Bifurcations of zeros in translated families of functions and applications

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Abstract. In this paper we study the creation of zeros in a certain type of families of functions. The families studied are given by the difference of two basic functions with a translation made in the argument of one of these functions. The problem is motivated by applications in the 16-th Hilbert problem and its ramifications. Here, we apply the results obtained to the study of bifurcations of critical periods in the Loud family of quadratic centers.

1 Introduction and main results

This paper is motivated by the study of bifurcations of critical points of the period function in a neighborhood of a polycycle. A key problem in theses studies is the breaking of separatrices of the polycycle. It appears also in the study of limit cycles corresponding to fixed points of the Poincaré return map of a family of planar vector fields. Contrary to the situation in the study of limit cycles, here by breaking a polycycle it is replaced by a polycycle with less vertices. The simplest situation is when a polycycle with two vertices is broken and a saddle loop polycycle is created.

The cyclicity (i.e. number of limit cycles appearing by perturbation) of hyperbolic polycycles has been extensively studied in [1, 20, 24, 16, 17, 18, 15, 4] among others.

On the other hand, in our study of critical points of the period function of Loud systems [9, 6, 7, 10, 13] we gave a conjectural bifurcation diagram. We could not prove it in full generality due in part to some phenomena of breaking of separatrices of the polycycle bounding the period annulus. Here we deal with this problem. The simplest setting is the breaking of one separatrix (or two separatrices in the presence of symmetry).

This problem leads to the following type of equation

$$\Delta(s;\varepsilon,\mu) := F_1(s;\mu) - F_2(s+\varepsilon;\mu) = 0, \tag{1}$$

where $s = 0$ corresponds to the polycycle, $s \geq 0$ parametrizes the monodromic region, $\varepsilon \approx 0$ is the parameter controlling the breaking of the separatrix and μ regroups all other parameters of the family, which we study in a neighborhood of a parameter value μ_0.

We study the family of functions $\Delta_\nu(s) = \Delta(s,\varepsilon;\mu)$, for the parameter $\nu = (\varepsilon,\mu)$ in a neighborhood of $\nu_0 = (0,\mu_0)$ and $s \geq 0$ close to 0.

1.1 General results

Given a family of functions the notion of *cyclicity* has been defined by Roussarie [19]. It counts the maximal number of zeros, counted with multiplicity, born in the family from the origin. A value $\nu_0 \in V$ is a bifurcation value if and only if there exists a sequence (ν_n, s_n) with $s_n > 0$ and $\nu_n \in \operatorname{Int} V$ converging to $(\nu_0, 0)$ such that $\Delta_{\nu_n}(s_n) = 0$ for all n.

Definition 1.1. [19] Let $\{\Delta_\nu\}_{\nu \in V}$ be a continuous family of smooth functions on $(0, s_0)$ and fix $\nu_0 \in V \subset \mathbb{R}^k$, i.e. let $(s, \nu) \longmapsto \Delta_\nu(s)$ be a continuous mapping on $(0, s_0) \times V$. For each $\delta, \rho > 0$ let $N(\delta, \rho)$ be the supremum of the number of isolated zeros counted with multiplicity of Δ_ν in $(0, \delta)$ for $\nu \in B_\rho(\nu_0) \cap \operatorname{Int} V$. We define the *cyclicity* $Z(F_\nu, \nu_0) \in \mathbb{N} \cup \{0, \infty\}$ as the infimum of $N(\delta, \rho)$ varying $\delta, \rho > 0$. We say that a $\nu_0 \in V$ is a *regular value* of the parameter if $Z(\Delta_\nu, \nu_0) = 0$ and that it is a *bifurcation value* of the parameter otherwise.

In Theorem A, we study the cyclicity of a family of functions of the form (1). The Theorem has two parts. First part is more general and relatively simple. It gives cyclicity $Z = 0$. We include it here for completenes and because we use it in applications. The second part gives a stronger conclusion ($Z = 1$), but under more specific hypothesis. It presents the main part of Theorem A.

We say that a family of functions $F(s; \mu)$ tends to $L(\mu)$ as $s \to 0^+$ with a uniformly positive (resp. negative) sign at μ_0 if for every $\epsilon > 0$, there exists $\delta > 0$ and a neighborhood W of μ_0 such that for all $\mu \in W$ and every $0 < s < \delta$, we have $0 < F(s; \mu) - L(\mu) < \epsilon$ (resp. $-\epsilon < F(s; \mu) - L(\mu) < 0$). For example, the family $F(s; \mu) = s(s - \mu)$ tends uniformly to $L(\mu) \equiv 0$ but not with uniform sign at $\mu_0 = 0$.

In order to obtain our principal results on higher cyclicity ($Z > 0$), in the second part of Theorem A we particularize the family of functions (1), by requiring that the functions $F_i(s, \mu)$, $i = 1, 2$, are of the form

$$F_i(s, \mu) = f_i(s, \mu)(c_i(\mu) + \psi_i(s, \mu)), \quad c_i(\mu_0) \neq 0. \tag{2}$$

with

(a) $c_i(\mu)$, $f_i(s; \mu)$ and $\psi_i(s; \mu)$ continuous functions, with f_i and ψ_i defined on $(0, s_0) \times U$,

(b) for each $\mu \in U$, $\psi_i(\,\cdot\,; \mu)$ and $f_i(\,\cdot\,; \mu)$ smooth functions on $(0, s_0)$,

(c) $f_i(s; \mu)$, $\psi_i(s; \mu)$ and $\mathscr{D}\psi_i(s; \mu)$ tend to zero as $s \to 0^+$ uniformly on compact sets in μ, where $\mathscr{D} = s\partial_s$ is the Euler differential operator.

Definition 1.2. We write $f_1 \prec_{\mu_0} f_2$ (i.e. f_1 *precedes* f_2 at μ_0) if

$$\frac{f_2(s; \mu)}{f_1(s; \mu)} \longrightarrow 0 \text{ as } s \to 0^+ \text{ uniformly on a neighborhood of } \mu_0.$$

We say that f_1 and f_2 are *orderable* at μ_0 if $f_1 \prec_{\mu_0} f_2$ or $f_2 \prec_{\mu_0} f_1$. Finally, we write $f_1 \sim_{\mu_0} f_2$ if

$$\frac{f_2(s; \mu)}{f_1(s; \mu)} \longrightarrow 1 \text{ as } s \to 0^+ \text{ uniformly on a neighborhood of } \mu_0.$$

For a given $\mu_0 \in U$, we define the set of admissible functions at μ_0 as

$$\mathcal{A}_{\mu_0} := \Big\{ s^{\lambda(\mu)} : \lambda \in \mathscr{C}^0(U) \text{ with } \lambda(\mu_0) > 0 \Big\} \bigcup \Big\{ s\omega(s; \alpha(\mu)) : \alpha \in \mathscr{C}^0(U) \text{ with } \alpha(\mu_0) = 0 \Big\}, \tag{3}$$

where recall that

$$\omega(s; \alpha) := \begin{cases} \frac{s^{-\alpha} - 1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases} \tag{4}$$

is the *Ecalle-Roussarie compensator* [20]. Notice also that the admissible functions $f_i(s; \mu)$ tend to zero as $s \to 0^+$ with a uniformly positive sign.

Theorem A. *Consider a family of functions* $\Delta_{\varepsilon,\mu}(s) = \Delta(s, \varepsilon, \mu)$ *given by* (1)*, for* $\nu = (\varepsilon, \mu) \in V$*, where* $V = [0, \varepsilon_0) \times U$*, for some* $\varepsilon_0 > 0$ *and* U *is an open neighborhood of* $\mu_0 \in \mathbb{R}^{k-1}$*.*

(i) We assume that the functions $F_i(s; \mu)$ *tend uniformly on compact neighborhoods of* μ_0 *to continuous real valued functions* $L_i(\mu)$*,* $i = 1, 2$*,* $s \to 0^+$*.*

  (a) If $L_1(\mu_0) \neq L_2(\mu_0)$ *then* $\nu_0 = (0, \mu_0)$ *is not a bifurcation value, i.e.* $Z(\Delta_\nu, \nu_0) = 0$*.*

  (b) Assume that $F_i(s, \mu)$ *tend uniformly to* $L_i(\mu)$ *as* $s \to 0^+$ *on compact neighborhoods of* μ_0 *with uniform sign. If* $L_1(\mu_0) = L_2(\mu_0)$ *but* $F_i - L_i$ *is uniformly of the opposite sign, then* $\nu_0 = (\varepsilon, \mu_0)$ *is not a bifurcation value of* $\Delta_\nu(s)$*, i.e.* $Z(\Delta_\nu, \nu_0) = 0$*.*

(ii) Assume that F_i *are given in* (2) *where* $f_i \in \mathcal{A}_{\mu_0}$ *(see* (3)*) and* ψ_i*,* c_i*,* $i = 1, 2$*, are as above. Suppose that* f_1 *and* f_2 *are orderable at* μ_0*. Then, the following assertions are equivalent:*

  (a) $\nu_0 = (0, \mu_0)$ *is a bifurcation value for the family* $\{\Delta_\nu\}_{\nu \in V}$*,*

  (b) $f_1 \prec_{\mu_0} f_2$ *and* $c_1(\mu_0)c_2(\mu_0) > 0$*,*

  (c) $Z(\Delta_\nu, \nu_0) = 1$*.*

The proof of Theorem A shows the following:

Corollary 1.3. *Under the assumptions of (ii) in Theorem A, if* $\nu_0 = (0, \mu_0)$ *is a bifurcation value for the family* $\{\Delta_\nu\}_{\nu \in V}$ *then there exist* $\delta > 0$*, a neighbourhood* B *of* ν_0 *and a continuous function* $v : B \longrightarrow \mathbb{R}$ *with* $v(0, \mu_0) = 0$ *such that* $\Delta(s; \nu) = 0$ *with* $s \in (0, \delta)$ *and* $\nu \in B \cap \mathrm{Int}V$ *if and only if* $s = \sigma(\nu)$*, with*

$$\sigma(\varepsilon, \mu) = \sigma_0(\varepsilon, \mu)(1 + v(\varepsilon, \mu)) \quad \textit{and} \quad \sigma_0(\varepsilon; \mu) := f_1^{-1}\left(\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon; \mu); \mu\right).$$

The following example is two-fold. It illustrates the appearance of the *translated families problem* in the context of creation of limit cycles. It is the model example of the displacement function (controling limit cycles) of a generic polycycle with two hyperbolic vertices and two separatrices being broken [17]. It also gives an example where the uniform sign condition is not fulfilled.

Example 1.4. Fix $r_2 > r_1 > 0$, $\mu_0 = 0$ and consider the equation (1) with $F_1(s; \mu) = s^{r_1} + \mu$ and $F_2(s; \mu) = s^{r_2}$ which gives

$$\Delta(s; \varepsilon, \mu) := s^{r_1} + \mu - (s + \varepsilon)^{r_2} = 0.$$

In that case we have $L(\mu) = \mu$ which changes sign at $\mu = 0$. Let us sketch the argument showing that $Z = 2$. Consider the equation given by the derivative $\Delta'(s; \varepsilon, \mu) = r_1 s^{r_1 - 1} - r_2(s + \varepsilon)^{r_2 - 1} = 0$. We apply Theorem A (ii) to $\Delta'(s; \varepsilon, \mu) = 0$ and obtain that there exists a unique solution $s = \sigma(\varepsilon) > 0$ tending to 0 as $\varepsilon \to 0^+$. By Rolle's theorem, this shows that $Z \leq 2$. On the other hand choosing conveniently $\mu = \mu(\varepsilon)$ we obtain a tangency point $(\sigma(\varepsilon), \sigma(\varepsilon)^{r_1} + \mu(\varepsilon))$ between the graphs of the functions $F_1(s; \mu)$ and $F_2(s + \varepsilon; \mu)$, showing that $Z = 2$.

1.2 Critical Periods in Loud systems

This work was initially motivated by the study of the bifurcation diagram of the period function of the dehomogenized Loud family of quadratic centers

$$\begin{aligned} \dot{x} &= -y + xy, \\ \dot{y} &= x + Dx^2 + Fy^2. \end{aligned} \tag{5}$$

This study was started by Chicone and Jacobs [2]. They focused on the bifurcations of critical periods near the inner boundary (the center itself) of the period annulus by means of the period constants obtained by the Taylor expansion at the origin. In [7], we developed a technique to compute the first coefficient of a uniform asymptotic expansion of the period function at the outer boundary (a polycycle) of the period annulus under the hypothesis that the polycycle only contains linearizable saddles. In [9], we applied this technique to the Loud system obtaining a Jordan curve on which there is bifurcation of critical periods near the outer boundary and an open dense set of regular values. Unfortunately, a union Γ_U of straight segments on which the character (bifurcation or regular) remained unknown for different reasons.

In particular, along the line $\{F + D = 0, F \notin [0, 1]\}$ a heteroclinic connection between hyperbolic saddles bounding the period annulus occurs. The connection is broken when leaving this line in the space of parameters. We denote by ε the breaking parameter, i.e. the signed distance between the two separatrices measured on a transverse section. The study of the bifurcation of critical periods in that case, leads to the type of bifurcations studied in Theorem A. We prove:

Theorem B. *Consider the period function of the period annulus of the Loud family (5) containing the origin. Take a parameter (D, F), with $F + D = 0$ and $F \notin \{0, 1, 1/2\}$.*

(a) If $F \notin [3/2, 2]$, then $Z = 0$.

(b) If $F \in [3/2, 2)$, then $Z = 1$.

(c) If $F = 2$, then $Z = 2$.

The following theorem gives a more precise description of the bifurcations occuring at $(D, F) = (-2, 2)$, for the critical period function T' in the Loud Family.

Theorem C. *(a) There is a curve Γ of double critical periods bifurcating from the polycycle which arrives to the point $(D, F) = (-2, 2)$. It is contained in the sector $\{F \geq 2, F + D \geq 0\}$ and Γ is given by the graph of a continuous positive function $\varepsilon = f(F)$ with $f(2) = 0$, where $\varepsilon = (D + F)U(D, F)$ is the breaking parameter, with $U(-2, 2) > 0$.*

(b) The curve Γ has a flat tangency with the line $F = 2$, more precisely, there exist $k_2 > k_1 > 0$ such that

$$e^{-\frac{k_2}{f(F)}} < F - 2 < e^{-\frac{k_1}{f(F)}}$$

for $F - 2 > 0$ small enough.

(c) Moreover, crossing the curve Γ from above, two simple critical periods bifurcate from the double critical period.

The exponentially flat behavior of the double bifurcation curve Γ explains why it is hard to find numerically two critical periods near the point $(D, F) = (-2, 2)$, see [9, Figure 1].

2 Proof of Theorem A

The proof of (i) is easy. We include it here for completeness and as we use it in applications. The claim (ii) of Theorem A is proved using the implicit function theorem after a convenient blow-up. We use the classical version (Theorem A.1) of the implicit function theorem which requires differentiability only with respect to the variable that we want to isolate.

Proposition 2.1. *Fix $\mu_0 \in U$ and consider the family $\{\Delta_\nu\}$ in (1) taking $f_1, f_2 \in \mathcal{A}_{\mu_0}$. Assume that f_1 and f_2 are orderable at μ_0. If there exists a sequence $(\nu_n, s_n)_{n\in\mathbb{N}} = (\varepsilon_n, \mu_n, s_n)_{n\in\mathbb{N}}$, with $\varepsilon_n, s_n > 0$ and $\mu_n \in U$, converging to $(0, \mu_0, 0)$ such that $\Delta_{\nu_n}(s_n) = 0$, for all n, then $f_1 \prec_{\mu_0} f_2$ and $\lim_{n\to\infty} \frac{f_1(s_n;\mu_n)}{f_2(\varepsilon_n;\mu_n)} = \frac{c_2(\mu_0)}{c_1(\mu_0)}$.*

The claim of the proposition is intuitively clear. In Δ we have a competition between two functions essentially $c_1 f_1$ and $c_2 f_2$ both tending to zero. One operates a translation by ε in f_2. The only way that the two contributions can cancel is that f_2 be smaller than f_1 (i.e. $f_1 \prec f_2$) and that we take $\varepsilon > 0$ (i.e. $s+\varepsilon$ is further away from the origin), so that we increase the contribution of f_2.

Proof. By contradiction, suppose that $f_2 \prec_{\mu_0} f_1$. Then, since $f_2(\,\cdot\,;\mu)$ is monotonous increasing (see Lemma A.6), we have that $0 \leqslant \frac{f_1(s;\mu)}{f_2(s+\varepsilon;\mu)} \leqslant \frac{f_1(s;\mu)}{f_2(s,\mu)}$, for any $\varepsilon \geqslant 0$, which, on account of $f_2 \prec_{\mu_0} f_1$, implies that $\lim_{s\to 0^+} \frac{f_1(s;\mu)}{f_2(s+\varepsilon;\mu)} = 0$, uniformly on $\nu = (\varepsilon,\mu)$ in a neighbourhood of $\nu_0 = (0,\mu_0)$. On account of this, by applying Lemma A.2, we can assert that

$$0 = \frac{\Delta(s_n;\nu_n)}{f_2(s_n+\varepsilon_n;\mu_n)} = \left(\frac{f_1(s_n;\mu_n)}{f_2(s_n+\varepsilon_n;\mu_n)}\big(c_1(\mu_n)+\psi_1(s_n;\mu_n)\big) - \big(c_2(\mu_n)+\psi_2(s_n+\varepsilon_n;\mu_n)\big)\right)$$

tends to $-c_2(\mu_0)$, as $n \longrightarrow +\infty$. Thus $c_2(\mu_0) = 0$, which contradicts the assumption $c_1(\mu_0)c_2(\mu_0) \neq 0$. Therefore $f_1 \prec_{\mu_0} f_2$.

