THE EULER-JACOBI FORMULA AND THE PLANAR QUADRATIC-QUARTIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. The Euler-Jacobi formula provides an algebraic relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the planar quadratic–quartic polynomial differential systems when these systems have eight finite singular points.

1. Introduction and statement of the main results

Consider the planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

in $\mathbb{R}^2$ where $P(x, y)$ and $Q(x, y)$ are real polynomials of degree n and m respectively. Assuming that system (1) has nm finite sigular points using the Euler-Jacobi formula we obtain an algebraic relation between the finite singular points of the polynomial differential system (1) and the topological indices of these finite singular points. A proof of the Euler-Jacobi formula can be found in [1].

It also follows from Bezout's Theorem that in the complex projective plane, and taking into account all the multiplicities of the singular points, if the number of singular points is finite, then it is at most nm. The Euler-Jacobi formula deals with the case in which all the singular points have multiplicity one and are located in the finite part of the projective space. In the two-dimensional case this formula can be enunciated as follows. Consider a system of two real polynomials of degrees n and m respectively in the variables x and y. If the set of zeroes of that system (that we denote by A) contains exactly nm elements, then the Jacobian determinant

$$J = \det \begin{pmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{pmatrix}$$

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evaluated at each zero does not vanish and for any polynomial R of degree less than or equal to $n+m-3$ we have

$$\sum_{a\in A} \frac{R(a)}{J(a)} = 0. \tag{2}$$

Using the Euler-Jacobi formula we want to characterize the number and distribution of the singular points of the quadratic–quartic polynomial differential systems, i.e. of system (1) with $n = 2$ and $m = 4$.

Consider the polynomial vector field $X = (P, Q)$ associated with the differential system (1) where the degree of P is 2 and the degree of Q is 4. We will call it a *quadratic–quartic polynomial differential system.* We denote by $A_X = A$ the set of $\{p \in \mathbb{R}^2 : X(p) = 0\}$ of finite singular points. Given a finite subset B of $\mathbb{R}^2$, we denote by $\hat{B}$ its convex hull, by ∂B its boundary, and by $\#B$ its cardinal.

Set $A_0 = A$ and for $i \geq 1$ $A_i = A_{i-1} \setminus (A_{i-1} \cap \partial \hat{A}_{i-1})$. There is an integer q such that $A_{q+1} = \emptyset$.

We say that A has the *configuration* $(K_0; K_1; K_2; \ldots; K_q)$ where K_i is the natural positive number defined by

$$K_i = \#(A_i \cap \partial \hat{A}_i)$$

We say that A has configuration $(K_0; K_1; K_2; \ldots; K_r; *)$ if we do not specify for the values of K_i for i between $r+1$ and q.

We are also interested in the study of the (topological) indices of the singular points of X. We say that the singular points of X which belong to $A_i \cap \partial \hat{A}_i$ are on the *i-th level*.

We recall that if we assume that $\#A_X = 8$ then the determinant of the Jacobian matrix J is non-zero at any singular point of the vector field X, consequently topological indices of the singular points are ± 1, and in this case we substitute the number K_i corresponding to the i-th level by the vector $(n_i^1+, n_i^2-, \ldots, n_i^{m_i-1}+, n_i^{m_i}-)$ where n_i^j are positive integers such that $\sum_j n_i^j = K_i$. More precisely, when $A_i \cap \partial \hat{A}_i$ is a polygon, these numbers take into account the number of consecutive points with positive and negative indices, viewing the ith level oriented counterclockwise: n_i^1 corresponds to the string with largest number of consecutive points with positive and negative indices. If there are several strings with the same number of points we choose one such that the next string (that has points with negative indices) is as large as possible. We continue the process for n_i^2 and so on. Furthermore, when $A_i \cap \partial \hat{A}_i$ is a segment, the numbers take into account the number of consecutive points with positive indices, beginning at one of its endpoints.

With this notation we can state the main result of the paper. We denote by $i_X(a)$ the index of a singular point $a \in A$ of a planar quadratic–quartic polynomial vector field X.

Theorem 1. *For planar quadratic–quartic polynomial differential systems such that $\#A_X = 8$, the following statements hold.*

(a) *either* $\left|\sum_{a\in A} i_X(a)\right| = 2$ *or* $\sum_{a\in A} i_X(a) = 0$.

