-c_1)} &= \frac{D_2}{D_1}, \\ \frac{m_3(r_{23}^{-3}-c_1)}{m_1(r_{12}^{-3}-c_1)} &= \frac{D_3}{D_1}, \\ \frac{m_4(r_{24}^{-3}-c_1)}{m_1(r_{12}^{-3}-c_1)} &= \frac{D_4}{D_1}. \end{aligned}$$

Since $m_1=1$, using (7) and (11), we have

$$\tag{12} \begin{aligned} m_2 &= \frac{D_2 r_{23}^3 r_{24}^3 (r_{13}^3 - r_{14}^3)}{D_1 r_{13}^3 r_{14}^3 (r_{23}^3 - r_{24}^3)}, \\ m_3 &= \frac{D_3 r_{23}^3 r_{34}^3 (r_{12}^3 - r_{14}^3)}{D_1 r_{12}^3 r_{14}^3 (r_{23}^3 - r_{34}^3)}, \\ m_4 &= \frac{D_4 r_{24}^3 r_{34}^3 (r_{12}^3 - r_{13}^3)}{D_1 r_{12}^3 r_{13}^3 (r_{24}^3 - r_{34}^3)}. \end{aligned}$$

Substituting these masses into the first six Dziobek equations (4), and taking only the numerators of these six equations because the denominators do not vanish, we have

$$\tag{13} \begin{aligned} e_{1,2} =& D_2(c_2 D_1^3 r_{12}^3 r_{13}^3 r_{14}^2 (r_{24}^3 - r_{23}^3) - r_{23}^3 r_{24}^3 (c_1 r_{12}^3 - 1)(r_{13}^3 - r_{14}^3)), \\ e_{1,3} =& D_3(c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 (r_{34}^3 - r_{23}^3) - r_{23}^3 r_{34}^3 (c_1 r_{13}^3 - 1)(r_{12}^3 - r_{14}^3)), \\ e_{2,3} =& D_2 D_3 (r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{23}^3 - 1)(r_{12}^3 r_{14}^3)(r_{14}^3 - r_{13}^3) - \\ & c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^6 (r_{23}^3 - r_{24}^3)(r_{23}^3 - r_{34}^3)), \\ e_{1,4} =& D_4(c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 (r_{34}^3) - r_{24}^3 - r_{24}^3 r_{34}^3 (c_1 r_{14}^3 - 1)(r_{12}^3 - r_{13}^3)), \\ e_{2,4} =& D_2 D_4 (-r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{24}^3 - 1)(r_{12}^3 - r_{13}^3)(r_{13}^3 - r_{14}^3) - \\ & c_2 D_1^2 r_{12}^3 r_{13}^6 r_{14}^3 (r_{23}^3 - r_{24}^3)(r_{24}^3 - r_{34}^3)), \\ e_{3,4} =& D_3 D_4 (c_2 D_1^2 r_{12}^6 r_{13}^3 r_{14}^3 (r_{23}^3 - r_{34}^3)(r_{34}^3 - r_{24}^3) - \\ & r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{34}^3 - 1)(r_{12}^3 - r_{13}^3)(r_{12}^3 - r_{14}^3)). \end{aligned}$$

Notice that $D_i (i = 1, 2, 3, 4)$ is non-zero, so we can eliminate D_i from equations (13). First, we solve the first two equations with respect to c_1 and c_2, and then we substituted c_1 and c_2 in the last four equations of (13). We obtain

$$
\begin{aligned}
e_{2,3} &= \frac{D}{d} r_{23}^6 r_{24}^3 r_{34}^3 (r_{14}^3 - r_{12}^3)(r_{14}^3 - r_{13}^3), \\
e_{1,4} &= 0, \\
e_{2,4} &= \frac{D}{d} r_{23}^3 r_{24}^6 r_{34}^3 (r_{13}^3 - r_{12}^3)(r_{13}^3 - r_{14}^3), \\
e_{3,4} &= \frac{D}{d} r_{23}^3 r_{24}^3 r_{34}^6 (r_{13}^3 - r_{12}^3)(r_{12}^3 - r_{14}^3),
\end{aligned} \tag{14}
$$

where $D = 0$ is the Dziobek equation (9), and

$$
d = r_{12}^3 (r_{13}^3 r_{23}^3 (r_{24}^3 - r_{34}^3) + r_{14}^3 r_{24}^3 (r_{34}^3 - r_{23}^3)) + r_{13}^3 r_{14}^3 r_{34}^3 (r_{23}^3 - r_{24}^3),
$$

is the denominator which comes from the denominator of c_1 and c_2.