At this point we claim that if $f_1, f_2 \in \mathcal{A}_{\mu_0}$ with $f_1 \prec_{\mu_0} f_2$, then $f_2^{-1} \prec_{\mu_0} f_1^{-1}$. There are three different cases to consider, namely:

(1) $f_1(s;\mu) = s^{\lambda_1(\mu)}$ and $f_2(s;\mu) = s^{\lambda_2(\mu)}$,

(2) $f_1(s;\mu) = s^{\lambda_1(\mu)}$ and $f_2(s;\mu) = s\omega(s;\alpha(\mu))$,

(3) $f_1(s;\mu) = s\omega(s;\alpha(\mu))$ and $f_2(s;\mu) = s^{\lambda_2(\mu)}$.

The claim is obvious in the first case. In the second case, by (c) in Lemma A.4, the assumption $f_1 \prec_{\mu_0} f_2$ implies $\lambda_1(\mu_0) < 1$. On the other hand, by (c) in Lemma A.6,

$$f_2^{-1}(s) \sim_{\mu_0} \frac{s\kappa(\alpha(\mu))}{\omega(s;\alpha'(\mu))}, \text{ where } \alpha'(\mu) := \frac{\alpha(\mu)}{1-\alpha(\mu)}.$$

On account of this, and by applying (c) in Lemma A.4 once again, $f_2^{-1} \prec_{\mu_0} f_1^{-1} = s^{1/\lambda_1(\mu)}$ if and only if $\lambda_1(\mu_0) < 1$. So the claim follows in the second case. The third case follows exactly the same way.

Let us write $f_1(s_n;\mu_n) = r_n \sin\theta_n$ and $f_2(\varepsilon_n;\mu_n) = r_n\cos\theta_n$ with $r_n > 0$ and $\theta_n \in [0,\pi/2]$. Then, due to $\lim_{s\to 0^+} f_i(s;\mu) = 0$ uniformly on $\mu \approx \mu_0$, by Lemma A.2, we can assert that $\lim_{n\to\infty} r_n = 0$. In addition, since $f_1 \prec_{\mu_0} f_2$, on account of $0 \leqslant f_1^{-1}(r_n\sin\theta_n) \leqslant f_1^{-1}(r_n)$ and the previous claim, we get

$$\lim_{n\to\infty} \frac{f_1^{-1}(r_n\sin\theta_n)}{f_2^{-1}(r_n)} = 0. \tag{6}$$

Here and in what follows we omit the dependence of μ when there is no risk of confusion. We write

$$f_2(s_n+\varepsilon_n) = f_2\left(f_2^{-1}(r_n)A_n\right) \text{ with } A_n := \frac{f_2^{-1}(r_n\cos\theta_n)}{f_2^{-1}(r_n)} + \frac{f_1^{-1}(r_n\sin\theta_n)}{f_2^{-1}(r_n)}.$$

Suppose at this point that $\theta_\star$ is an accumulation point of the sequence $(\theta_n)_{n\in\mathbb{N}}$. If $f_2(s;\mu) = s^{\lambda(\mu)}$ then, taking (6) into account,

$$\begin{aligned} 1 =& \frac{f_2(s_n+\varepsilon_n)(c_2+\psi_2(s_n+\varepsilon_n))}{f_1(s_n)(c_1+\psi_1(s_n))} \\ =& \frac{\left(\cos^{1/\lambda}\theta_n + \frac{f_1^{-1}(r_n\sin\theta_n)}{f_2^{-1}(r_n)}\right)^{\lambda}\left(c_2+\psi_2(f_1^{-1}(r_n\sin\theta_n)+f_2^{-1}(r_n\cos\theta_n))\right)}{\sin\theta_n\left(c_1+\psi_1(f_1^{-1}(r_n\sin\theta_n))\right)} \longrightarrow \frac{c_2(\mu_0)\cos\theta_\star}{c_1(\mu_0)\sin\theta_\star} \end{aligned}$$

as n tends to $+\infty$. Therefore $\frac{c_2(\mu_0)\cos\theta_\star}{c_1(\mu_0)\sin\theta_\star}=1$. Consider now the case $f_2(s;\mu)=s\omega(s;\alpha(\mu))$. Set $\alpha_n=\alpha(\mu_n)$ and $\alpha_n'=\alpha'(\mu_n)$ for shortness. Note then the sequences (α_n) and (α_n') tend to zero, as $n\to+\infty$. From (6) and applying Lemma A.6,

$$\lim_{n\to\infty}A_n=\lim_{n\to\infty}\frac{\frac{r_n\cos\theta_n}{\omega(r_n\cos\theta_n;\alpha_n')}}{\frac{r_n}{\omega(r_n;\alpha_n')}}=\lim_{n\to\infty}\frac{\omega(r_n;\alpha_n')\cos\theta_n}{(\cos\theta_n)^{-\alpha_n'}\omega(r_n;\alpha_n')+\omega(\cos\theta_n;\alpha_n')}$$
$$=\lim_{n\to\infty}\frac{(\cos\theta_n)^{1+\alpha_n'}}{1+\frac{(\cos\theta_n)^{\alpha_n'}\omega(\cos\theta_n;\alpha_n')}{\omega(r_n;\alpha_n')}}=\lim_{n\to\infty}\frac{(\cos\theta_n)^{1+\alpha_n'}}{1+\frac{\omega(\cos\theta_n;-\alpha_n')}{\omega(r_n;\alpha_n')}}=\cos\theta_\star.$$

If $\cos\theta_\star\neq0$ then the last equality follows by (b) in Lemma A.4, whereas for $\cos\theta_\star=0$, it follows easily due to $\omega(s;\alpha)>0$, for $\alpha\approx0$ and $s>0$ small enough. Then, by using Lemma A.6 and $\lim_{n\to+\infty}A_n=\cos\theta_\star$, we get that

$$1=\frac{f_2(s_n+\varepsilon_n)(c_2+\psi_2(s_n+\varepsilon_n))}{f_1(\varepsilon_n)(c_1+\psi_1(s_n))}$$
$$=\frac{1}{\sin\theta_n}\left(A_n^{1-\alpha_n}+\frac{f_2^{-1}(r_n)}{r_n}f_2(A_n)\right)\frac{c_2+\psi_2(f_1^{-1}(r_n\sin\theta_n)+f_2^{-1}(r_n\cos\theta_n))}{c_1+\psi_1(f_1^{-1}(r_n\sin\theta_n))}\longrightarrow\frac{c_2(\mu_0)\cos\theta_\star}{c_1(\mu_0)\sin\theta_\star},$$

as n tends to $+\infty$. Consequently, also in this case, $\frac{c_2(\mu_0)\cos\theta_\star}{c_1(\mu_0)\sin\theta_\star}=1$. So far we have proved that $\frac{\cos\theta_\star}{\sin\theta_\star}=\frac{c_1(\mu_0)}{c_2(\mu_0)}$, which in particular shows that the accumulation point $\theta_\star\in[0,\pi/2]$ is unique. Furthermore, we can also assert that $\frac{f_1(s_n;\mu_n)}{f_2(\varepsilon_n;\mu_n)}=\frac{\sin\theta_n}{\cos\theta_n}\longrightarrow\frac{\cos\theta_\star}{\sin\theta_\star}=\frac{c_1(\mu_0)}{c_2(\mu_0)}$, as $n\to+\infty$. Hence, the result is proved. ■

Lemma 2.2. *Fix $\mu_0\in U$ and consider $f_1,f_2\in\mathcal{A}_{\mu_0}$ and $h\in\mathscr{C}^0(U)$ verifying $f_1\prec_{\mu_0}f_2$ and $h(\mu_0)>0$. Then $s\prec_{\mu_0}f_1^{-1}\big(h(\mu)f_2(s;\mu);\mu\big)$.*

Proof. There are three different cases to consider, namely:

(1) $f_1(s;\mu)=s^{\lambda_1(\mu)}$ and $f_2(s;\mu)=s^{\lambda_2(\mu)}$,

(2) $f_1(s;\mu)=s^{\lambda_1(\mu)}$ and $f_2(s;\mu)=s\omega(s;\alpha(\mu))$,

(3) $f_1(s;\mu)=s\omega(s;\alpha(\mu))$ and $f_2(s;\mu)=s^{\lambda_2(\mu)}$.

In the first case, due to $f_1\prec_{\mu_0}f_2$, it is necessary that $\lambda_1(\mu_0)<\lambda_2(\mu_0)$, and then

$$\frac{f_1^{-1}(h(\mu)f_2(s;\mu);\mu)}{s}=h(\mu)^{\frac{1}{\lambda_1(\mu)}}s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}-1}\longrightarrow0,\ \text{as } s\to0^+, \text{ uniformly on } \mu\approx\mu_0.$$

In the second case, by (c) in Lemma A.4, $\lambda_1(\mu_0)<1$, and then

$$\frac{f_1^{-1}(h(\mu)f_2(s;\mu);\mu)}{s}=\Big(h(\mu)s^{1-\lambda_1(\mu)}\omega(s;\alpha(\mu))\Big)^{\frac{1}{\lambda_1(\mu)}}\longrightarrow0,\ \text{as } s\to0^+, \text{ uniformly on } \mu\approx\mu_0,$$

thanks to (c) in Lemma A.4 again. Finally, in the third case, $\lambda_2(\mu_0)>1$ and

$$\lim_{s\to0^+}\frac{f_1^{-1}(h(\mu)f_2(s;\mu);\mu)}{s}=\lim_{s\to0^+}\frac{\kappa(\alpha(\mu))h(\mu)s^{\lambda_2(\mu)}}{s\omega(h(\mu)s^{\lambda_2(\mu)};\alpha(\mu)/(1-\alpha(\mu)))}$$
$$=\lim_{x\to0^+}\frac{\kappa(\alpha(\mu))h(\mu)^{\frac{1}{\lambda_2(\mu)}}x^{1-\frac{1}{\lambda_2(\mu)}}}{\omega(x;\alpha(\mu)/(1-\alpha(\mu)))}=0,\ \text{uniformly on } \mu\approx\mu_0.$$

In the first equality above we use (c) in Lemma A.6, in the second one, Remark A.5 and in the third one, (b) in Lemma A.4. This proves the result. ■

Proof of Theorem A. (i) If $L_1(\mu_0) \neq L_2(\mu_0)$, since $F_i(s;\mu)$ tends to $L_i(\mu)$ as $s \to 0$ uniformly on compact neighborhoods of μ_0 and $L_i(\mu)$ is continuous at μ_0 then we get that for any compact neighborhood K of μ_0, there exist $E > 0$ and $\delta > 0$, such that $F_1(s;\mu) - F_2(s+\varepsilon;\mu)$ is of the sign of $L_1(\mu_0) - L_2(\mu_0)$, for $0 < s < \delta$, $0 < \varepsilon < E$, $\mu \in K$. Hence Δ_ν has no zeros for the above values of (s, ε, μ) and $Z(\Delta_\nu, (0, \mu_0)) = 0$.

If $L_1(\mu_0) = L_2(\mu_0)$, but $F_i - L_i$ are uniformly of the opposite sign, then in a neighborhood of the origin Δ_ν corresponds to a sum of two terms of the same sign. Hence, in that neighborhood Δ_ν is of the (uniform) sign of $F_1(s,\mu)$ and so $Z(\Delta_\nu, (0, \mu_0)) = 0$.

(ii) Let us prove the equivalence of the statements (a), (b) and (c): That (a) implies (b) follows by applying Proposition 2.1 and using that f_1 and f_2 are positive functions.

The principal step of the proof is to show that (b) implies the statement of Corollary 1.3. To this end, as we explained in Remark A.7, we consider continuous extensions of the families $\{f_2(\,\cdot\,;\mu)\}_{\mu\approx\mu_0}$ and $\{f_1^{-1}(\,\cdot\,;\mu)\}_{\mu\approx\mu_0}$ to $(-s_2, s_2)$ and $(-s_1, s_1)$, respectively, in order to be able to apply the implicit function theorem. Taking this into account, note that if μ varies in a compact neighbourhood of μ_0, then there exists $\varepsilon_0 > 0$ such that

$$\sigma_0(\varepsilon;\mu) := f_1^{-1}\left(\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu);\mu\right)$$

defines a continuous family of functions on $(-\varepsilon_0, \varepsilon_0)$ with $\sigma_0(0;\mu) = 0$. In order to study the roots of $\Delta(s;\varepsilon,\mu) = 0$, see (1), we shall make the *generalized blow-up*

$$s = \sigma_0(\varepsilon;\mu)(1+v).$$

Our goal is to obtain an *equivalent equation*, $G(\varepsilon, v;\mu) = 0$, with the function G verifying the hypothesis in Theorem A.1. With this end in view, some computations show that

$$f_1(s;\mu)|_{s=\sigma_0(\varepsilon;\mu)(1+v)} = \frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu) g_1(\varepsilon, v;\mu) \tag{7}$$

with

$$g_1(\varepsilon, v;\mu) := \begin{cases} (1+v)^{\lambda_1(\mu)}, & \text{if } f_1(s;\mu) = s^{\lambda_1(\mu)}, \\ (1+v)^{1-\alpha(\mu)} + \frac{\sigma_0(\varepsilon;\mu)}{\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu)} f_1(1+v;\mu), & \text{if } f_1(s;\mu) = s\omega(s;\alpha(\mu)), \end{cases}$$

where in the second case we use (a) in Lemma A.6. It is to be noted moreover that in this second case the function $g_1(\varepsilon, v;\mu)$ is a priori not defined at $\varepsilon = 0$. However, setting $\kappa_1(\alpha) := \frac{|\alpha|-\alpha}{2} \alpha^{\frac{|\alpha|+\alpha}{2(1-\alpha)}}$, notice that

$$\lim_{\varepsilon\to 0} \frac{\sigma_0(\varepsilon;\mu)}{\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu)} = \lim_{\varepsilon\to 0} \frac{f_1^{-1}\left(\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu);\mu\right)}{\frac{c_2(\mu)}{c_1(\mu)} f_2(\varepsilon;\mu)} = \lim_{x\to 0} \frac{f_1^{-1}(x;\mu)}{x} = \lim_{x\to 0} \frac{\kappa(\alpha(\mu))}{\omega(x;\alpha'(\mu))} = \kappa_1(\alpha(\mu))$$

uniformly on $\mu \approx \mu_0$. In the first equality above we use the definition of σ_0, in the second one we take Remark A.5 into account, whereas in the third and forth ones we use (c) in Lemma A.6 and (b) in Lemma A.4. This provides a continuous extension of the function $g_1(\varepsilon, v;\mu)$ to $\varepsilon = 0$ that, on account of $\kappa_1(0) = 0$, verifies $g_1(0,0;\mu_0) = 1$. In what follows, by abuse of notation, we refer to this extension as g_1.

On the other hand, by Lemma 2.2, $\frac{\sigma_0(\varepsilon;\mu)}{\varepsilon}$ extends to a continuous function $\sigma_1(\varepsilon;\mu)$ on $\varepsilon = 0$ such that $\sigma_1(0;\mu) = 0$. Then some computations show that

$$f_2(s+\varepsilon;\mu)|_{s=\sigma_0(\varepsilon;\mu)(1+v)} = f_2(\varepsilon;\mu) g_2(\varepsilon, v;\mu) \tag{8}$$

with

$$g_2(\varepsilon, v;\mu) := \begin{cases} (1+\sigma_1(\varepsilon;\mu)(1+v))^{\lambda_2(\mu)}, & \text{if } f_2(s;\mu) = s^{\lambda_2(\mu)}, \\ (1+\sigma_1(\varepsilon;\mu)(1+v))^{1-\alpha(\mu)} + \frac{\varepsilon}{f_2(\varepsilon;\mu)} f_2\left(1+\sigma_1(\varepsilon;\mu)(1+v)\right), & \text{if } f_2(s) = s\omega(s;\alpha(\mu)), \end{cases}$$

where, again, in the second case we use (a) in Lemma A.6. Also in this case, thanks to (b) in Lemma A.4,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{f_2(\varepsilon;\mu)}=\lim_{\varepsilon\to 0}\frac{1}{\omega(\varepsilon;\alpha(\mu))}=\frac{|\alpha(\mu)|-\alpha(\mu)}{2}\text{ uniformly on }\mu\approx\mu_0.$$

This provides a continuous extension of $g_2(\varepsilon,v;\mu)$ to $\varepsilon=0$, verifying $g_2(0,0;\mu_0)=1$.

