PHASE PORTRAITS OF THE COMPLEX ABEL POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we characterize the phase portraits of the complex Abel polynomial differential equations

$$\dot{z} = (z-a)(z-b)(z-c),$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$. We give the complete description of their phase portraits in the Poincaré disc (i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity) modulo topological equivalence.

1. Introduction and statement of the main results

Numerous problems of applied mathematics, or in physics, chemistry, economics, ... are modeled by polynomial differential systems. Excluding linear systems the quadratic polynomial differential systems are the ones with the lowest degree of complexity, and the large bibliography on them proves their relevance, see the books [1, 12, 13] and the surveys [3, 4]. After the quadratic polynomial differential systems come the cubic ones, which also have many applications. Among the cubic polynomial differential systems we emphasize the Abel systems, see for instance the papers [2, 6, 8] where the Abel systems are applied to modelize problems from Ecology, control theory for electrical circuits and cosmology, respectively.

In this paper we characterize the phase portraits of the complex Abel differential equations

$$\dot{z} = (z-a)(z-b)(z-c), \tag{1}$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ and the dot means derivative with respect to the time $t \in \mathbb{R}$. We write $z = x + iy$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, with $x, y \in \mathbb{R}$ and $a_i, b_i, c_i \in \mathbb{R}$ for $i = 1, 2$. The complex differential equation (1) becomes the real differential system

$$\begin{aligned}
\dot{x} = {} & -a_1b_1c_1 + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 \\
& - a_2c_2 - b_2c_2)x - (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 + a_1c_2 + b_1c_2)y \\
& - (a_1 + b_1 + c_1)x^2 + 2(a_2 + b_2 + c_2)xy + (a_1 + b_1 + c_1)y^2 + x^3 - 3xy^2, \\
\dot{y} = {} & -a_2b_1c_1 - a_1b_2c_1 - a_1b_1c_2 + a_2b_2c_2 + (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 \\
& + a_1c_2 + b_1c_2)x + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 - a_2c_2 - b_2c_2)y \\
& - (a_2 + b_2 + c_2)x^2 - 2(a_1 + b_1 + c_1)xy + (a_2 + b_2 + c_2)y^2 + 3x^2y - y^3.
\end{aligned} \tag{2}$$

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