In conclusion the tangential trapezoid central configurations must satisfy $(e_{2,3}, e_{2,4}, e_{3,4}) = (0, 0, 0)$.

Substituting (10) into $e_{2,3} = 0$, $e_{2,4} = 0$, $e_{3,4} = 0$, we get that these three equations are satisfied if and only if the following equations

$$
\begin{aligned}
E_{2,3} =& DW_1 W_3 = 0, \\
E_{2,4} =& DW_2 W_3 = 0, \\
E_{3,4} =& DW_1 W_2 = 0,
\end{aligned} \tag{15}
$$

have solutions respectively, where

$$
\begin{aligned}
W_1 =& 1 - \sqrt{a^2 + b^2}, \\
W_2 =& 1 - \sqrt{a^2 + \frac{(b-1)^2 (a^2 + 2(b + \sqrt{a^2 + b^2}))^2}{(a^2 + 4b - 4)^2}}, \\
W_3 =& \sqrt{a^2 + b^2} - \sqrt{a^2 + \frac{(b-1)^2 (a^2 + 2(b + \sqrt{a^2 + b^2}))^2}{(a^2 + 4b - 4)^2}},
\end{aligned}
$$

and $D = 0$ is the Dziobek equation (9).

The solutions of $\{E_{2,3}, E_{2,4}, E_{3,4}\} = 0$ and of Dziobek $=0$ having positive masses are the tangential trapezoid central configurations of the four-body problem.

First we solve equations $\{E_{2,3}, E_{2,4}, E_{3,4}\} = 0$. Using Mathematica, we obtain that

$$
a = \frac{\sqrt{3}}{2}, \quad b = -\frac{1}{2}, \quad m_2 = 1.5124659.., \quad m_3 = 1, \quad m_4 = 1.5124659...
$$

Next we find solutions of Dziobek $=0$. In order to solve Dziobek $=0$, we use Mathematica. We plot the graphic of Dziobek $=0$ when $a = 1, 0.95, 0.9$, see Figures 3, 4, and 5.

When $a = 1$ we solve Dziobek $= 0$ and get $b = 0$, and from (12) we obtain $m_1 = m_2 = m_3 = m_4 = 1$.

When $a = 0.95$ we solve Dziobek $= 0$ and get the three solutions

$$b = -0.3122499.., \quad b = 0.3122499.., \quad b = 1.02234...$$

Then from (12) we have

$$\begin{aligned}
m_2 =&0.49701.., & m_3 =&1, & m_4 =&0.49701.., & \text{for}(a = 0.95,\, b = -0.31224..),\\
m_2 =&2.01201.., & m_3 =&1, & m_4 =&2.01201.., & \text{for } (a = 0.95,\, b = 0.31224..),\\
m_2 =&1.09839.., & m_3 =&0.7557.., & m_4 =&0.63245.., & \text{for}(a = 0.95,\, b = 1.02234..).
\end{aligned}$$

When $a = 0.9$ we solve Dziobek $= 0$ and get the three solutions

$$b = -0.4358899.., \quad b = 0.4358899.., \quad b = 1.05179...$$

Therefore from (12) we have

$$\begin{aligned}
m_2 =&0.20834.., & m_3 =&1, & m_4 =&0.20834.., & \text{for}(a = 0.9,\, b = -0.43588..),\\
m_2 =&4.79976.., & m_3 =&1, & m_4 =&4.79976.., & \text{for}(a = 0.9,\, b = 0.43588..),\\
m_2 =&1.14928.., & m_3 =&0.51298.., & m_4 =&0.27025.., & \text{for}(a = 0.9,\, b = 1.05179..).
\end{aligned}$$

When a goes from 1 to $\frac{\sqrt{3}}{2}$, the solutions for Dziobek $= 0$ has two symmetric solutions in the interval $(-\frac{1}{2},\, \frac{1}{2})$ and one solution in the interval $(1,\, \frac{11}{10})$ with positive masses, and we have

$$\begin{aligned}
m_1 = m_3 = 1,\, m_2 = m_4 > 1 \quad &\text{for} \quad 0 \leq b < \frac{1}{2},\\
m_1 = m_3 = 1,\, m_2 = m_4 < 1 \quad &\text{for} \quad -\frac{1}{2} < b \leq 0,\\
m_2 > m_1 = 1 > m_3 > m_4 \quad &\text{for} \quad 1 \leq b < \frac{11}{10}.
\end{aligned}$$