Now, taking (7) and (8) into account, from (1), it follows that

$$\Delta(s;\varepsilon,\mu)|_{s=\sigma_0(\varepsilon;\mu)(1+v)}=c_2(\mu)f_2(\varepsilon;\mu)G(\varepsilon,v;\mu),$$

where

$$G(\varepsilon,v;\mu):=g_1(\varepsilon,v;\mu)\left(1+\frac{\psi_1\big(\sigma_0(\varepsilon;\mu)(1+v);\mu\big)}{c_1(\mu)}\right)-g_2(\varepsilon,v;\mu)\left(1+\frac{\psi_2\big(\varepsilon(1+\sigma_1(\varepsilon;\mu)(1+v));\mu\big)}{c_2(\mu)}\right).$$

We claim, cf. Definition 1.1, that $\Delta(\hat s;\hat\varepsilon,\hat\mu)=0$ with $\hat s\in(0,\delta)$, $\hat\varepsilon\in(0,\rho)$ and $\|\hat\mu-\mu_0\|<\rho$, if and only if $G(\hat\varepsilon,\hat v;\hat\mu)=0$, with $\hat s=\sigma_0(\hat\varepsilon;\hat\mu)(1+\hat v)$, for some $\hat v\approx 0$. The sufficiency is obvious due to $\sigma_0(0;\mu)=0$. To show the necessity, note first that $c_2(\mu_0)\neq 0$, by assumption, whereas $f_2(\varepsilon;\mu)=0$, if and only if $\varepsilon=0$. Accordingly, $G(\hat\varepsilon,\hat v;\hat\mu)=0$, and so it only remains to show that if we take an arbitrary sequence $(s_n,\varepsilon_n,\mu_n)_{n\in\mathbb{N}}$, with $\Delta(s_n;\varepsilon_n,\mu_n)=0$, for all n, such that $\lim_{n\to+\infty}(s_n,\varepsilon_n,\mu_n)=(0,0,\mu_0)$, then $v_n:=\frac{s_n}{\sigma_0(\varepsilon_n;\mu_n)}-1$ tends to zero, as $n\to+\infty$. To this end, let us set $a_n:=f_1(s_n;\mu_n)$ and $b_n:=\frac{c_2(\mu_n)}{c_1(\mu_n)}f_2(\varepsilon_n;\mu_n)$, so that $\lim_{n\to\infty}\frac{a_n}{b_n}=1$, by Proposition 2.1. Then it follows that $\frac{f_1^{-1}(a_n;\mu_n)}{f_1^{-1}(b_n;\mu_n)}=\frac{s_n}{\sigma_0(\varepsilon_n;\mu_n)}$ tends to 1, as $n\to+\infty$. This is clear in case that $f_1(s;\mu)=s^{\lambda(\mu)}$, whereas for $f_1(s;\mu)=s\omega(s;\alpha(\mu))$ we have that

$$\lim_{n\to\infty}\frac{f_1^{-1}(a_n;\mu_n)}{f_1^{-1}(b_n;\mu_n)}=\lim_{n\to\infty}\frac{a_n}{b_n}\frac{\omega(b_n;\alpha_n')}{\omega(a_n;\alpha_n')}=\lim_{n\to\infty}\frac{\omega(a_n\frac{b_n}{a_n};\alpha_n')}{\omega(a_n;\alpha_n')}=\lim_{n\to\infty}\left(\left(\frac{b_n}{a_n}\right)^{-\alpha_n'}+\frac{\omega(\frac{b_n}{a_n};\alpha_n')}{\omega(a_n;\alpha_n')}\right)=1,$$

where $\alpha_n':=\frac{\alpha(\mu_n)}{1-\alpha(\mu_n)}$ tends to 0, as $n\to+\infty$. (In the first and third equalities above we use (c) and (a) in Lemma A.6, respectively, and in the last one (b) in Lemma A.4.) This shows that $\lim_{n\to+\infty}v_n=0$, as desired, and completes the proof of the claim.

Note at this point that $(\varepsilon,v,\mu)\longmapsto G(\varepsilon,v;\mu)$ is a continuous function in a neighborhood of $(\varepsilon,v,\mu)=(0,0,\mu_0)$ with $G(0,0;\mu_0)=0$. Moreover, $\partial_v g_i(\varepsilon,v;\mu)$, $i=1,2$, are continuous at $\varepsilon=0$ as well. In addition, by Lemma A.3, $\psi_i(s;\mu)$ and $\phi_i(s;\mu):=s\partial_s\psi_i(s;\mu)$ form both continuous families of functions on $(-s_0,s_0)$, with $\psi_i(0;\mu)=\phi_i(0;\mu)=0$ for $i=1,2$. Therefore the functions

$$\partial_v\psi_1\big(\sigma_0(\varepsilon;\mu)(1+v)\big)=\frac{\phi_1(\sigma_0(\varepsilon;\mu)(1+v))}{1+v}$$

and

$$\partial_v\psi_2\big(\varepsilon(1+\sigma_1(\varepsilon;\mu)(1+v))\big)=\phi_2\big(\varepsilon(1+\sigma_1(\varepsilon;\mu)(1+v))\big)\frac{\sigma_1(\varepsilon;\mu)}{1+\sigma_1(\varepsilon;\mu)(1+v)}$$

are continuous and vanish at $\varepsilon=0$. Accordingly

$$\begin{aligned}\partial_v G(\varepsilon,v;\mu)&=\partial_v g_1(\varepsilon,v;\mu)\left(1+\frac{\psi_1\big(\sigma_0(\varepsilon;\mu)(1+v);\mu\big)}{c_1(\mu)}\right)+\frac{g_1(\varepsilon,v)\partial_v\psi_1\big(\sigma_0(\varepsilon;\mu)(1+v);\mu\big)}{c_1(\mu)}\\&-\partial_v g_2(\varepsilon,v;\mu)\left(1+\frac{\psi_2(\varepsilon(1+\sigma_1\big(\varepsilon;\mu)(1+v));\mu\big)}{c_2(\mu)}\right)-\frac{g_2(\varepsilon,v)\partial_v\psi_2(\varepsilon(1+\sigma_1(\varepsilon;\mu)(1+v);\mu)}{c_2(\mu)}\end{aligned}$$

is a continuous function in a neighborhood of $(\varepsilon, v, \mu) = (0, 0, \mu_0)$ with

$$\partial_v G(0, 0; \mu_0) \in \{\lambda_1(\mu_0) - \lambda_2(\mu_0), \lambda_1(\mu_0) - 1, 1 - \lambda_2(\mu_0)\},$$

which is negative in the corresponding case. Here we use that, by assumption, $f_1 \prec_{\mu_0} f_2$, with

$$f_1(s;\mu) = \begin{cases} s^{\lambda_1(\mu)}, \\ s\omega(s;\alpha(\mu)), \end{cases} \quad \text{and} \quad f_2(s;\mu) = \begin{cases} s^{\lambda_2(\mu)}, \\ s\omega(s;\alpha(\mu)), \end{cases}$$

and we apply (c) in Lemma A.4. Therefore $\partial_v G(0, 0; \mu_0) < 0$. We can now apply Theorem A.1 to the function $(\varepsilon, v, \mu) \longmapsto G(\varepsilon, v; \mu)$ at the point $(0, 0, \mu_0)$ in order to conclude that there exists a continuous function $v(\varepsilon, \mu)$, with $v(0, \mu_0) = 0$ and such that $G(\varepsilon_1, v_1; \mu_1) = 0$, if and only if $v_1 = v(\varepsilon_1, \mu_1)$, provided $(\varepsilon_1, v_1, \mu_1)$ is close enough to $(0, 0, \mu_0)$. Taking $\sigma(\varepsilon;\mu) := \sigma_0(\varepsilon;\mu)\big(1 + v(\varepsilon;\mu)\big)$, the combination of this with the previous claim proves the statement of Corollary 1.3.

In order to prove that (b) implies (c), we must check that the unique zero of $\Delta(s, \varepsilon;\mu)$ which has the form $s = \sigma(\varepsilon;\mu)$ obtained before is of multiplicity 1. We have that

$$\partial_s\Delta = \partial_s f_1(s)(c_1 + \psi_1(s)) + f_1(s)\partial_s\psi_1(s) - \partial_s f_2(s+\varepsilon)(c_2 + \psi_2(s+\varepsilon)) - f_2(s+\varepsilon)\partial_s\psi_2(s+\varepsilon)$$

We treat three cases:

1. $f_i(s) = s^{\lambda_i}$, $i = 1, 2$,
2. $f_1(s) = s^{\lambda_1}$, $f_2(s) = s\omega_\alpha(s)$, with $\lambda_1(\mu_0) < 1$,
3. $f_1(s) = s\omega_\alpha(s)$ and $f_2(s) = s_2^\alpha(\mu_0)$, with $\lambda_2(\mu_0) > 1$.

In the first case, using (7) and (8), we get the evaluation of $\frac{\partial_s\Delta}{(\lambda_1-1)s^{\lambda_1-1}}$ at $s = \sigma(\varepsilon)$:

$$\left.\frac{\partial_s\Delta}{(\lambda_1 - 1)s^{\lambda_1-1}}\right|_{s=\sigma(\varepsilon)} = c_1 + \psi_1(\sigma(\varepsilon)) + \frac{\sigma(\varepsilon)}{\lambda_1\frac{c_2}{c_1}f_2(\varepsilon)g_1}\left[\frac{c_2}{c_1}f_2(\varepsilon)g_1\partial_s\psi_1(\sigma(\varepsilon)) - \lambda_2\frac{f_2(\varepsilon)g_2}{\sigma(\varepsilon)+\varepsilon}(c_2 + \psi_2(\sigma(\varepsilon)+\varepsilon)) - \frac{f_2(\sigma(\varepsilon)+\varepsilon)}{\sigma(\varepsilon)+\varepsilon}\mathscr{D}\psi_2(\sigma(\varepsilon)+\varepsilon)\right].$$

It tends to $c_1(\mu) \neq 0$, as $\varepsilon \to 0^+$, uniformly on $\mu \approx \mu_0$, because

(i) g_i are bounded,

(ii) $\psi_i(s)$ and $s\partial_s\psi_i(s)$ tend to zero, as $s \to 0^+$, uniformly on compact sets of μ,

(iii) $\frac{\sigma(\varepsilon)}{\sigma(\varepsilon)+\varepsilon} = \frac{\sigma(\varepsilon)/\varepsilon}{\sigma(\varepsilon)/\varepsilon+1} \to 0$, as $\varepsilon \to 0$, thanks to the fact $\sigma_1(\varepsilon) = \sigma_0(\varepsilon)/\varepsilon \to 0$, as $\varepsilon \to 0$,

(iv) $\frac{f_2(\sigma(\varepsilon)+\varepsilon)}{f_2(\varepsilon)} = \left(\frac{\sigma(\varepsilon)}{\varepsilon} + 1\right)^{\lambda_2} \to 1$, as $\varepsilon \to 0$.

Consider now the second case: $f_1(s) = s^{\lambda_1}$ and $f_2(s) = s\omega_\alpha(s)$ (with $\lambda_1(\mu_0) < 1$). Then

$$\left.\frac{\partial_s\Delta}{(\lambda_1 - 1)s^{\lambda_1-1}}\right|_{s=\sigma(\varepsilon)} = c_1 + \psi_1(\sigma(\varepsilon)) + \frac{\sigma(\varepsilon)}{\lambda_1\frac{c_2}{c_1}f_2(\varepsilon)g_1}\left[\frac{c_2}{c_1}f_2(\varepsilon)g_1\partial_s\psi_1(\sigma(\varepsilon)) - \left((1-\alpha)\frac{f_2(\varepsilon)g_2}{\sigma(\varepsilon)+\varepsilon} - 1\right)(c_2 + \psi_2(\sigma(\varepsilon)+\varepsilon)) - \frac{f_2(\sigma(\varepsilon)+\varepsilon)}{\sigma(\varepsilon)+\varepsilon}\mathscr{D}\psi_2(\sigma(\varepsilon)+\varepsilon)\right].$$

It tends to $c_1(\mu)$, as $\varepsilon \to 0^+$, uniformly on $\mu \approx \mu_0$, because

(i)
$$\frac{\sigma(\varepsilon)}{f_2(\varepsilon)}\left((1-\alpha)\frac{f_2(\varepsilon)g_2}{\sigma(\varepsilon)+\varepsilon}-1\right)=\frac{\sigma(\varepsilon)}{\sigma(\varepsilon)+\varepsilon}\left(\frac{(1-\alpha)f_2(\varepsilon)g_2}{f_2(\varepsilon)}-\frac{(\sigma(\varepsilon)+\varepsilon)}{f_2(\varepsilon)}\right)\to 0,$$
as $\varepsilon\to 0^+$, thanks to the fact that $\frac{\sigma(\varepsilon)+\varepsilon}{f_2(\varepsilon)}=\frac{\varepsilon\left(1+\frac{\sigma(\varepsilon)}{\varepsilon}\right)}{\varepsilon\omega_\alpha(\varepsilon)}=\frac{1+\frac{\sigma(\varepsilon)}{\varepsilon}}{\omega_\alpha(\varepsilon)}\to\frac{|\alpha|-\alpha}{2}\approx 0$, as $\varepsilon\to 0^+$, uniformly on $\mu\approx\mu_0$.

(ii) $\frac{f_2(\sigma(\varepsilon)+\varepsilon)}{f_2(\varepsilon)}=\frac{\sigma(\varepsilon)+\varepsilon}{\varepsilon}\frac{\omega_\alpha(\varepsilon(1+(1+v(\varepsilon))\sigma_1(\varepsilon)))}{\omega_\alpha(\varepsilon)}=\frac{\sigma(\varepsilon)+\varepsilon}{\varepsilon}\frac{u^{-\alpha}\omega_\alpha(\varepsilon)+\omega_\alpha(u)}{\omega_\alpha(\varepsilon)}\to 0$, as $\varepsilon\to 0$, where $u=1+(1+v(\varepsilon))\sigma_1(\varepsilon)\to 1$, as $\varepsilon\to 0$.

In the third case $f_1(s)=s\omega_\alpha(s)$ and $f_2(s)=s^{\lambda_2}$ (with $\lambda_2(\mu_0)>1$), we consider $\frac{\partial_s\Delta}{(1-\alpha)\omega-1}\Big|_{s=\sigma(\varepsilon)}$. We get

$$\begin{aligned}\frac{\partial_s\Delta}{(1-\alpha)\omega-1}\Big|_{s=\sigma(\varepsilon)}=&c_1+\psi_1(\sigma(\varepsilon))+\frac{\sigma(\varepsilon)}{(1-\alpha)\frac{c_2}{c_1}f_2(\varepsilon)g_1-\sigma(\varepsilon)}\left[\frac{c_2}{c_1}f_2(\varepsilon)g_1\partial_s\psi_1(\sigma(\varepsilon))\right.\\&\left.-\lambda_2\frac{f_2(\varepsilon)g_2}{\sigma(\varepsilon)+\varepsilon}(c_2+\psi_2(\sigma(\varepsilon)+\varepsilon))-\frac{f_2(\sigma(\varepsilon)+\varepsilon)}{\sigma(\varepsilon)+\varepsilon}\mathscr{D}\psi_2(\sigma(\varepsilon)+\varepsilon)\right].\end{aligned}$$

It tends to $c_1(\mu)$, as $\varepsilon\to 0^+$, uniformly on $\mu\approx\mu_0$, because

(i) $\frac{\sigma(\varepsilon)}{f_2(\varepsilon)}\sim\frac{f_1^{-1}(x)}{x}\Big|_{x=\frac{c_2}{c_1}f_2(\varepsilon)}\sim\frac{1}{\omega_\alpha(x)}\Big|_{x=\frac{c_2}{c_1}f_2(\varepsilon)}$ tends to $\frac{|\alpha|-\alpha}{2}\approx 0$, as $\varepsilon\to 0^+$, uniformly on $\mu\approx\mu_0$.

(ii) $\frac{f_2(\sigma(\varepsilon)+\varepsilon)}{(1-\alpha)f_2(\varepsilon)-\sigma(\varepsilon)}=\frac{(\sigma(\varepsilon)+\varepsilon))^{\lambda_2}}{(1-\alpha)\varepsilon^{\lambda_2}-\sigma(\varepsilon)}=\frac{\left(1+\frac{\sigma(\varepsilon)}{\varepsilon}\right)^{\lambda_2}}{1-\alpha-\frac{\sigma(\varepsilon)}{f_2(\varepsilon)}}$ is bounded, as $\varepsilon\to 0^+$, uniformly on $\mu\approx\mu_0$.

By definition (d) implies (a). This completes the proof of the result. ■

3 Proof of Theorem B

Taking into account the symmetry $(x,y)\to(x,-y)$ of the Loud system, it suffices to consider half of the period, i.e. the time between local transverse sections $\Sigma_\pm$ at the outer boundary placed on $y=0$ with $\pm x>0$. The singular point $S_D=(-1/D,0)$ will be allowed to belong to one of the sections $\Sigma_\pm$. For convenience we introduce an auxiliary transverse section Σ_0 on $x=0$ and $y>0$. The outer boundary of the period annulus intersects the three transverse sections. We study the time function T and its derivative of orbits from Σ_+ to Σ_- parametrized by points on Σ_0.

Let us explain it in more detail using some facts proved in [9]. The invariant algebraic curves of the Loud system are the line at infinity L_∞, the line $L_1=\{x=1\}$ and the conic $C=\{y^2/2=a(D,F)x^2+b(D,F)x+c(D,F)\}$, where

$$a(D,F)=\frac{D}{2(1-F)},\quad b(D,F)=\frac{(D-F+1)}{(1-F)(1-2F)},\quad c(D,F)=\frac{(F-D-1)}{2F(1-F)(1-2F)}.$$

When $D+F=0$ the conic C becomes degenerated. More precisely, it is the union of the two straight lines $y=\frac{Fx-1}{\pm\sqrt{F(F-1)}}$ passing through the point S_D, which is a hyperbolic saddle, for $D\notin[-1,0]$. In that case the outer boundary of the period annulus is contained in the triangle $C\cup L_1$ and $C\cup L_\infty$, for $F<0$ and $F>1$ respectively. The other two symmetric vertices $Q^\pm$ are also hyperbolic saddles at finite distance, for $F<0$, or at infinity, for $F>1$. For $D+F\neq 0$, the local separatrices of the saddle point S_D disconnect from those of $Q^\pm$, i.e. breaking of the saddle connections occurs, for $D+F=0$.

