

ON THE DISTRIBUTION OF THE ZEROS OF SOME POLYNOMIAL MAPS $(P, Q, R): \mathbb{R}^3 \rightarrow \mathbb{R}^3$

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ABSTRACT. Consider a map $(P, Q, R): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of degree (l, m, n) with $l \leq m \leq n$, $1 \leq l, m, n \leq 2$ and we assume that it has lmn different real zeros. We study the distribution of these lmn zeros in the space \mathbb{R}^3 for the degrees $(l, m, n) = (1, 1, 2)$, $(l, m, n) = (1, 2, 2)$ or $(l, m, n) = (2, 2, 2)$.

1. INTRODUCTION

Consider three real polynomials $P(x, y)$, $Q(x, y, z)$ and $R(x, y, z)$ of degrees l , m and n respectively with $l \leq m \leq n$ and $1 \leq l, m, n \leq 2$. Then we say that the polynomial map $(P, Q, R): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has *degree* (l, m, n) . We assume that the map (P, Q, R) has exactly lmn different real zeros.

The main objective of the paper is to study the distribution of these lmn zeros of the polynomial map (P, Q, R) in the space \mathbb{R}^3 when $(l, m, n) = (1, 1, 2)$, $(l, m, n) = (1, 2, 2)$ or $(l, m, n) = (2, 2, 2)$. Note that when $(l, m, n) = (1, 1, 1)$ the polynomial map has a unique zero and consequently it does not have sense to study its distribution. When l, m or n increases then the number of zeros and the complexity of their distributions increase very fast.

We recall that all the papers in the literature concerning the distribution of the zeros of polynomial maps are for polynomials in one variable (see for instance [1, 2, 4, 5, 6, 7, 9, 13, 16, 17, 18, 19, 23, 24, 25, 26, 27, 28]). A first study of the distribution of zeros for polynomial maps in \mathbb{R}^2 has been made in [14]. As far as the authors are concerned this is the first time in the literature where the distribution of the zeros of a class of polynomial maps in \mathbb{R}^3 is studied. Polynomial maps with at least one positive real zero has been studied in [29].

We introduce some definitions. Given a finite subset B of \mathbb{R}^3 , we denote by \hat{B} , $\partial\hat{B}$ and $\#B$ its convex hull, the boundary of the convex hull, and its cardinal, respectively. We also denote by A the set of lmn zeros of the polynomial map (P, Q, R) .

Set $A_0 = A$ and $A_{i+1} = A_i \cap \partial\hat{A}_i$ for $i \geq 0$. Note that there exists a positive integer q such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$. Then we say that A has the *distribution* $(K_0; K_1; K_2; \dots; K_q)$ if $K_i = \#A_i$ and we say that the zeros of the polynomial map $(P, Q, R): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which belong to A_i are on the i -th level.

A first easy result when $l = 1$ is the following.

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