We find that $m_2 = m_4$ as $(a, b) \to (\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $m_2 = m_4 \to 0$ as $(a, b) \to (\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

In summary studying the graphic of $D = 0$ with a varying from 1 to $\frac{\sqrt{3}}{2}$, we obtain the statements of Theorem 2. □

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References

[1] Albouy, A., Symétrie des configurations centrales de quatre corps, C. R. Acad. Sci. Paris, **320** (1995), 217–220.

[2] Albouy, A. Recherches sur le problème des n corps, Notes Scientifiques et Techniques du Bureau des Longitudes, Pairs, 1997, pp. 78.

[3] Albouy, A., Chenciner, A., Le problème des n corps et les distances mutuelles, Invent. Math. **131** (1998), 151–184.

[4] Albouy, A. and Fu, Y., Euler configurations and quasi polynomial systems, Regul. Chaotic Dyn. **12** (2007), 39–55.

[5] Albouy, A., Fu, Y. and Sun, S., Symmetry of planar four body convex central configurations, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **464** (2008), 1355–1365.

[6] Albouy, A. and Kaloshin, V., Finiteness of central configurations of five bodies in the plane, Ann. of Math. (2) **176** (2012), 535–588.

[7] Álvarez, M. and Llibre, J., The symmetric central configurations of the 4–body problem with masses $m_1 = m_2 \neq m_3 = m_4$, Appl. Math. Comp. **219** (2013), 5996–6001.

[8] Álvarez, M. and Llibre, J., Hjelmslev quadrilateral central configurations, Physics Letters A **383** (2018), 103–109.

[9] Álvarez, M. and Llibre, J., Equilic quadrilateral central configurations, Commun. Nonlinear Sci. Numer. Simul. **78** (2019), 104872, 7 pp.

[10] Alvarez-Ramirez, A., Santos, A.A., and Vidal, C., On co-circular central configurations in the four and five body-problem for homogeneous force law, J.Dynam. Differential Equations **25** (2013), no.2, 269–290.

[11] Arenstorf, R.F., Central configurations of four bodies with one inferior mass, Cel. Mechanics **28** (1982), 9–15.

[12] Barros, J.F. and Leandro, E.S.G., The set of degenerate cetral configurations in the planar restricted four-body problem, SIAM Journal on Mathematical Analysis **43** (2011), 634–661.

[13] Barros, J.F. and Leandro, E.S.G., Bifurcations and enumeration of classes of relative equilibria in the planar restricted four-body problem, SIAM Joural on Mathematical Analysis **46** (2014), 1185–1203.

[14] Bernat, J., Llibre, J. and Perez Chavela, E., On the planar central configurations of the 4-body problem with three equal masses, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **16** (2009), 1–13.

[15] Corbera, M. and Llibre, J., Central configurations of the 4-body problem with masses $m_1 = m_2 > m_3 = m_4 = m > 0$ and m small, Appl. Math. Comput. **246** (2014), 121–147.

[16] Corbera, M., Cors, J.M.,Llibre, J. and Perez-Chavela, E., Trapezoid central configurations, Appl. Math. and Comput. **346** (2019), 127–142.

[17] Cors J.M. and Roberts G.E., Four-body co-circular central configurations, Nonlinearity **25** (2012), 343-370.

[18] Davis, C., Geyer, S., Johnson, W. and Xie, Z., Inverse problem of central configurations in the collinear 5-body problem, J. Math. Phys. **59** (2018), no. 5, 052902, 18 pp.

[19] Dziobek, O., Ueber einen merkwürdigen Fall des Vielkörperproblems, Astro. Nach. **152** (1900), 32–46.

[20] Érdi, B. and Czirják, Z., Central configuration of four bodies with an axis of symmetry, Celestial Mech.Dynam. Astronom. **125** (2016), no.1, 33–70.

[21] Euler, L., De moto rectilineo trium corporum se mutuo attahentium, Novi Comm. Acad. Sci. Imp. Petrop., **11** (1767), 144–151.

[22] Fernandes, A.C., Garcia, B.A., Llibre, J. and Mello, L.F., New central configurations of the (n+1)-body problem, J. Geom. Phys. **124** (2018), 199–207.