If $F+D$ is small and $F>1$ (resp. $F<0$), then the period annulus is bounded by

1. a homoclinic loop through S_D (resp. $C \cup L_1$), if $F + D < 0$,
2. a triangle $C_0 \cup L_1 \ni S_D$ if $F + D = 0$,
3. $C \cup L_\infty$ (resp. a homoclinic loop through S_D), if $F + D > 0$.

Blowing up the point S_D in the Loud family we obtain an exceptional divisor E with two saddles points with hyperbolicity ratios $1/2$. In this situation the bifurcation along $F + D = 0$ becomes the breaking of the heteroclinic connections between the saddle points on E and $C \cap L_1$, see Figure 1.

We introduce two classes of functions suitable for dealing with the remainder terms in our study. In order to formalize both notions we consider families of smooth functions on $s > 0$. More concretely, consider $K \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ and an open subset U of $\mathbb{R}^N$. We say that a function $\psi(s; \mu)$ belongs to the class $\mathscr{C}^K_{s>0}(U)$ if there exist an open neighbourhood Ω of $\{s = 0\} \times U$ in $\mathbb{R}^{N+1}$ such that $(s, \mu) \mapsto \psi(s; \mu)$ is $\mathscr{C}^K$ on $\Omega \cap \big((0, +\infty) \times U\big)$.

Definition 3.1. Let $\mathscr{D} := s\partial_s$ be the *Euler operator* and consider some $\hat{\mu} \in U$. We say that $\psi(s; \mu) \in \mathscr{C}^K_{s>0}(U)$ belongs to the class $\mathcal{I}_K(\hat{\mu})$ if for each $k = 0, 1, \ldots, K$ there exists a neighbourhood V of $\hat{\mu}$ such that

$$\lim_{s \to 0^+} \mathscr{D}^k \psi(s; \mu) = 0 \text{ uniformly on } \mu \in V.$$

If W is a (not necesarily open) subset of U then we define $\mathcal{I}_K(W) = \bigcap_{\hat{\mu} \in W} \mathcal{I}_K(\hat{\mu})$.

Definition 3.2. Given some $L \in \mathbb{R}$ and $\hat{\mu} \in U$, we say that $\psi(s; \mu) \in \mathscr{C}^K_{s>0}(U)$ is (L, K)-*flat with respect to s at $\hat{\mu}$*, and we write $\psi \in \mathcal{F}^K_L(\hat{\mu})$, if for each $\nu = (\nu_0, \ldots, \nu_N) \in \mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu| = \nu_0 + \cdots + \nu_N \leqslant K$ there exist a neighbourhood V of $\hat{\mu}$ and $C, s_0 > 0$ such that

$$\left| \frac{\partial^{|\nu|} \psi(s; \mu)}{\partial s^{\nu_0} \partial \mu_1^{\nu_1} \cdots \partial \mu_N^{\nu_N}} \right| \leqslant C s^{L - \nu_0} \text{ for all } s \in (0, s_0) \text{ and } \mu \in V.$$

If W is a (not necessarily open) subset of U, then define $\mathcal{F}^K_L(W) := \bigcap_{\hat{\mu} \in W} \mathcal{F}^K_L(\hat{\mu})$.

The first definition is suitable for performing the derivation-division algorithm and the second is well-adapted when derivation with respect to the parameters is needed. The second notion is more general and the precise relationship is given by the following result, see [12, Lemma A.6]:

Lemma 3.3. *The inclusion $\mathcal{F}^K_{L+\varepsilon}(W) \subset s^L \mathcal{I}_K(W)$ holds for any $\varepsilon > 0$.*

Proof of Theorem B. Let us prove (a). By [9, Theorem A], the segment $F + D = 0$, $F \in [\frac{3}{2}, 2]$ is entirely composed of local bifurcation values at the outer boundary and the segment $F + D = 0$, $F \in (0, 1) \setminus \{1/2\}$ is entirely composed of local regular values at the outer boundary. It only remains to show that the points of $F + D = 0$, with $F \in (-\infty, 0) \cup (1, \frac{3}{2}) \cup (2, +\infty)$ are local regular values at the outer boundary.

Taking into account the symmetry $(x, y) \to (x, -y)$ of the Loud system, it suffices to consider half of the period, i.e. the time between local transverse sections $\Sigma_\pm$ at the outer boundary placed on $y = 0$. At the level $D + F = 0$ upper half of the boundary of the period annulus is then given by a saddle connection of two saddles $P_\pm$ and the other separatrix of one of them. By convention, we denote the saddle more to the left by P_- (see Figure 1 for the case $F > 1$ and [9, Figure 4] for the case $F < 0$). Note that one of these saddle points $P_\pm$ is on one of the transverse sections $\Sigma_\pm$. For $D + F \neq 0$, the saddle connection breaks. We denote by S_- the local separatrix of P_- and S_+ the local separatrix of P_+. We also introduce an auxiliary transverse section Σ_0 on $x = 0$ and $y > 0$.

Let us denote by $S_\pm \cap \Sigma_0 = \{(0, \zeta_\pm)\}$, $\sigma_\pm(s) = (0, \zeta_\pm(1 - s))$, $\sigma(s) = (0, \zeta(1 - s))$, $\zeta = \min(\zeta_+, \zeta_-)$,

$$\varepsilon := \zeta_+ - \zeta_- = (D + F) U(D, F), \quad \text{with} \quad U(D, F) > 0 \quad \text{for } F > 1.$$

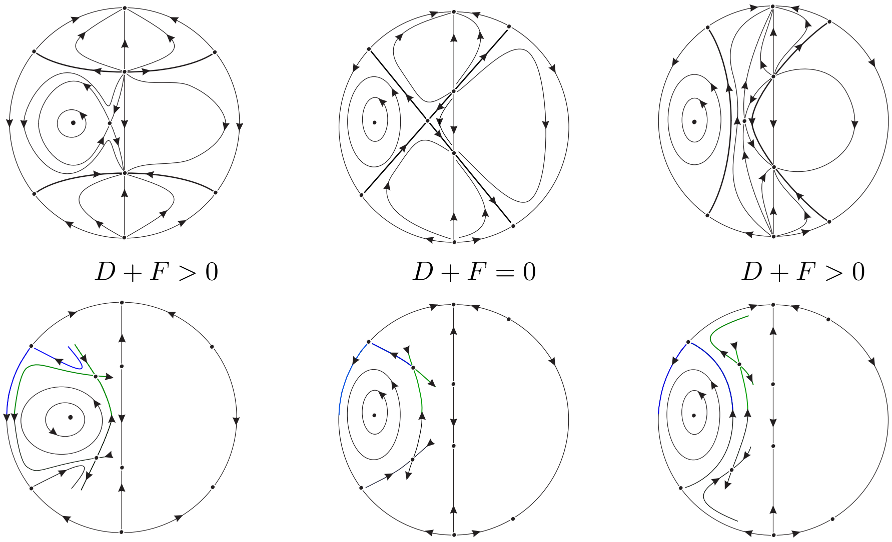

Figure 1: In the first row, phase portrait of (5), with $D<-1$ and $F>1$ in the Poincaré disc, where we use thick lines to draw the conic. In the second row, the phase portrait near the period annulus after blowing-up the saddle point $S_D=(-1/D,0)$.

Indeed, the Loud systems (5) possesses the Darboux first integral

$$H(x,y)=(1-x)^{-2F}\left(\frac{y^2}{2}-(a(D,F)x^2+b(D,F)x+c(D,F))\right).$$

If $F>1$ then $\{(0,\pm\zeta_+)\}=\{H(x,y)=H(-1/D,0)\}\cap\{x=0\}$ and $\{(0,\pm\zeta_-)\}=\{H(x,y)=0\}\cap\{x=0\}$. A straightforward computation shows that

$$\varepsilon=(D+F)\frac{(F-1)}{2\,F(2F-1)}\frac{1}{\sqrt{(F-1)\,F}}\left(\frac{F-1}{F}\right)^{-2\,F}+o(D+F) \tag{9}$$

is analytic in a neighborhood of any point (D,F) with $D+F=0$. Let $T_\pm(s)$ be the (signed) Dulac time of the trajectory starting at the point $\sigma_\pm(s)\in\Sigma_0$ ending in $\Sigma_\pm$ and let $T(s)$ be the period of the orbit passing through $\sigma(s)$. By definition T_+ is negative because we follow the trajectory in opposite sense. Then

$$\frac{1}{2}T(s)=\begin{cases} T_-(s)-T_+(s+\varepsilon), & \text{if} \quad \varepsilon\geq 0\\ T_-(s-\varepsilon)-T_+(s), & \text{if} \quad \varepsilon\leq 0\end{cases}$$

$$\frac{1}{2}T'(s)=\begin{cases} T'_-(s)-T'_+(s+\varepsilon), & \text{if} \quad \varepsilon\geq 0\\ T'_-(s-\varepsilon)-T'_+(s), & \text{if} \quad \varepsilon\leq 0.\end{cases} \tag{10}$$

We will apply (i) of Theorem A to the equation $T'(s)=0$. We must study the behaviour of the derivative of the Dulac time of a hyperbolic saddle at finite distance or at infinity. By [13, Theorem A] and Lemma 3.3, it follows that the derivative of the Dulac time of a finite saddle of the family has the form

$$\tau'(s;\mu)=\frac{1}{s}\left[-c_0(\mu)+\psi(s;\mu)\right], \quad \text{with} \quad c_0(\mu)>0 \tag{11}$$

and $\psi\in\mathcal{I}_1$, i.e. $\psi\to 0$, $s\psi'\to 0$ as $s\to 0^+$ uniformly on μ. Moreover, the derivative of the Dulac time of the saddle point at infinity of hyperbolicity ratio $r(\mu)>0$, $r(\mu)\neq 1$, of the Loud family is given by

$$\tau'(s;\mu)=s^{\lambda-1}(\lambda c_1(\mu)+\psi(s;\mu)), \tag{12}$$

with $\lambda = \min(1, r(\mu))$ and $\psi \in \mathcal{I}_1$.

In the case $F < 0$, we have $P_- = (-1/D, 0) \in \Sigma_-$ is a finite saddle point invariant by the symmetry $\rho(x,y) = (x, -y)$ so that T_- is half of the Dulac time of P_- between the transverse sections Σ_0 and $\rho(\Sigma_0)$. $P_+ = (1, \sqrt{-(D+1)/F})$ is finite saddle point. By (11), $T'_\pm(s) = \pm s^{-1}(c_\pm + \psi_\pm(s))$, with $c_\pm > 0$ and $\psi_\pm \in \mathcal{I}_1$. We cannot apply directly (i) of Theorem A. Instead of considering the equation $\frac{1}{2}T'(s) = 0$, we replace it by the equivalent equation $1/T'_-(s) - 1/T'_+(s+\varepsilon) = 0$, if $\varepsilon = \zeta_+ - \zeta_- \geq 0$ and similarly if $\varepsilon \leq 0$. Now we can apply assertion (b) in case (i) of Theorem A and conclude that there is no bifurcation.

If $F > 1$, then P_- is a saddle at infinity with hyperbolicity ratio $r(\mu) = \frac{1}{2(F-1)}$, whereas $P_+ = (-1/D, 0) \in \Sigma_+$ is a finite saddle. By analogous arguments as in the study of T_- for $F < 0$, we have $T'_+(s) = s^{-1}(c_+ + \psi(s))$, with $c_+ > 0$. The asymptotic expansion for T_- depends on $r(\mu) > 1$, or $r(\mu) < 1$, which corresponds to $F \in (1, 3/2)$ or $F > 3/2$ respectively. In the first case, thanks to (12), we have $T'_-(s) = c_1(\mu) + \psi(s;\mu)$. Hence, $\lim\limits_{s\to 0^+} T'_+(s) = \infty \neq c_1 = \lim\limits_{s\to 0^+} T'_-(s)$. Using the same trick as in the $F < 0$ case, we conclude by assertion (a) in case (i) of Theorem A that there is no bifurcation.

It only remains to study the case $F > 3/2$ in which $0 < r(\mu) < 1$. In that case [13, Theorem A] implies that $T'_+(s) = \frac{1}{s}[c_0(\mu) + \psi_+(s;\mu)]$ and $T'_-(s) = s^{r(\mu)-1}(r(\mu)c_1(\mu) + \psi_-(s;\mu))$, with $c_i(\mu)$ continuous functions, $c_0(\mu) > 0$ and $\psi_\pm \in \mathcal{I}_1$. Thus, $\lim\limits_{s\to 0^+} T'_\pm(s;\mu) = \infty$. We obtain a new translation family constructed from

$$\frac{1}{T'_+(s;\mu)} = s(c_0(\mu) + \psi_+(s;\mu))^{-1} = s(c_0^{-1}(\mu) + \hat{\psi}_+(s;\mu))$$

and

$$\frac{1}{T'_-(s;\mu)} = s^{1-r(\mu)}(r(\mu)c_1(\mu) + \psi_-(s;\mu))^{-1} = s^{1-r(\mu)}(r(\mu)^{-1}c_1^{-1}(\mu) + \hat{\psi}_-(s;\mu))$$

with $\hat{\psi}_\pm \in \mathcal{I}_1$. By Lemma A.8, the coefficient $c_1(\mu)$ is positive for $F \in (3/2, 2)$ and negative for $F > 2$. Applying Theorem A, we obtain that $Z = 1$, for $F \in (3/2, 2)$ and $Z = 0$, for $F > 2$.

Let us prove now (b), with $F = 3/2$. Since the set of bifurcation values is closed, we have $Z \geq 1$, for $F = 3/2$. It remains to show that $Z \leq 1$. To prove (b) we take $\alpha = 1 - r(\mu) \approx 0$. By [13, Theorem A] and Lemma 3.3,

$$T_-(s) = \bar{c}_0 + \bar{c}_1 s\omega + \bar{c}_2 s + s^{1+\delta}\bar{I}(s) \Rightarrow T'_-(s) = c_1\omega + c_2 + s^\delta I(s), \quad \text{where} \quad \bar{I} \in \mathcal{I}_2, I \in \mathcal{I}_1,$$

$$T_+(s) = c_0 \log s + d + s^\delta \bar{I}_+(s) \Rightarrow T'_+(s) = s^{-1}(c_0 + s^\delta I_+(s)), \quad \text{where} \quad \bar{I}_+ \in \mathcal{I}_2, I_+ \in \mathcal{I}_1$$

and $c_0 > 0$. Since $f_1(s) = \omega_\alpha(s) \prec s^{-1} = f_2(s)$ it follows as in the beginning of the proof of Proposition 2.1, that a necessary condition for $Z \geq 1$ is that $\varepsilon \geq 0$ in equation (10). Then

$$T'(s) = T'_-(s) - T'_+(s+\varepsilon) = c_1\omega_\alpha(s) + c_2 + s^\delta I(s) - (s+\varepsilon)^{-1}(c_0 + (s+\varepsilon)^\delta I_+(s+\varepsilon))$$

and it suffices to control the zeros of

$$\begin{aligned}\Delta(s;\varepsilon,\alpha) = \frac{(s+\varepsilon)}{\omega_\alpha(s)}T'(s) &= (s+\varepsilon)\left(c_1 + \frac{c_2}{\omega} + \frac{s^\delta I(s)}{\omega}\right) - \frac{c_0}{\omega} - \frac{(s+\varepsilon)^\delta}{\omega}I_+(s+\varepsilon) \\ &= \varepsilon c_1 + (\varepsilon c_2 - c_0)\frac{1}{\omega} + R, \end{aligned} \tag{13}$$

where

$$R = c_1 s + c_2\frac{s}{\omega} + (s+\varepsilon)\frac{s^\delta I(s)}{\omega} - \frac{(s+\varepsilon)^\delta}{\omega}I_+(s+\varepsilon).$$

Since

$$\mathscr{D}\Delta = (\varepsilon c_2 - c_0)\mathscr{D}\left(\frac{1}{\omega}\right) + \mathscr{D}R,$$

to prove that $Z \leq 1$ it suffices to see that

$$\frac{\mathscr{D}R}{\mathscr{D}\left(\frac{1}{\omega}\right)} \longrightarrow 0 \quad \text{as} \quad (s,\varepsilon,\alpha) \to (0^+,0^+,0).$$

We have

$$\mathscr{D}R = c_1 s + c_2\left(\frac{s}{\omega} + s\mathscr{D}\left(\frac{1}{\omega}\right)\right) + \left(\frac{(\delta+1)s^{\delta+1}+\delta\varepsilon s^{\delta}}{\omega} + (s+\varepsilon)s^{\delta}\mathscr{D}\left(\frac{1}{\omega}\right)\right)I(s) + \\ + \frac{(s+\varepsilon)s^{\delta}}{\omega}\mathscr{D}I(s) - \left(\frac{\delta(s+\varepsilon)^{\delta-1}s}{\omega} + (s+\varepsilon)^{\delta}\mathscr{D}\left(\frac{1}{\omega}\right)\right)I_+(s+\varepsilon) - \frac{(s+\varepsilon)^{\delta}}{\omega}sI_+'(s+\varepsilon).$$

Using that $\mathscr{D}\left(\frac{1}{\omega}\right) = s^{-\alpha}\omega^{-2}$ we obtain that

$$\frac{\mathscr{D}R}{\mathscr{D}\left(\frac{1}{\omega}\right)} = c_1 s^{1+\alpha}\omega^2 + c_2(s^{1+\alpha}\omega + s) + \left(s^{\delta+\alpha}((\delta+1)s+\delta\varepsilon)\omega + (s+\varepsilon)s^{\delta}\right)I(s) + \\ + s^{\delta+\alpha}(s+\varepsilon)\omega\mathscr{D}I(s) - \left(\delta(s+\varepsilon)^{\delta-1}s^{1+\alpha}\omega + (s+\varepsilon)^{\delta}\right)I_+(s+\varepsilon) - (s+\varepsilon)^{\delta-1}s^{1+\alpha}\omega[(\mathscr{D}I_+)(s+\varepsilon)].$$

All the summands, except perhaps the last two, tend to zero, as $s \to 0^+$. The last two summands tend to zero, as $(s,\varepsilon) \to (0^+,0^+)$ and α is small enough. Indeed, $I_+(s+\varepsilon)$ and $(\mathscr{D}I_+)(s+\varepsilon)$ tends to 0 as $(s,\varepsilon) \to (0^+,0^+)$ due to $I_+ \in \mathcal{I}_1$ and on the other hand

$$(s+\varepsilon)^{\delta-1}s^{1+\alpha}\omega_\alpha(s) = s^{\frac{\delta}{2}}\omega_\alpha(s)\left[(s+\varepsilon)^{\delta-1}s^{1+\alpha-\frac{\delta}{2}}\right].$$

The first factor tends to zero, as $s \to 0^+$, and making the substitution $s = r\cos\theta$, $\varepsilon = r\sin\theta$ in the second factor, we deduce that

$$\left[(s+\varepsilon)^{\delta-1}s^{1+\alpha-\frac{\delta}{2}}\right] = r^{\frac{\delta}{2}+\alpha}(\cos\theta+\sin\theta)^{\delta-1}(\cos\theta)^{1+\alpha-\frac{\delta}{2}}$$

tends to zero, as $r \to 0^+$, uniformly in $\theta \in \left[0,\frac{\pi}{2}\right]$, if α is small enough. Here, we have taken $\delta \in (0,1)$.

