POLYNOMIAL DIFFERENTIAL SYSTEMS WITH INVARIANT ALGEBRAIC CURVES OF ARBITRARY DEGREE FORMED BY LEGENDRE POLYNOMIALS

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ABSTRACT. In 1891 Poincaré asked: *Given $m \geq 2$, is there a positive integer $M(m)$ such that if a polynomial differential system of degree m has an invariant algebraic curve of degree $\geq M(m)$, then it has a rational first integral?* Brunella and Mendes repeated the same open question in 2000, and Lins-Neto in 2002. Between the years 2001 and 2003 three different families of quadratic polynomial differential systems provided a negative answer to this question. One of the answers used the Hermite polynomials. Recently a new negative answer was provided for polynomial differential systems of arbitrary degree using the Laguerre polynomils.

In this paper we provide another new negative answer but using for first time the Legendre polynomials. So the orthogonal polynomials plays a role in the Poincaré's question. Moreover we classify the phase portraits of these polynomial differential systems having invariant algebraic curves of arbitrary degree based on the Legendre polynomials.

1. Introduction and statement of the main results

A *polynomial differential system of degree m* in $\mathbb{R}^2$ is a differential system the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where P and Q are polynomials in the variables x and y with coefficients in $\mathbb{R}$ and $m = \max\{\deg P, \deg Q\}$. The polynomial vector field associated to the differential system (1) is

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Let $f = f(x,y) \in \mathbb{R}[x,y]$, i.e. f is a polynomial in the variables x and y with coefficients in $\mathbb{R}$. When the equality

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

holds for the polynomial differential system (1) and for some polynomial $K \in \mathbb{R}[x,y]$, we say that $f = 0$ is an invariant algebraic curve $f = 0$ of system (1) with cofactor K, because this curve is formed by trajectories of system (1).

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