(b) *If* $\left|\sum_{a\in A} i_X(a)\right| = 2$*, then only the following configurations are possible*
   (i) $(4;3;1) = (4+;3-;+)$ *or* $(4-;3+;-)$,
   (ii) $(3;5) = (3+;+,2-,+,-)$ *or* $(3-;+,-,+,-,+)$,
   and there exist examples of such configurations.

(c) *If* $\sum_{a\in A} i_X(a) = 0$*, then only the following configurations are possible*
   (i) $(8) = (+,-,+,-,+,-,+,-)$,
   (ii) $(4;4) = (+,-,+,-;+,-,+,-)$,
   and there exist examples of such configurations.

The proof of Theorem 1 is given in section 2. We recall that for the planar quadratic-quadratic polynomial differential systems this theorem is the well-known Berinskii's Theorem proved in [2] and reproved in [4] using the Euler-Jacobi formula. The case quadratic–cubic was also proved in [4], but the case cubic-cubic is much more difficult and is still open with only some partial results.

2. Proof of Theorem 1

In the proof of Theorem 1 we will use the following auxiliary result proved in [3].

Lemma 2. *Let $X = (P,Q)$ be a polynomial vector field with* $\max(\deg P, \deg Q) = n$*. If X has n singular points on a straight line $L(x,y) = 0$, this line is an isocline. If X has $n+1$ singular points on $L(x,y) = 0$ then this line is full of singular points.*

2.1. **Proof of statement (a) of Theorem 1.** First of all we observe that if a configuration exists for quadratic-quartic polynomial vector field X with $\#A_X = 8$ then it is possible to construct the same configuration but interchanging points with index $+1$ with points with index -1. For doing that it is enough to take $Y = (-P,Q)$ instead of $X = (P,Q)$. So we can restrict ourselves to the cases in which $\sum_{a\in A} i_X(a) \geq 0$.

During the proof we will denote by $\{p_1,\ldots,p_8\}$ the set of points of A if there is no information about their indices, by $\{p_1^+,\ldots,p_k^+\}$ the set of points of A with positive index, and by $\{p_1^-,\ldots,p_l^-\}$ the set of points of A with negative index. Also we will denote by $L_{i,j}^{u,v}$ the straight line $L_{i,j}^{u,v}(x,y)=0$ through the points p_i^u and p_j^v where $u,v\in\{+,-,\emptyset\}$, $i\in\{1,\dots,k\}, j\in\{1,\dots,l\}$, and by L_i^u a straight line through a point $p_i^u\in\partial\hat{A}$ such that for all $q\in A$ we have $L_i^u(A)\geq 0$ and it is zero only at q.

It was proved in [6, 5] that in the case of quadratic-quartic polynomial vector fields either $\left|\sum_{a\in A} i_X(a)\right| = 2$, or $\sum_{a\in A} i_X(a) = 0$, and by the above explanation we can only consider that either $\sum_{a\in A} i_X(a) = 2$, or $\sum_{a\in A} i_X(a) = 0$. This proves statement (a).

2.2. **Proof of statement (b) of Theorem 1.** We prove Theorem 1 in the case in which $\sum_{a\in A} i_X(a) = 2$. In this case we have that 5 points have positive index and three points have negative index. First we will show that there are no singular points with index -1 in $\partial\hat{A}$ and so that $\#(A\cap\partial\hat{A})\leq 5$, or equivalently, that the unique possible configurations are $(K+,*)$ with $K\leq 5$. Indeed, assume first that $p_1^-\in\partial\hat{A}$ and denote by p_1^-, p_2^-, p_3^- the points with negative index and by p_i^+ for $i=1,\dots,5$ the points with positive index. Consider the cubic $C(x,y) = L_1^-(x,y)(L_{2,3}^{--}(x,y))^2$. Since, by the definitions of L_1^- and $L_{2,3}^{--}$ we have $C(P_i^+)\geq 0$ for $i=1,\dots,5$ applying the Euler-Jacobi formula we get

$$\sum_{i=1,\dots,5}\frac{C(P_i^+)}{J(p_i^+)} = 0$$

which is a contradiction because $C(p_i^+)\geq 0$, $J(p_i^+)=1$ for $i=1,\dots,5$, and all of them cannot be zero because it is known that since the maximum degree of P and Q is four, if the five singular points are on the straight line $L_{23}^{--}(x,y)=0$ by Lemma 2 this straight line is full of singular points, which is not the case. This contradiction implies that $\#(A\cap\partial\hat{A})\leq 5$ and the configuration of A must be $(K+;*)$ with $K\leq 5$.