Assertion (c) will be proved in the next section together with the proof of Theorem C. ■

Remark 3.4. From (13), we deduce that, as $s \to 0^+$, the curve

$$C_s = \{(\varepsilon,\alpha) \,:\, \Delta(s;\varepsilon,\alpha) = 0\} = \{(\varepsilon,\alpha) \,:\, \varepsilon c_1 + c_\alpha(\varepsilon c_2 - c_0) + (\varepsilon c_2 - c_0)\left(\frac{1}{\omega_\alpha(s)} - c_\alpha\right) + R(s;\varepsilon,\alpha) = 0\}$$

tends to the bifurcation curve at the outer boundary of the period annulus

$$\Gamma_B = \lim_{s\to 0^+} C_s = \{(\varepsilon,\alpha) \,:\, \varepsilon c_1(\varepsilon,\alpha) + c_\alpha(\varepsilon c_2(\varepsilon,\alpha) - c_0(\varepsilon,\alpha)) = 0\}.$$

Since $c_\alpha = \frac{|\alpha|-\alpha}{2}$ and $c_0(\varepsilon,\alpha), c_1(\varepsilon,\alpha), c_2(\varepsilon,\alpha)$ are $\mathscr{C}^\infty$ we obtain a corner in Γ_B at the point $(-3/2, 3/2)$. If $\alpha > 0$, the linear part of the equation defining Γ_B is $\varepsilon c_1(0,0) = 0$ and if $\alpha < 0$ the linear part is $\varepsilon c_1(0,0) + \alpha c_0(0,0) = 0$.

4 Study of the point $(D,F) = (-2,2)$

In this section we will prove assertion (c) in Theorem B and Theorem C. We will use the notations introduced in the proof of Theorem B in the previous section.

Proof of assertion (c) in Theorem B and Theorem C. If $F \approx 2$ then the hyperbolicity ratio of P_- is $r = r(F) = \frac{1}{2(F-1)} \approx \frac{1}{2}$. Defining $\bar{\alpha} := 1 - 2r(F) = \frac{F-2}{F-1}$ and applying Theorem A of [13] we obtain

$$T_-(s) = T_{00} + T_{01}s^r + T_{101}s\omega_{\bar{\alpha}}(s) + T_{100}s + \mathcal{F}^\infty_{3/2-\delta}(s)$$

for every $\delta > 0$. On the other hand, half of the Dulac time of the finite singular point at $(-1/D, 0)$ has the form

$$T_+(s) = c_0 \log s + T^+_{00} - T^+_{101} s \log s + T^+_{100} s + \mathcal{F}^\infty_{2-\delta}(s)$$

with $c_0 > 0$. Let $\varepsilon = (D+F)U(D,F)$ be the breaking parameter of the connexion, where $U(D,F) > 0$ by (9). Since $f_1(s) = s^{r-1} \prec s^{-1} = f_2(s)$ it follows again as in the beginning of the proof of Proposition 2.1, that no bifurcation occurs for $\varepsilon \leq 0$, see equation (10). Therefore we must study the zeros of the function

$$T'(s) = T'_-(s) - T'_+(s+\varepsilon), \quad \text{with } \varepsilon \geq 0.$$

To this end we define

$$\begin{aligned}
\bar{\Delta}(s,D,F) &:= s^{1-r}(s+\varepsilon)T'(s) = (s+\varepsilon)\left[rT_{01} + T_{101}(1-\bar{\alpha})s^{1-r}\omega_{\bar{\alpha}}(s) + (T_{100} - T_{101})s^{1-r} + \mathcal{F}^\infty_{1-\delta}(s)\right] \\
&\qquad + s^{1-r}\left[-c_0 + T^+_{101}(s+\varepsilon)\log(s+\varepsilon) - (T^+_{100} - T^+_{101})(s+\varepsilon) + \mathcal{F}^\infty_{2-\delta}(s+\varepsilon)\right] \\
&= \varepsilon r T_{01} + \varepsilon T_{101}(1-\bar{\alpha})s^{1-r}\omega_{\bar{\alpha}}(s) + \left[-c_0 + \varepsilon(T_{100} - T_{101} - T^+_{100} + T^+_{101})\right]s^{1-r} \\
&\quad + T^+_{101}s^{1-r}(s+\varepsilon)\log(s+\varepsilon) + \mathcal{F}^\infty_{1-\delta}(s) + s^{1-r}\mathcal{F}^\infty_{2-\delta}(s+\varepsilon).
\end{aligned}$$

Let us work from now on with the variables $z = s^{1-r}$, $\alpha = \frac{\bar{\alpha}}{1-r} = \frac{F-2}{F-\frac{3}{2}}$ and $\varepsilon = (D+F)U(D,F)$. Note that the map $(D,F) \mapsto (\varepsilon, \alpha)$ is a local diffeomorphism at the point $(D,F) = (-2,2)$. By Lemma A.8, we have that $T_{01}(D,F) = -\alpha U_{01}(\varepsilon,\alpha)$ with $U_{01}(0,0) > 0$ and $T_{101}(-2,2) > 0$. Dividing $\bar{\Delta}$ by minus the coefficient in s^{1-r} and taking into account that $\omega_{\bar{\alpha}}(s) = \omega_{\bar{\alpha}}(z^{\frac{1}{1-r}}) = \frac{1}{1-r}\omega_\alpha(z)$ we obtain the function

$$\Delta(z,\varepsilon,\alpha) := -\varepsilon\alpha c_1(\varepsilon,\alpha) + \varepsilon c_2(\varepsilon,\alpha) z\omega_\alpha(z) - z + \underbrace{c_3(\varepsilon,\alpha)zh(z,\varepsilon,\alpha) + g(z,\varepsilon,\alpha) + zf(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha)}_{R(z,\varepsilon,\alpha)} \tag{14}$$

which controls the positive zeros of $T'(s)$, where $c_1(\varepsilon,\alpha) = rU_{01}(\varepsilon,\alpha)$ and $c_2(\varepsilon,\alpha) = \frac{(1-\bar{\alpha})T_{101}}{1-r}$ are $\mathscr{C}^\infty$, with $\bar{c}_1 = c_1(0,0)$ and $\bar{c}_2 = c_2(0,0)$ positive, $h(z,\varepsilon,\alpha) = (z^{2-\alpha}+\varepsilon)\log(z^{2-\alpha}+\varepsilon)$ and $f, g \in \mathcal{F}^\infty_{2-\delta}$. In fact, $c_3(\varepsilon,\alpha)h(z,\varepsilon,\alpha) + f(z^{2-\alpha}+\varepsilon) = h_0(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha)$ where

$$h_0(z,\varepsilon,\alpha) := c_3(\varepsilon,\alpha)z\log z + f(z,\varepsilon,\alpha) \in \mathcal{F}^\infty_{1-\delta}. \tag{15}$$

Let us prove (c) in Theorem B. We claim that the functions $1, z\omega_\alpha(z)$ and

$$\mathscr{R}(z;\varepsilon,\alpha) = -z + R(z;\varepsilon,\alpha)$$

form an extended complete Chebyshev system, for $z \in (0,\epsilon)$, see [8]. Since the Wronskian $\mathscr{W}(1, z\omega_\alpha(z)) > 0$ for $z > 0$ small, it suffices to see that $W(\mathscr{R}(z;\varepsilon,\alpha))$ does not vanish on $(0,\epsilon)$, where $W(\rho)$ is defined as the Wronskian $\mathscr{W}(1, z\omega_\alpha(z), \rho) = \partial_z(z\omega_\alpha(z))\partial_z^2\rho - \partial_z^2(z\omega_\alpha(z))\partial_z\rho$. Since the Wronskian is linear we compute the contribution of each summand of $\mathscr{R}$ separately.

(a) Since $\partial_z(z\omega_\alpha(z)) = (1-\alpha)\omega_\alpha(z) - 1$ and $\partial_z^2(z\omega_\alpha(z)) = (1-\alpha)z^{-1-\alpha}$ we have that $W(-z) = (1-\alpha)z^{-1-\alpha}$.

(b) Since $g \in \mathcal{F}^\infty_{2-\delta}$ and $\omega_\alpha(z) \in \mathcal{F}^\infty_{-\delta}$, we have that $\partial_z^r g \in \mathcal{F}^\infty_{2-r-\delta}$,

$$W(g) \in ((1-\alpha)\omega_\alpha(z) - 1)\mathcal{F}^\infty_{-\delta} - (1-\alpha)z^{-1-\alpha}\mathcal{F}^\infty_{1-\delta} \subset \mathcal{F}^\infty_{-2\delta}$$

and $\frac{W(g)}{W(-z)} \in \mathcal{F}^\infty_{1-3\delta}$ tends to zero, as $z \to 0$, uniformly on $(\varepsilon,\alpha) \approx (0,0)$.

(c) Since $\partial_z(zh_0(z^{2-\alpha}+\varepsilon)) = h_0(z^{2-\alpha}+\varepsilon) + (2-\alpha)z^{2-\alpha}(\partial_z h_0)(z^{2-\alpha}+\varepsilon)$ and

$$\partial_z^2(zh_0(z^{2-\alpha}+\varepsilon)) = (2-\alpha)(3-\alpha)z^{1-\alpha}(\partial_z h_0)(z^{2-\alpha}+\varepsilon) + (2-\alpha)^2 z^{3-2\alpha}(\partial_z^2 h_0)(z^{2-\alpha}+\varepsilon),$$

with $h_0 \in \mathcal{F}_{1-\delta}^\infty$, one can check that

$$\frac{W(zh_0(z^{2-\alpha}+\varepsilon))}{W(-z)} = -h_0(z^{2-\alpha}+\varepsilon) + P_1(\omega_\alpha(z))z^2(\partial_z h_0)(z^{2-\alpha}+\varepsilon) + P_2(\omega_\alpha(z))z^{4-\alpha}(\partial_z^2 h_0)(z^{2-\alpha}+\varepsilon),$$

$P_i(\omega)$ being degree 1 polynomials, and consequently

$$\left|\frac{W(zh_0(z^{2-\alpha}+\varepsilon))}{W(-z)} + h_0(z^{2-\alpha}+\varepsilon)\right| \le C_1\frac{|z|^{2-\delta}}{|z^{2-\alpha}+\varepsilon|^\delta} + C_2\frac{|z|^{4-2\delta}}{|z^{2-\alpha}+\varepsilon|^{1+\delta}}.$$

The limit as $(z,\varepsilon,\alpha) \to (0,\varepsilon_0,\alpha_0)$ of this upper bound is equal to zero. This is clear for $\varepsilon_0 > 0$ and the the case $\varepsilon_0 = 0$ follows by using the quasihomogeneous blow-up $z = r\sin\theta$, $\varepsilon = r^{2-\alpha}\cos\theta$, with $\theta \in [0,\pi/2]$ and the fact that the functions $\frac{\sin^a\theta}{(\sin^b\theta+\cos\theta)^c}$ are bounded in $\theta \in [0,\pi/2]$, for $a,b,c>0$. On account of this and Lemma A.2 we get that $\frac{W(zh_0(z^{2-\alpha}+\varepsilon))}{W(-z)} + h_0(z^{2-\alpha}+\varepsilon)$ tends to 0 as $z \to 0$ uniformly on $(\varepsilon,\alpha) \approx (0,0)$. Consequently, as moreover $h_0 \in \mathcal{F}_{1-\delta}^\infty$ we deduce that $\lim_{z\to 0}\frac{W(zh_0(z^{2-\alpha}+\varepsilon))}{W(-z)} = -h_0(\varepsilon)$ uniformly on $(\varepsilon,\alpha) \approx (0,0)$.

Hence

$$\frac{W(\mathscr{R})}{W(z)} \to -1 + h_0(\varepsilon) < 0, \quad \text{as } z \to 0, \text{ uniformly on } (\varepsilon,\alpha) \approx (0,0), \text{ with } h_0(0) = 0. \tag{16}$$

Consequently, $1, z\omega_\alpha(z)$ and $\mathscr{R}(z;\varepsilon,\alpha)$ form a Chebyshev system in a suitable interval $(0,\epsilon)$. Hence, $F(z;\varepsilon,\alpha)$ has at most 2 zeros in $(0,\epsilon)$ and $Z_{T'} \le 2$.

On the other hand $Z_{T'} \ge 2$ will follow once we prove assertion (a) in Theorem C, i.e. the existence of the double bifurcation curve Γ arriving to the point $(-2,2)$. This will complete the proof of assertion (c) in Theorem B.

In order to prove assertion (a) in Theorem C we consider the system

$$\begin{cases} \Delta(z,\varepsilon,\alpha) = 0 \\ \partial_z\Delta(z,\varepsilon,\alpha) = 0 \end{cases} \tag{17}$$

where $\Delta(z,\varepsilon,\alpha)$ is given in (14) and for convenience we define

$$G(z,\varepsilon,\alpha) = \partial_z\Delta(z,\varepsilon,\alpha) = \varepsilon(1-\alpha)c_2(\varepsilon,\alpha)\omega_\alpha(z) - (1+\varepsilon c_2(\varepsilon,\alpha)) + S(z,\varepsilon,\alpha),$$

where

$$S(z,\varepsilon,\alpha) = c_3(\varepsilon,\alpha)(h(z,\varepsilon,\alpha) + z\partial_z h(z,\varepsilon,\alpha)) + zf_1(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha) + g_1(z,\varepsilon,\alpha).$$

The system (17) will implicitly define a curve in the (ε,α)-plane, which gives the curve Γ, coming back to the (D,F)-plane by the local diffeomorphism $(D,F) \mapsto (\varepsilon,\alpha)$.

Let us define

$$H(z,\varepsilon,\alpha) = (\alpha-1)\Delta(z,\varepsilon,\alpha) + zG(z,\varepsilon,\alpha) = \alpha\varepsilon(1-\alpha)c_1(\varepsilon,\alpha) - (\alpha+\varepsilon c_2(\varepsilon,\alpha))z + zV_0(z,\varepsilon,\alpha),$$

where $V_0 = (\alpha-1)R_0 + S$,

$$R_0(z,\varepsilon,\alpha) = c_3(\varepsilon,\alpha)h(z,\varepsilon,\alpha) + f(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha) + g_0(z,\varepsilon,\alpha),$$

$f \in \mathcal{F}_{2-\delta}^\infty$, $g_0 = g/z \in \mathcal{F}_{1-\delta}^\infty$, $f_1 = (2-\alpha)\partial_z f \in \mathcal{F}_{1-\delta}^\infty$ and $g_1 = \partial_z g \in \mathcal{F}_{1-\delta}^\infty$. Hence,

$$V_0(z,\varepsilon,\alpha) = c_3(\varepsilon,\alpha)(\alpha h(z,\varepsilon,\alpha) + z\partial_z h(z,\varepsilon,\alpha)) + f_0(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha) + zf_1(z^{2-\alpha}+\varepsilon,\varepsilon,\alpha) + f_2(z,\varepsilon,\alpha), \tag{18}$$

with $f_0 = (\alpha-1)f \in \mathcal{F}^{\infty}_{2-\delta}$ and $f_1, f_2 = (\alpha-1)g_0 + g_1 \in \mathcal{F}^{\infty}_{1-\delta}$. We are interested in the solutions of the system $\Delta(z,\varepsilon,\alpha) = G(z,\varepsilon,\alpha) = 0$, for $z > 0$. For technical reasons we will rather study the system $G(z,\varepsilon,|\alpha|) = H(z,\varepsilon,|\alpha|) = 0$ which is equivalent to the preceding one on $\alpha \geq 0$. In order to avoid writing the absolute value of α we will make all the manipulations on $\alpha \geq 0$ and they must be extended by parity to $\alpha < 0$. Note that the functions R_0, S, V_0 extend continuously in a neighborhood of $(z,\varepsilon,\alpha) = (0,0,0)$ thanks to [12, Lemma A.1].