[23] Fernandes, A.C., Llibre, J. and Mello, L.F., Convex central configurations of the 4-body problem with two pairs of equal masses, Arch. Rational Mech. Anal. **226** (2017), 303–320.

[24] Gannaway, J.R., Determination of all central configurations in the planar 4-body problem with one inferior mass, Ph. D., Vanderbilt University, Nashville, USA, 1981.

[25] Hagihara, Y., Celestial Mechanics, vol. 1, MIT Press, Massachusetts, 1970.

[26] Hampton, M. and Moeckel, R., Finiteness of relative equilibria of the four-body problem, Invent. Math. **163** (2006), no.2, 289–312.

[27] Josefson, M., More characterizations of tangential quadrilaterals, Forum Geometricorum **11** (2011), 65–82.

[28] Largange, J.L., Essai surle probléme des trois corps, recueil des pièces qui ont remporté le prix de l'Académie royale des Sciences de Paris,tome IX,1772, reprinted in Ouvres, Vol.6(Gauthier-Villars,Paris,1873), pp 229–324.
[29] Leandro, E.S.G., Finiteness and bifurcation of some symmetrical classes of central configurations, Arch. Rational Mech. Anal. **167** (2003), 147–177.
[30] Llibre, J., Posiciones de equilibrio relativo del problema de 4 cuerpos, Publica cions Matemàtiques UAB**3** (1976), 73–88.
[31] Llibre, J., On the number of central configurations in the N-body problem, Celestial Mech. Dynam. Astronom. **50** (1991), 89–96.
[32] Long, Y., Admissible shapes of 4-body non-collinear relative equilibria, Adv. Nonlinear Stud. **3** (2003), no. 4, 495–509.
[33] Long, Y. and Sun, S., Four-Body Central Configurations with some Equal Masses, Arch. Rational Mech. Anal. **162** (2002), 24–44.
[34] MacMillan, W.D. and Bartky, W., Permanent Configurations in the Problem of Four Bodies, Trans. Amer. Math. Soc. **34** (1932), no. 4, 838–875.
[35] Moeckel, R., On central configurations, Mathematische Zeitschrift **205** (1990), no. 4, 499–517.
[36] Moeckel, R., Generic finiteness for Dziobek configurations, Trans. Amer. Math. Soc. **353** (2001), 4673–4686.
[37] Meyer, K.R., Bifurcation of a central configuration, Cel.Mech. **40** (1987), 273–282.
[38] Moulton, F.R., The straight line solutions of n bodies, Ann. of Math. **12** (1910), 1–17.
[39] Palmore, J.I., Classifying relative equilibria II, Bull. Amer. Math. Soc. **81**(1975), 71–73.
[40] Pedersen, P., Librationspunkte im restringierten Vierkörperproblem, Danske Vid. Selsk. Math.Fys. **21** (1944), 1–80.
[41] Pérez-Chavela E., Santoprete, M., Convex four-body central configurations with some equal masses, Arch. Rational Mech. Anal. **185** (2007), 481–494.
[42] Piña, E., Lonngi, P., Central configuration for the planar Newtonian four-body problem, Celest. Mech. Dyn. Astron. **108** (2010), 73–93.
[43] Saari, D.G., On the role and properties of central configurations, Celestial Mech., **21** (1980), 9–20.
[44] Schmidt, D.S., Central configurations in $\mathbb{R}^2$ and $\mathbb{R}^3$, Contemporary Math. **81**(1980), 59–76.
[45] Simó, C., Relative equilibrium solutions in the four-body problem, Cel. Mechanics **18** (1978), 165–184.
[46] Smale, S., Topology and mechanics II. The planar n-body problem, Invent.Math. **11** (1970), 45–64.
[47] Wintner, A., The Analytical Foundations of Celestial Mechanics, Princeton Math. Series 5, Princeton University Press, Princeton, NJ, 1941.
[48] Xia, Z., Central configurations with many small masses, J. Differential Equations **91** (1991), 168–179.
[49] Xia, Z., Convex central configurations for the n-body problem, J. Differential Equations **200** (2004), 185–190.
[50] Xie, Z., Isosceles trapezoid central configurations of the Newtonian four-body problem, Proc. R. Soc. Edinb., Sect. A, Math. **142** (2012), 665–672.

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