We would like to solve the system $G(z,\varepsilon,\alpha) = H(z,\varepsilon,\alpha) = 0$ by applying the implicit function theorem. Unfortunately, the hypothesis are not fulfilled. Therefore, we replace $c_i(\varepsilon,\alpha)$ by $\bar{c}_i$ and S, V by zero and we are able to find explicitly all the solutions of the system $G = H = 0$ thus obtained. There is a unique solution, which is given by

$$\begin{aligned} z &= z_0(\alpha) := (k\alpha)^{\frac{1}{1-\alpha}} \sim k\alpha, \\ \varepsilon &= \varepsilon_0(\alpha) := \frac{1/\bar{c}_2}{(1-\alpha)\omega_\alpha(z_0(\alpha)) - 1} = \frac{k^{\frac{\alpha}{1-\alpha}}}{\bar{c}_2} \frac{\alpha^{\frac{1}{1-\alpha}}}{1-\alpha-(k\alpha)^{\frac{\alpha}{1-\alpha}}} \sim \frac{-1/\bar{c}_2}{1+\log(k\alpha)}, \end{aligned} \tag{19}$$

where $k = \frac{\bar{c}_1}{\bar{c}_2}$. Here $\sim$ means that the quotient of the two functions tends to 1 as $\alpha \to 0$.

Now, the idea is to find the solutions of the original system $G = H = 0$ as small perturbations of the above particular solution, i.e. in the following form

$$z = z_1(\alpha,u) := z_0(\alpha)(1+u), \quad \varepsilon = \varepsilon_1(\alpha,u,v) := \frac{(1+v)/\bar{c}_2}{(1-\alpha)\omega_\alpha(z_1(\alpha,u)) - 1}, \tag{20}$$

with (u,v) near $(0,0)$. Notice that $z_1(\alpha,u)$ and $\varepsilon_1(\alpha,u,v)$ tend to zero, as $\alpha \to 0^+$, uniformly on $u, v \approx 0$, so that they define continuous functions in a neighborhood of $(\alpha,u,v) = (0,0,0)$ and $z_1(\alpha_0,u)$ and $\varepsilon_1(\alpha_0,u,v)$ are $\mathscr{C}^1$ in u,v for every $\alpha_0 \geq 0$ small enough. Indeed, $z_1(0,u) = 0$ and $\omega_\alpha(z_1(\alpha,u)) = \omega_\alpha(z_0(\alpha)) + z_0(\alpha)^{-\alpha}\omega_\alpha(1+u)$ so that $\varepsilon_1(0,u,v) = 0$. Moreover, $z_1(\alpha,u) > 0$ and $\varepsilon_1(\alpha,u,v) > 0$, for all $u,v \approx 0$ and $\alpha > 0$ small enough. Setting

$$c_i(\varepsilon,\alpha) = \bar{c}_i(1 + C_i(\varepsilon,\alpha)), \quad C_i(0,0) = 0, \tag{21}$$

we define the functions

$$G_1(\alpha,u,v) := G(z_1(\alpha,u),\varepsilon_1(\alpha,u,v),\alpha) = v + (1+v)C_2(\varepsilon_1(\alpha,u,v),\alpha) + S(z_1(\alpha,u),\varepsilon_1(\alpha,u,v),\alpha)$$

and

$$\begin{aligned} H_1(\alpha,u,v) &:= \frac{1}{\alpha\varepsilon_1(\alpha,u,v)} H(z_1(\alpha,u),\varepsilon_1(\alpha,u,v),\alpha) \\ &= \bar{c}_1(1+C_1)(1-\alpha) - \bar{c}_2(1+C_2)(1+u)k^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} \\ &\quad - \frac{z_1(\alpha,u)}{\varepsilon_1(\alpha,u,v)} + \frac{z_1(\alpha,u)V_0(z_1(\alpha,u),\varepsilon_1(\alpha,u,v),\alpha)}{\alpha\varepsilon_1(\alpha,u,v)} \end{aligned} \tag{22}$$

Putting

$$\Phi(\alpha,u,v) = (G_1(\alpha,u,v), H_1(\alpha,u,v)), \tag{23}$$

we will show that the implicit function problem $\Phi(\alpha,u(\alpha),v(\alpha)) = (0,0)$, with $(u(0),v(0)) = (0,0)$ has a unique solution $(u,v) = (u(\alpha),v(\alpha))$. By (20), this will define the curve $\varepsilon(\alpha) = \varepsilon_1(\alpha,u(\alpha),v(\alpha))$, and ultimately the curve Γ in the (D,F) parameter space, along which $z = z_0(\alpha)(1+u(\alpha))$ is a double critical period of the corresponding Loud system.

It remains to verify the hypothesis of the implicit function Theorem A.1, for (23). By (21), it follows that G_1 is continuous in a neighborhood of $(\alpha,u,v) = (0,0,0)$. In order to prove the continuity at $(0,0,0)$ of the second component of Φ, given by H_1, it suffices to show that $H_1(\alpha,u,v)$ tends to 0, for $\alpha \to 0^+$, uniformly on $u,v \approx 0$. This has to be verified only for the last two terms in (22). For the before last term, it follows from Lemma 4.1.a. Indeed, we have in particular, $\frac{(z_1(\alpha,u))^{1-\delta}}{\varepsilon_1(\alpha,u,v)} = \frac{(k\alpha)^{\frac{1-\delta}{1-\alpha}}(1+u)^{1-\delta}}{\varepsilon_1(\alpha,u,v)}$ also tends to zero, as $\alpha \to 0^+$, uniformly on $u, v \approx 0$, if δ is small enough. For the last term, since $\frac{z_1(\alpha,u)}{\alpha} = k^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}(1+u)$ is bounded for $\alpha \geq 0$, the function $\frac{z_1(\alpha,u)}{\alpha}\frac{V_0(z_1(\alpha,u),\varepsilon_1(\alpha,u,v))}{\varepsilon_1(\alpha,u,v)}$ also tends to zero, as $\alpha \to 0^+$, uniformly on $u, v \approx 0$, thanks to Lemma 4.1.b. This shows that by putting $\Phi(0,0,0) = 0$, and extending by parity in α, the function Φ extends to a continuous function in a neighborhood of $(0,0,0)$.

It follows by composition of differentiable functions and the fact that and $z_1(\alpha_0, u)$ $\varepsilon_1(\alpha_0, u, v)$ are $\mathscr{C}^1$ in u, v, for every $\alpha_0 \geq 0$ small enough.

Finally, we have that $G_1(0, u, v) = v$, and $H_1(0, u, v) = -\bar{c}_1 u$, with $\bar{c}_1 \neq 0$. Hence, $\frac{\partial \Phi}{\partial(u,v)}(0,0,0)$ is surjective. Applying Goursat's implicit function Theorem A.1, we obtain continuous functions $u = u(\alpha)$ and $v = v(\alpha)$, with $u(0) = v(0) = 0$, defining thus the curve Γ.

Let us prove assertion (b). It can be checked, using (19) and assertion (a) in Lemma A.4, that

$$\lim_{\alpha \to 0^+} (-\log \alpha)\varepsilon_1(\alpha, u, v) = \frac{1+v}{\bar{c}_2}$$

uniformly on (u, v) in a neighborhood of $(0, 0)$. Consequently,

$$\lim_{\alpha \to 0^+} \frac{\varepsilon_1(\alpha, u(\alpha), v(\alpha))}{\frac{-1/\bar{c}_2}{\log \alpha}} = 1, \tag{24}$$

which gives us the asymptotic behavior of Γ at the point $(-2, 2)$ with respect to the axis $\{\alpha = 0\} = \{F = 2\}$. Indeed, it follows from (24) that, for any $\delta > 0$, for $\alpha > 0$ sufficiently close to 0, we have

$$1 - \delta < -\bar{c}_2 \varepsilon_1 \log \alpha < 1 + \delta.$$

Hence,

$$e^{\frac{-1-\delta}{\bar{c}_2 \varepsilon_1}} < \alpha < e^{\frac{-1+\delta}{\bar{c}_2 \varepsilon_1}}.$$

In order to prove (c) of Theorem C, we consider the family of functions $\Delta_{\varepsilon,\alpha}(z)$ given by (14). For parameters (ε, α) on the curve Γ given as a graph of $\varepsilon = \varepsilon(\alpha)$ let $z = z(\alpha)$ be the corresponding double zero i.e. $\Delta(z(\alpha), \varepsilon(\alpha), \alpha) = 0$ and

$$\frac{\partial}{\partial z}\Delta|_{(z(\alpha),\varepsilon(\alpha),\alpha)} = 0. \tag{25}$$

The idea is simple. First we give the proof modulo two technical claims, whose proof will be given at the end. We consider the function $\Delta_{\varepsilon(\alpha_1),\alpha_1}$, for $\alpha_1 > 0$ small enough in order to satisfy (26) and (28). We know that $\Delta_{\varepsilon(\alpha_1),\alpha_1}$ has a double zero at $z(\alpha_1)$.

$$\text{Claim 1:} \qquad \frac{\partial^2 \Delta_{\varepsilon(\alpha_1),\alpha_1}}{\partial z^2}(z(\alpha_1)) < 0. \tag{26}$$

Hence, the function $\Delta_{\varepsilon(\alpha_1),\alpha_1}$ is strictly concave in a neighborhood of $z(\alpha_1)$. It follows that it has the value 0 as a strict local maximum at $z(\alpha_1)$. Moreover, by (26), applying the implicit function theorem to equation

$$\frac{\partial}{\partial z}\Delta(z, \varepsilon, \alpha) = 0 \tag{27}$$

it follows that (27) has a unique solution $z = z(\varepsilon, \alpha)$ in a neighborhood of $z(\alpha_1)$, for (ε, α) in a neighborhood of $(\varepsilon(\alpha_1), \alpha_1)$. Moreover, by continuity of $\frac{\partial^2 \Delta}{\partial z^2}$, it follows from (26) that for parameters in a small neighborhood of $(\varepsilon(\alpha_1), \alpha_1)$, the solution $z = z(\varepsilon, \alpha)$ of the implicit function problem (27) is a local maximum of $\Delta_{\varepsilon,\alpha}$.

We consider the function $\Delta_{\varepsilon(\alpha_1),\alpha}$, for α in an neighborhood of α_1. Note that here $\varepsilon = \varepsilon(\alpha_1)$ is fixed and the point $(\varepsilon(\alpha_1), \alpha)$ leaves the curve Γ.

$$\text{Claim 2:} \qquad \frac{\partial \Delta_{\varepsilon(\alpha_1),\alpha}(z(\varepsilon(\alpha_1), \alpha))}{\partial \alpha}|_{\alpha=\alpha_1} < 0. \tag{28}$$

From Claim 2 it will follow that decreasing the value of α from $\alpha = \alpha_1$ close to α_1, the maximal value of $\Delta_{\varepsilon(\alpha_1),\alpha}$ will increase and will hence become positive (it is zero for $\alpha = \alpha_1$). From local concavity of $\Delta_{\varepsilon(\alpha_1),\alpha_1}$, for $\delta > 0$ arbitrarily small, we have that $\Delta_{\varepsilon(\alpha_1),\alpha_1}(z(\alpha_1) \pm \delta) < 0$. The condition $\Delta_{\varepsilon(\alpha_1),\alpha}(z(\alpha_1) \pm \delta) < 0$ will be preserved for $\alpha < \alpha_1$ sufficiently close. By continuity of $\Delta_{\varepsilon(\alpha_1),\alpha}$ and sign change it follows that there exist two zeros $z_\alpha^\pm$ of $\Delta_{\varepsilon(\alpha_1),\alpha}$ verifying

$$z(\alpha_1) - \delta < z_\alpha^- < z(\varepsilon(\alpha_1), \alpha) < z_\alpha^+ < z(\alpha_1) + \delta.$$

The proof of assertion (b) of Theorem C will be completed, once we prove the two claims (26) and (28). We first prove claim (26). We perform the standard division-derivation procedure. Due to (16) we know that

$$\frac{\partial \left(\frac{\partial \Delta / \partial z}{\partial (z\omega_\alpha)/\partial z} \right)}{\partial z} < 0,$$

for z, ε, α sufficiently small strictly positive. Using now the formula for the derivative of a quotient, (25) and the fact that $\frac{\partial (z\omega_\alpha)}{\partial z} > 0$, then Claim 1 (26) follows.

Let us now prove Claim 2. We consider the function $\eta(\alpha) = \Delta(z(\varepsilon(\alpha_1), \alpha), \varepsilon(\alpha_1), \alpha)$, for fixed α_1 (and hence fixed $\varepsilon(\alpha_1)$. We have to prove that $\frac{d\eta}{d\alpha}(\alpha_1) < 0$. Recall that $\frac{\partial}{\partial z}\Delta(z(\varepsilon(\alpha_1), \alpha_1), \varepsilon(\alpha_1), \alpha_1) = 0$. Hence, by the chain rule it follows that $\frac{d\eta}{d\alpha}(\alpha_1) = \frac{\partial}{\partial \alpha}\Delta(z(\varepsilon(\alpha_1), \alpha_1), \varepsilon(\alpha_1), \alpha)|_{\alpha=\alpha_1}$.

Now, using (14) and putting $\varepsilon_1 = \varepsilon(\alpha_1)$ and $z_1 = z(\alpha_1)$, we have

$$\begin{aligned} \frac{1}{\varepsilon_1}\frac{\partial}{\partial \alpha}\Delta(z(\varepsilon_1, \alpha_1), \varepsilon_1, \alpha)|_{\alpha=\alpha_1} &= -c_1(\varepsilon_1, \alpha_1) + c_2(\varepsilon_1, \alpha_1) z_1 \omega_{\alpha_1}(z_1) \\ &\quad + c_2(\varepsilon_1, \alpha_1) z_1 \frac{\partial \omega_\alpha(z_1)}{\partial \alpha}\Big|_{\alpha=\alpha_1} + \frac{1}{\varepsilon_1}\frac{\partial R(z_1, \varepsilon_1, \alpha)}{\partial \alpha}|_{\alpha=\alpha_1}. \end{aligned} \tag{29}$$

Knowing that $\varepsilon_1 > 0$, it will be enough to prove that the expression (29) is negative, for α_1 sufficienty small. Recall that the first term in the right hand side of (29) tends to $-\bar{c}_1 < 0$. We show that all other terms will tend to zero, for $\alpha_1 \to 0$. Recall that $z^\delta \omega \to 0$, for $\delta > 0$, see assertion (c) of Lemma A.4. This solves the second term.

For the third term, recalling that $|\frac{\partial \omega_\alpha}{\partial \alpha}/\omega^2|$ is bounded (see assertion (d) of Lemma A.4), or that $|\partial_\alpha \omega(z; \alpha)| \le C z^{-\delta}$ using [12, Lemma A.4(b)], it follows equally that the third term tends to zero. Finally, in order to prove that $\frac{1}{\varepsilon_1}\frac{\partial R(z_1, \varepsilon_1, \alpha)}{\partial \alpha}|_{\alpha=\alpha_1}$ tends to zero as $\alpha_1 \to 0$, we use (20) for $(u, v) = (0, 0)$ i.e. (19). By continuity, this will be enough in order to show that (29) is negative for $\alpha_1 > 0$ small enough. We have that the growth of $1/\varepsilon_1$ is bounded by $C \log \alpha_1$, see (19) and (20). On the other hand, all the terms of $\frac{\partial R(z_1, \varepsilon_1, \alpha)}{\partial \alpha}|_{\alpha=\alpha_1}$ are bounded by $C_1 z_1$, for some $C_1 > 0$. Indeed, $R(z, \varepsilon, \alpha) = z h_0(z^{2-\alpha} + \varepsilon, \varepsilon, \alpha) + g(z, \varepsilon, \alpha)$ with $h_0 \in \mathcal{F}_{1-\delta}$ defined in (15) and $g \in \mathcal{F}_{2-\delta}$. Then $|\partial_\alpha g| \le C z^{2-\delta}$ and

$$\partial_\alpha (h_0(z^{2-\alpha} + \varepsilon, \varepsilon, \alpha)) = z^{2-\alpha}(-\log z)\partial_z h_0(z^{2-\alpha} + \varepsilon, \varepsilon, \alpha) + \partial_\alpha h_0(z^{2-\alpha} + \varepsilon, \varepsilon, \alpha)$$

is bounded in absolute value by

$$C|z^{2-\alpha} + \varepsilon|^{-\delta} z^{2-\alpha}(-\log z) + C'|z^{2-\alpha} + \varepsilon|^{1-\delta} \le C'' z^{1-\alpha}|z^{2-\alpha} + \varepsilon|^{-\delta} + C'|z^{2-\alpha} + \varepsilon|^{1-\delta} \le C'''.$$

In order to verify the last inequality it suffices to check that the first summand in the second term tends to 0 as $(z, \varepsilon) \to (0^+, 0^+)$ uniformly on $\alpha \approx 0$ making the weighted blow-up $z = r \sin\theta$, $\varepsilon = r^{2-\alpha}\cos\theta$, $\theta \in [0, \pi/2]$. Now from $z_1 \sim \alpha_1$, it follows that $\frac{1}{\varepsilon_1}\frac{\partial R(z_1, \varepsilon_1, \alpha)}{\partial \alpha}|_{\alpha=\alpha_1}$ is bounded by $C\alpha_1 \log \alpha_1$ which tends to zero as $\alpha_1 \to 0$. ∎

Lemma 4.1. *(a) If $\delta > 0$ is small enough, then $\frac{\alpha^{1-\delta}}{\varepsilon_1(\alpha,u,v)}$ tends to zero, as $\alpha \to 0^+$ uniformly on $u, v \approx 0$.*

(b) $\frac{1}{\varepsilon_1(\alpha,u,v)} V_0(z_1(\alpha,u), \varepsilon_1(\alpha,u,v), \alpha)$ tends to zero, as $\alpha \to 0^+$ uniformly on $u, v \approx 0$.

Proof. In order to see assertion (a) we write

$$\begin{aligned}
\frac{\alpha^{1-\delta}}{\varepsilon_1(\alpha,u,v)} &= \frac{\bar{c}_2 \alpha^{1-\delta}}{1+v}((1-\alpha)\omega_\alpha(z_0(\alpha)(1+u)) - 1) \\
&= \frac{\bar{c}_2}{1+v}\alpha^{1-\delta}\left[(1-\alpha)\omega_\alpha(z_0(\alpha)) + (1-\alpha)z_0(\alpha)^{-\alpha}\omega_\alpha(1+u) - 1\right] \\
&= \frac{\bar{c}_2(1-\alpha)}{1+v}\left[\frac{(k\alpha)^{\frac{-\alpha}{1-\alpha}} - 1}{\alpha^\delta} + \alpha^{1-\delta-\frac{\alpha}{1-\alpha}} k^{\frac{-\alpha}{1-\alpha}}\omega_\alpha(1+u) - \frac{\alpha^{1-\delta}}{1-\alpha}\right].
\end{aligned}$$

The terms $\frac{\bar{c}_2(1-\alpha)}{1+v}$ and $k^{\frac{-\alpha}{1-\alpha}}\omega_\alpha(1+u)$ are uniformly bounded in $\alpha, u, v \approx 0$. The terms $\alpha^{1-\delta-\frac{\alpha}{1-\alpha}}$ and $\frac{\alpha^{1-\delta}}{1-\alpha}$ tend to zero as $\alpha \to 0$. To deal with the remaining term we apply L'Hôpital's rule obtaining that

$$\begin{aligned}
\lim_{\alpha\to 0^+} \frac{(k\alpha)^{\frac{-\alpha}{1-\alpha}} - 1}{\alpha^\delta} &= \lim_{\alpha\to 0^+} \frac{e^{-\frac{\alpha}{1-\alpha}\log(k\alpha)} - 1}{\alpha^\delta} = -\lim_{\alpha\to 0^+} \frac{(k\alpha)^{\frac{-\alpha}{1-\alpha}}\left(\frac{1+2\alpha}{(1-\alpha)^2}\log(k\alpha) + \frac{1}{1-\alpha}\right)}{\delta\alpha^{\delta-1}} \\
&= \frac{-1}{\delta}\lim_{\alpha\to 0^+} \alpha^{1-\delta-\frac{\alpha}{1-\alpha}}\left(\frac{1+2\alpha}{(1-\alpha)^2}\log(k\alpha) + \frac{1}{1-\alpha}\right) = 0
\end{aligned}$$

provided $\delta < 1$.

Proof of claim (b): According to (18), it suffices to see that

(i) $\frac{\alpha}{\varepsilon_1} h(z_1, \varepsilon_1, \alpha)$,

(ii) $\frac{(z\partial_z h)(z_1,\varepsilon_1,\alpha)}{\varepsilon_1} = (2-\alpha)\frac{z_1^{2-\alpha}}{\varepsilon_1}\left(\log(z_1^{2-\alpha} + \varepsilon_1) + 1\right) = (2-\alpha)\left(\frac{z_1}{\varepsilon_1}\right)^{2-\alpha}\varepsilon_1^{2-\alpha}\left(\log\left(\varepsilon_1\left(1 + \frac{z_1^{2-\alpha}}{\varepsilon_1}\right)\right) + 1\right)$,

(iii) $\left|\frac{f_0(z_1^{2-\alpha}+\varepsilon_1,\varepsilon_1,\alpha)}{\varepsilon_1}\right| \leq C\frac{(z_1^{2-\alpha}+\varepsilon_1)^{2-\delta}}{\varepsilon_1} = C\varepsilon_1^{1-\delta}\left(1 + \frac{z_1^{2-\alpha}}{\varepsilon_1}\right)^{2-\delta}$,

(iv) $\left|\frac{z_1 f_1(z_1^{2-\alpha}+\varepsilon_1,\varepsilon_1,\alpha)}{\varepsilon_1}\right| \leq C\frac{z_1}{\varepsilon_1}(z_1^{2-\alpha} + \varepsilon_1)^{1-\delta} = C\frac{z_1}{\varepsilon_1^\delta}\left(1 + \frac{z_1^{2-\alpha}}{\varepsilon_1}\right)^{1-\delta} = Cz_1^{1-\delta}\left(\frac{z_1}{\varepsilon_1}\right)^\delta\left(1 + \frac{z_1^{2-\alpha}}{\varepsilon_1}\right)^{1-\delta}$,

(v) $\left|\frac{f_2(z_1,\varepsilon_1,\alpha)}{\varepsilon_1}\right| \leq C\frac{z_1^{1-\delta}}{\varepsilon_1}$

tend to zero, as $\alpha \to 0^+$, uniformly on $u, v \approx 0$, using the previous assertion (a) and the fact that $f_0 \in \mathcal{F}_{2-\delta}^\infty$ and $f_1, f_2 \in \mathcal{F}_{1-\delta}^\infty$. ■

5 The state of art of the conjectural bifurcation diagram of the critical periods in Loud system

We resume our study of the bifurcation diagram of the period function of the quadratic centers that we initiate in [7]. Let us explain succinctly the results we have obtained so far on this issue. The *dehomogeneized* Loud's family (5) of quadratic reversible centers is $\{X_\mu\}_{\mu\in\mathbb{R}^2}$, where

$$X_\mu := -y(1-x)\partial_x + (x + Dx^2 + Fy^2)\partial_y \text{ with } \mu := (D, F),$$

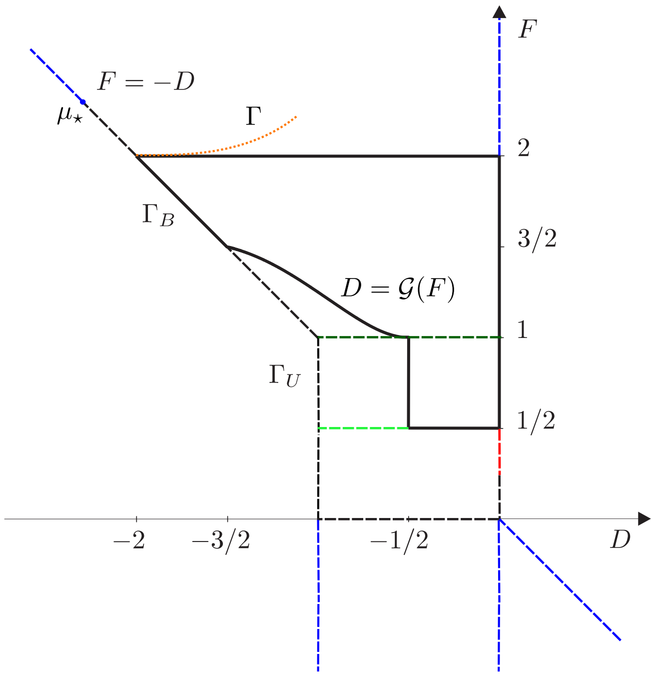

Figure 2: Bifurcation diagram of the period function at the polycycle according to [9] and, in colour, the subsequent improvements due to [5, 10, 11, 25, 26], where $\mu_\star = (-F_\star, F_\star)$ with $F_\star \approx 2.34$. The curve that joins $\left(-\frac{3}{2}, \frac{3}{2}\right)$ and $\left(-\frac{1}{2}, 1\right)$ is the graphic of an analytic function $D = \mathscr{G}(F)$. The double bifurcation curve Γ of Theorem C appears starting from the point $(-2, 2)$.

Following this notation, let Γ_U be the union of dotted straight lines in Figure 2, whatever its colour is. Consider also the bold curve Γ_B. (Here the subscripts B and U stand for bifurcation and unspecified respectively.) Then [9, Theorem A] shows that the open set $\mathbb{R}^2 \setminus (\Gamma_B \cup \Gamma_U)$ corresponds to regular values and that the ones in Γ_B are bifurcation values (cf. Definition 1.1). We also conjectured that any parameter in Γ_U is regular, except for the segment $\{0\} \times \left[0, \frac{1}{2}\right]$ in the vertical axis, that should consist of bifurcation parameters. Since the formulation of this conjecture there have been some progress in the study of the parameters in Γ_U:

- From the results in [5, 26] it follows that the parameters in blue are indeed regular. In these papers the authors determine a region M in the parameter plane for which the corresponding center has a globally monotonic period function. The parameters that we draw in blue are inside the interior of M, which prevents the bifurcation of critical periodic orbits.

- By [10, Theorem C] it follows that the parameters in dark green are regular as well. In that paper the authors give an asymptotic expansion of the Dulac time of an unfolding of a saddle-node. The techniques used in [9] enable only to study an unfolding of a hyperbolic saddle.

- Theorem B in [11] shows that the parameters in red, more precisely the segment $\{0\} \times \left[\frac{1}{4}, \frac{1}{2}\right]$, are bifurcation values of the period function at the polycycle. To this end, after blowing up the polycycle, the authors show that any neighbourhood of a parameter $\hat{\mu} \in \{0\} \times \left[\frac{1}{4}, \frac{1}{2}\right]$ contains two parameters, say μ_+ and μ_-, such that the derivative of the period function near the polycycle is positive for X_{μ_+} and negative for X_{μ_-}.

- More recently, by [25, Corollary B] it follows that the parameters in light green are regular as well.

Beyond the dichotomy regular vs bifurcation, a challenging problem is the study of the cyclicity of critical periods Z of the bifurcation parameters, i.e., to compute the *exact number* of critical periodic orbits that bifurcate from the polycycle. With respect to this problem see the results for the Loud family in [21, 22, 23].

What remains to study in the bifurcation diagram of the period function at the outer boundary is the following:

- Along the segment $\{F = 0, D \in [-1, 0]\}$, which is conjectured to be regular ($Z = 0$), a saddle-node bifurcation occurs. Unfortunately, we can not apply directly the results in [10] because in the bifurcations studied there the outer boundary of the period annulus had a part of the line at infinity for all values of the parameters. This is not the case for $F = 0$ as a separatrix bounding the period annulus bifurcates from the line at infinity.

- Along the segments $\{D = 0, F \in [0, \frac{1}{4}]\}$ and $\{D = -1, F \in [0, 1]\}$ bifurcations of degenerate (nilpotent) singularites at the outer boundary of the period annulus occur. Along the first one we conjecture that $Z = 2$ and that a curve of double critical periods arrives to the point $(0, 0)$. On the other hand, we also conjecture that the second one is regular ($Z = 0$). We think that, in order to show the regularity in this situation, it is necessary to make higher dimensional blow-ups.

A Appendix

We put in the appendix some classical results that we need, as well as some specific technicalities

A.1 Classical results

We start with Goursat's version of the implicit function theorem which requires continuous differentiability only with respect to the variable that we isolate.

Theorem A.1. *[3] Let X be an open subset of $\mathbb{R}^n$ and let W be an open subset of $\mathbb{R}^k$. Consider $(x_0, w_0) \in X \times W$ and $\Phi : X \times W \to \mathbb{R}^k$ be such that*

(a) $\Phi(x_0, w_0) = 0$;

(b) $\Phi(x, w)$ is continuous on $X \times W$;

(c) $\partial_w \Phi(x, \cdot)$ is continuous on W, for all $x \in X$;

(d) $\partial_w \Phi(x_0, w_0)$ is surjective.

Then there exist a neighborhood $X_1 \times W_1$ of (x_0, w_0) and a function $\phi : X_1 \to W_1$ such that $\phi(x_0) = w_0$ and for every $(x_1, w_1) \in X_1 \times W_1$ we have $\Phi(x_1, w_1) = 0$ if and only if $w_1 = \phi(x_1)$. Moreover, ϕ is continuous.

Lemma A.2. *Let $\{f_\mu\}_{\mu \in U}$ be a continuous family of functions on $(0, s_0)$ and let $K \subset U$ be a compact set. Then $\lim_{s \to 0^+} f_\mu(s) = \ell(\mu)$, uniformly on $\mu \in K$, if and only if $\lim_{(\mu,s) \to (\hat{\mu}, 0^+)} f_\mu(s) = \ell(\hat{\mu})$, for every $\hat{\mu} \in K$.*

Proof. Assume that $\lim_{s \to 0^+} f_\mu(s) = \ell(\mu)$, uniformly on $\mu \in K$. Then $\mu \longmapsto \ell(\mu)$ is continuous on K. Let us show now that, under the uniformity assumption, $f_\mu(s)$ tends to $\ell(\hat{\mu})$, as $(s, \mu) \longrightarrow (0^+, \hat{\mu})$. Consider a given $\varepsilon > 0$. Then, thanks to the claim, there exists a neighbourhood $\mathscr{U}$ of $\hat{\mu}$ such that $|\ell(\mu) - \ell(\hat{\mu})| < \varepsilon/2$

for all $\mu \in \mathscr{U}$. Furthermore, on account of the uniformity, there exists $\delta > 0$ such that $|f_\mu(s) - \ell(\mu)| < \varepsilon/2$ for all $s \in (0, \delta)$ and $\mu \in \mathscr{U}$. Consequently,

$$|f_\mu(s) - \ell(\hat{\mu})| \leqslant |f_\mu(s) - \ell(\mu)| + |\ell(\mu) - \ell(\hat{\mu})| < \varepsilon, \text{ for all } s \in (0, \delta) \text{ and } \mu \in \mathscr{U},$$

and so $\lim_{(\mu,s)\to(\hat{\mu},0^+)} f_\mu(s) = \ell(\hat{\mu})$, as desired. Suppose now that $\lim_{(\mu,s)\to(\hat{\mu},0^+)} f_\mu(s) = \ell(\hat{\mu})$, for every $\hat{\mu} \in K$. Then the map $(s, \mu) \longmapsto f_\mu(s)$ extends continuously to $[0, s_0/2] \times K$, which is compact. So the map is uniformly continuous, which clearly implies that $\lim_{s\to 0^+} f_\mu(s) = \ell(\mu)$ is uniform on K. This proves the result. ∎

It will be convenient in order to apply the implicit function theorem, to work with functions defined in an open neighborhood of the origin. For that reason, we extend monotone function $\hat{f}_\mu$ defined on a one-sided neighborhood of the origin to an odd function $\hat{f}_\mu$ defined in a full neighborhood of the origin.

Lemma A.3. *Let $\{f_\mu\}_{\mu\in U}$ be a continuous family of functions on $(0, s_0)$ with $\lim_{s\to 0^+} f_\mu(s) = 0$ uniformly on U. For each $\mu \in U$, we define*

$$\hat{f}_\mu(s) = \begin{cases} f_\mu(s), & \text{if } s \in (0, s_0), \\ 0, & \text{if } s = 0, \\ -f_\mu(-s), & \text{if } s \in (-s_0, 0). \end{cases}$$

Then $\{\hat{f}_\mu\}_{\mu\in U}$ is a continuous family of functions on $(-s_0, s_0)$. If in addition $s \longmapsto f_\mu(s)$ is monotonous on $(0, s_0)$, for all $\mu \in U$, then $\{\hat{f}_\mu^{-1}\}_{\mu\in U}$ is a continuous family of functions on $(-s_1, s_1)$, for some $s_1 > 0$.

Proof. The continuity of $(s, \mu) \longmapsto \hat{f}_\mu(s)$ at some $(\hat{s}, \hat{\mu}) \in (0, s_0) \times U$ is obvious, for $\hat{s} \neq 0$, whereas, for $\hat{s} = 0$, it follows by applying Lemma A.2. Suppose additionally that f_μ is monotonous on $(0, s_0)$ for all $\mu \in U$. Then $\hat{f}_\mu$ is monotonous on $(-s_0, s_0)$ for all $\mu \in U$. Accordingly $(s, \mu) \longmapsto (\hat{f}_\mu(s), \mu)$ is an injective continuous map from the open set $(-s_0, s_0) \times U \subset \mathbb{R}^k$ to $\mathbb{R}^k$. Then, by the Domain Invariance Theorem, it follows that there exists $s_1 > 0$ such that $\{\hat{f}_\mu^{-1}\}_{\mu\in U}$ is a continuous family of functions on $(-s_1, s_1)$. Hence, the result is proved. ∎

A.2 Technicalities

Recall (4) that ω is a deformation of the logarithmic function. The first claim of the following lemma is the deformation of the fomula for the logarithm of a product for the function ω.

Lemma A.4. *The following hold:*

(*a*) $\omega(ab; \alpha) = a^{-\alpha}\omega(b; \alpha) + \omega(a; \alpha)$,

(*b*) $\frac{1}{\omega(s;\alpha)} \to \frac{|\alpha|-\alpha}{2}$, *as $s \to 0^+$ uniformly on $\alpha \approx 0$,*

(*c*) *Let $\alpha \longmapsto \lambda(\alpha)$ be a continuous map at $\alpha = 0$. Then $1 \prec_0 s^{\lambda(\alpha)}\omega(s; \alpha)$, if and only if $\lambda(0) > 0$.*

(*d*) $|\frac{\partial\omega}{\partial\alpha}/\omega^2|$ *is bounded.*

Proof. The equality in (*a*) is straightforward taking the definition of $\omega(s; \alpha)$ into account. The assertion in (*b*) follows easily from the inequality $\omega(s; \alpha) \geqslant \inf(-\log s, 1/|\alpha|)$. Concerning (*c*), the sufficiency follows writing the compensator as $\omega(s; \alpha) = F(\alpha \log s) \log s$, where $F(x) := \frac{e^{-x}-1}{x}$, and using that $|F(x)| \leqslant e^{|x|}$ for all $x \in \mathbb{R}$. To show the necessity we use Lemma A.2, which implies that $\lim_{(\alpha,s)\to(\hat{\alpha},0^+)} s^{\lambda(\alpha)}\omega(s; \alpha) = 0$ for any $\hat{\alpha} \in [-\delta, \delta]$ with $\delta > 0$ sufficiently small. Clearly this is not possible if $\lambda(0) \leqslant 0$ because then $s^{\lambda(0)}\omega(s; 0) = s^{\lambda(0)} \log s$ tends to $-\infty$ as $s \longrightarrow 0^+$. Thus $\lambda(0) > 0$, and so (*c*) follows. Claim (d) follows from Lemma 4.1.1 in [8]. ∎

Remark A.5. If $\lim_{s\to0^+}\Psi_1(s;\mu)=L(\mu)$ and $\lim_{s\to0^+}\Psi_2(s;\mu)=0$, with both limits being uniform on μ, then $\lim_{s\to0^+}\Psi_1\big(\Psi_2(s;\mu);\mu\big)=L(\mu)$, uniformly on μ.

We shall deal with two types of families of admissible functions, $\{s^\lambda\}_{\lambda>0}$ and $\{s\omega(s;\alpha)\}_{\alpha\approx0}$, both defined in principle, for $s>0$. It is clear that each function $f_\lambda(s)=s^\lambda$ in the first family is monotonously increasing and that, by applying Lemma A.3, $\{f_\lambda\}_{\lambda>0}$ and $\{f_\lambda^{-1}\}_{\lambda>0}$ can be continuously extended to $(-s_0,s_0)$ for some $s_0>0$. It is obvious in addition that $f_\lambda^{-1}(s)=s^{1/\lambda}$. In the following result we show analogous properties for the second family.

Lemma A.6. *Set $f_\alpha(s)=s\omega(s;\alpha)$. Then the following hold:*

(*a*) $f_\alpha(ab)=a^{1-\alpha}f_\alpha(b)+bf_\alpha(a)$,

(*b*) *There exists $s_0>0$ and $\varepsilon>0$ such that $\{f_\alpha\}_{\alpha\in(-\varepsilon,\varepsilon)}$ is a continuous family of monotonous increasing functions on $(0,s_0)$ with $\lim_{s\to0^+}f_\alpha(s)=0$, uniformly on $\alpha\in(-\varepsilon,\varepsilon)$. In addition $\{f_\alpha^{-1}\}_{\alpha\in(-\varepsilon,\varepsilon)}$ is a continuous family of functions on $(0,s_0)$, with $\lim_{s\to0^+}f_\alpha^{-1}(s)=0$, uniformly on $\alpha\in(-\varepsilon,\varepsilon)$.*

(*c*) $f_\alpha^{-1}(s)\sim_0\frac{s\kappa(\alpha)}{\omega(s;\alpha/(1-\alpha))}$, *where* $\kappa(\alpha):=(1-\alpha)\,\alpha^{\frac{\alpha+|\alpha|}{2(1-\alpha)}}$.

Proof. The equality in (*a*) is straightforward taking the definition of $\omega(s;\alpha)$ into account. The monotonicity in (*b*) follows using that, by (*b*) in Lemma A.4, $f'_\alpha(s)=-1+(1-\alpha)\omega(s;\alpha)$ tends to $+\infty$ as $(\alpha,s)\to(0,0^+)$. The fact that $f_\alpha(s)$ tends to zero as $s\to0^+$ uniformly on α is a consequence of (*c*) in Lemma A.4. Taking this into account, the assertion concerning f_α^{-1} follows by applying Lemma A.3. In order to show (*c*), setting $\alpha':=\frac{\alpha}{1-\alpha}$, we first claim that

$$\Psi_1(s;\alpha):=\left.\frac{f_\alpha^{-1}(u)}{\frac{u}{\omega(u;\alpha')}}\right|_{u=f_\alpha(s)}=\frac{\omega\big(s\omega(s;\alpha);\alpha'\big)}{\omega(s;\alpha)}$$

tends to $\kappa(\alpha)$ as $s\longrightarrow0^+$ uniformly on $\alpha\approx0$. Note that (*c*) will follow once we prove this claim. Indeed, since $\lim_{s\to0^+}f_\alpha^{-1}(s)=0$ uniformly on $\alpha\approx0$ by (*b*), we get the desired conclusion, by applying Remark A.5, with $\Psi_2(s;\alpha)=f_\alpha^{-1}(s)$.

In order to prove the claim, we apply Lemma A.2. To this end note that $\kappa(\alpha_0)=(1-\alpha_0)\alpha_0^{\frac{\alpha_0}{1-\alpha_0}}$, if $\alpha_0>0$, $\kappa(0)=1$ and $\kappa(\alpha_0)=1-\alpha_0$, if $\alpha_0<0$. If $\alpha_0\neq0$, then, by definition,

$$\frac{\omega\big(s\omega(s;\alpha);\alpha'\big)}{\omega(s;\alpha)}=\frac{s^{-\alpha'}\left(\frac{s^{-\alpha}-1}{\alpha}\right)^{-\alpha'}-1}{s^{-\alpha}-1}\,\frac{\alpha}{\alpha'},$$

which clearly tends to $1-\alpha_0$, as $(\alpha,s)\longrightarrow(\alpha_0,0^+)$, in case that $\alpha_0<0$. If $\alpha_0>0$, then for convenience we write the above equality as

$$\begin{aligned}\frac{\omega\big(s\omega(s;\alpha);\alpha'\big)}{\omega(s;\alpha)}&=\frac{s^{-\alpha'+\alpha\alpha'}(1-s^\alpha)^{-\alpha'}-\alpha^{-\alpha'}}{s^{-\alpha}(1-s^\alpha)}\,\frac{\alpha^{1+\alpha'}}{\alpha'}=\frac{s^{-\alpha}(1-s^\alpha)^{-\alpha'}-\alpha^{-\alpha'}}{s^{-\alpha}(1-s^\alpha)}(1-\alpha)\alpha^{\alpha'}\\&=\frac{(1-s^\alpha)^{-\alpha'}-\alpha^{-\alpha'}s^\alpha}{1-s^\alpha}(1-\alpha)\alpha^{\alpha'},\end{aligned}$$

which tends to $(1-\alpha_0)\alpha_0^{\frac{\alpha_0}{1-\alpha_0}}$, as $(\alpha,s)\longrightarrow(\alpha_0,0^+)$. It only remains to see that $\frac{\omega(s\omega(s;\alpha);\alpha')}{\omega(s;\alpha)}\longrightarrow1$ as

$(\alpha, s) \longrightarrow (0, 0^+)$. With this aim in view, some manipulations show that

$$\frac{\omega\big(s\omega(s;\alpha);\alpha'\big)}{\omega(s;\alpha)} - 1 + \alpha = \frac{\big(s\omega(s;\alpha)\big)^{-\alpha'} - 1}{\alpha'}\frac{\alpha}{s^{-\alpha}-1} - 1 + \alpha = (1-\alpha)\left(\frac{s^{-\alpha'}\left(\frac{s^{-\alpha}-1}{\alpha}\right)^{-\alpha'} - 1}{s^{-\alpha}-1} - 1\right)$$

$$=(1-\alpha)\left(\frac{\left(\frac{1-s^{\alpha}}{\alpha}\right)^{-\alpha'} - s^{\alpha}}{1-s^{\alpha}} - 1\right) = (1-\alpha)\frac{\left(\frac{1-s^{\alpha}}{\alpha}\right)^{-\alpha'} - 1}{1-s^{\alpha}} = \frac{\omega\big(\omega(s;-\alpha);\alpha'\big)}{\omega(s;-\alpha)}, \tag{30}$$

where we take $\alpha' = \frac{\alpha}{1-\alpha}$ into account several times. Note that

$$\frac{\omega\big(\omega(s;-\alpha);\alpha'\big)}{\omega(s;-\alpha)} = \frac{\omega\big(x;\alpha'\big)}{x}, \text{ with } x = \omega(s;-\alpha) \longrightarrow +\infty, \text{ as } (\alpha, s) \longrightarrow (0, 0^+), \tag{31}$$

due to $\omega(s;\alpha) \geqslant \inf(-\log s, 1/|\alpha|)$. Moreover $\omega(s;\alpha) = F(\alpha \log s)\log s$, where recall that $F(x) = \frac{e^{-x}-1}{x}$ verifies that $|F(x)| \leqslant e^{|x|}$, for all $x \in \mathbb{R}$. Accordingly, for $x > 1$ and $\alpha' \in [-\frac{1}{2}, \frac{1}{2}]$, we can assert that

$$\left|\frac{\omega(x;\alpha')}{x}\right| \leqslant \frac{\log x}{x^{1-|\alpha'|}} \leqslant \frac{\log x}{x^{1/2}}.$$

Hence $\frac{\omega(x;\alpha')}{x} \longrightarrow 0$, as $x \longrightarrow +\infty$, uniformly on $\alpha' \in [-\frac{1}{2}, \frac{1}{2}]$. This, together with (31), implies that

$$\frac{\omega\big(\omega(s;-\alpha);\alpha'\big)}{\omega(s;-\alpha)} \longrightarrow 0 \text{ as } (\alpha, s) \longrightarrow (0, 0^+)$$

because $\alpha' = \frac{\alpha}{1-\alpha} \longrightarrow 0$ as $\alpha \longrightarrow 0$. Therefore, on account of (30), we finally obtain

$$\frac{\omega\big(s\omega(s;\alpha);\alpha'\big)}{\omega(s;\alpha)} - 1 = \frac{\omega\big(\omega(s;-\alpha);\alpha'\big)}{\omega(s;-\alpha)} - \alpha \longrightarrow 0 \text{ as } (\alpha, s) \longrightarrow (0, 0^+),$$

as desired. This proves the claim and so the result follows. ■

Remark A.7. By Lemma A.3, each family $\{s^{\alpha}\}_{\alpha>0}$ and $\{s\omega(s;\alpha)\}_{\alpha\approx 0}$ extends to a continuous family of homeomorphisms $\{\hat{f}_{\alpha}\}$ on $(-s_0, s_0)$ with $\hat{f}_{\alpha}(0) = 0$. Their respective inverses form also a continuous family of functions $\{\hat{f}_{\alpha}^{-1}\}_{\alpha}$ on $(-s_1, s_1)$ for some $s_1 > 0$. In the sequel, by an abuse of notation and when there is no risk of ambiguity, we will denote $\hat{f}$ and $\hat{f}^{-1}$ by f and f^{-1}, respectively.

Lemma A.8. *The Dulac time $T_-(s;F)$ of the Loud family (5) restricted to the line $D + F = 0$ between the transverse sections $\{x = 0\}$ parametrized by $\sigma(s) = \left(0, \frac{1-s}{\sqrt{F(F-1)}}\right)$ and $\{y = 0, x < 0\}$ satisfies the following:*

(a) For every $F_0 > 3/2$, there is $\delta > 0$, such that $T_-(s;F) = T_{00}(F) + T_{01}(F)s^{r(F)} + \mathcal{F}^{\infty}_{r(F_0)+\delta}(F = F_0)$ with $r(F) = \frac{1}{2(F-1)}$ and

$$T_{01}(F) = -F^{-\frac{F}{F-1}} 2^{\frac{1}{2(F-1)}} \sqrt{F(F-1)}\sqrt{\pi}\frac{\Gamma(1 - r(F))}{\Gamma(\frac{1}{2} - r(F))}.$$

In particular, $T_{01}(F) = -(F-2)U_{01}(F)$ with $U_{01}(2) > 0$.

(b) $T_-(s;F) = T_{00}(F) + T_{01}(F)s^{r(F)} + T_{101}s\omega\left(s; \frac{F-2}{F-1}\right) + T_{100}(F)s + \mathcal{F}^{\infty}_{3/2-\delta}(F = 2)$ with $T_{101}(2) > 0$.

Proof. We perform the projective change of coordinates $(u,v) = \phi(x,y) = (1 - \sqrt{F(F-1)}\frac{y}{1-Fx}, \frac{1}{1-Fx})$ which brings the Loud vector field $(x-1)y\partial_x + (x + F(y^2 - x^2))\partial_y$ into $X_F = \frac{1}{v}[P_F(u,v)u\partial_u + Q_F(u,v)v\partial_v]$ where $P_F(u,v) = \sqrt{\frac{F-1}{F}}(u-2)(v-1)$ and $Q_F(u,v) = \frac{1}{\sqrt{F(F-1)}}(u-1)((F-1)v+1)$ and the transverse sections $x = 0$ and $y = 0$ into $v = 1$ and $u = 1$ respectively. The hyperbolicity ratio of the saddle of X_F at $(u,v) = (0,0)$ is $r(F) = \frac{1}{2(F-1)} < 1$ for $F > 3/2$. This gives us the announced expansion in (a). Let us now compute the coefficient $T_{01}(F)$. In the coordinate chart (u,v) the parametrization $\sigma(s)$ translates into $\sigma(s) = (s,1)$ and we can take the parametrization $\tau(s) = (1,s)$ in the target transverse section. By applying [7, Theorem A], after some tedious but straightforward computations, we obtain that

$$\begin{aligned} T_{01}(F) &= F^{-\frac{F}{F-1}} \left(-\sqrt{F(F-1)} + \frac{1}{2}\sqrt{\frac{F}{F-1}} \int_0^1 \left((1 - \frac{u}{2})^{-1-r(F)} - 1 \right) \frac{du}{u^{1+r(\mu)}} \right) \\ &= -F^{-\frac{F}{F-1}} 2^{\frac{1}{2(F-1)}} \sqrt{F(F-1)}\sqrt{\pi} \frac{\Gamma(1-r(F))}{\Gamma(\frac{1}{2} - r(F))} \end{aligned}$$

thanks to the formula $\int_0^1 \left((1-\frac{u}{2})^{-a} - 1\right) \frac{du}{u^a} = \frac{2^{a-1}\sqrt{\pi}\Gamma(2-a)}{(1-a)\Gamma(\frac{3}{2}-a)} - \frac{1}{1-a}$ in which appears the Gamma function Γ. In particular, $\frac{\Gamma(1-r(F))}{\Gamma(\frac{1}{2}-r(F))} = \frac{\sqrt{\pi}}{2}(F-2) + O((F-2)^2)$.

To prove (b), it suffices to take $F \approx 2$ where the announced asymptotic expansion holds by [13, Theorem A]. In fact, we have $T_-(s) = T_{00} + T_{01}s^r + T_{10}s + T_{02}s^{2r} + \mathcal{F}_{2r_0+\delta}(r = r_0)$ if $r_0 > \frac{1}{2}$ and $T_{101} = (1-2r)T_{02}$, $T_{100} = T_{10} + T_{02}$ so that

$$T_{101}|_{F=2} = \lim_{r \to \frac{1}{2}} (1-2r)T_{02} = -\lim_{r \to \frac{1}{2}} (1-2r)T_{10}.$$

By [14, Theorem A]

$$T_{10} = -\sigma_{120} \left(\frac{\sigma_{121}}{\sigma_{120} Q(0,\sigma_{120})} + \frac{\sigma_{111}}{L_1(\sigma_{120})} \hat{B}_1(1/r - 1, \sigma_{120}) \right),$$

with

$$L_1(u) = \exp \int_0^u \left(\frac{P(0,z)}{Q(0,z)} + \frac{1}{r} \right) \frac{dz}{z} = (1 + (F-1)u)^{2F}$$

and

$$B_1(u) = L_1(u)\partial_1 Q^{-1}(0,u) = -\sqrt{F(F-1)}(1 + (F-1)u)^{2F-1}.$$

Since $1 - (1/r - 1) = \frac{1-2r}{-r}$ we have that

$$-\lim_{r \to \frac{1}{2}} (1-2r)T_{10} = -\frac{\sigma_{120}^2 \sigma_{111}}{2L_1(\sigma_{120})} B_1'(0)|_{F=2} = \frac{\sigma_{120}^2 \sigma_{111}}{L_1(\sigma_{120})} \sqrt{F(F-1)}(2F-1)(F-1)|_{F=2} > 0$$

using [14, Theorem B.1] in the first equality. ■

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