We now consider different cases.

$K=5$: Assume first that $K=5$, i.e, $\#(A\cap\partial\hat{A})=5$. Now, applying the Euler-Jacobi formula to $C(x,y) = L_{1,2}^{++}(x,y)L_{34}^{++}(x,y)L_5^+(x,y)$ we get to a contradiction. So this configuration is not possible.

$K=4$: Then $\#(A\cap\partial\hat{A})=4$. Write $\{p_1^+,p_2^+,p_3^+,p_4^+\} = A\cap\partial\hat{A}$ and take a conic $C_0(x,y)$ through them. Since all these points are in the boundary of a convex set, the remaining four singular points are in the same connected component of $\mathbb{R}^2\setminus\{C_0(x,y)=0\}$. Assume now that there is a point $p_5^+\in A_1$, i.e. in the 1st-level of A. Taking $L_{5,k}^{+-}$ where p_k^- is a point in $A_1\cap\partial\hat{A}_1$ contiguous with p_5^+, if the four singular points in A_1 are not on a straight line then applying the Euler-Jacobi formula with $C(x,y) = C_0(x,y)L_{5,K}^{+-}$, we get a contradiction. On the other hand, if the four singular points are on a straight line, say $L_0(x,y)=0$ then assuming without loss of generality that $p_1^+ \in \partial\hat{A}$, applying the Euler-Jacobi formula to $C(x,y)=L_0(x,y)L_{34}^{++}(x,y)L_1^+(x,y)$ we get a contradiction. So the configuration of A is $(4+;3-;+)$.

Note that the quadratic–quartic system (1) with

$$
\begin{aligned}
P(x,y) = &\ y^2-x^2+1,\\
Q(x,y) = &\ -x^4+\frac{\left(15\sqrt{2}+4\sqrt{15}\right)x^3y}{45\sqrt{2}+16\sqrt{15}}+\left(1-\frac{\sqrt{30}}{45\sqrt{2}+16\sqrt{15}}\right)x^2y^2\\
&-x^2y-\frac{4\left(15\sqrt{2}+4\sqrt{15}\right)xy^3}{3\left(45\sqrt{2}+16\sqrt{15}\right)}+\frac{4\sqrt{10}y^4}{45\sqrt{6}+48\sqrt{5}}+\frac{4y^3}{3}+y^2+1,
\end{aligned}
$$

has the singular points

$$(-2,\sqrt{3}),\ (-2,-\sqrt{3}),\ (-1,0),\ (1,0),\ (2,\sqrt{3}),\ (2,-\sqrt{3}),\ (3,-2\sqrt{2}),\ (4,\sqrt{15}),$$

in the configuration $(4+;3-;+)$.

$K=3$: Now assume that $\#(A\cap\partial\hat{A})=3$. Then we have the following possibilities: $(3+;5)$, $(3+;4;1)$ or $(3+;3;2)$. Since the polynomial P has degree 2, $P(x,y)=0$ is a conic and the eight finite singular points of system (1) are on this conic. Therefore any real conic (ellipse, parabola, hyperbola, two parallel straight lines, two straight lines intersecting in a point, one double straight line or one point) do not allow the configurations $(3+;4;1)$ or $(3+;3;2)$. Moreover, the configuration $(3+;5)$ only can be supported by a hyperbola.

Now we study the configuration $(3+;5)$. Since all the singular points lie in a hyperbola and in the 1st-level of A we must have five points, it is clear that one point is in one branch of the hyperbola and the other seven points in the other branch of the hyperbola. Denote by p_1^+ the point in one branch of the hyperbola and by $p_2,p_3,p_4,p_5,p_6,p_7,p_8$ the remaining points which are in the other branch of the hyperbola and ordered. Note that with this notation $p_2=p_2^+$ and $p_8=p_8^+$. Applying the Euler-Jacobi formula to $C(x,y)=L_{28}^{++}L_{37}L_{46}$ we get that p_1^+ and p_5 have different signs so $p_5=p_5^-$. Now applying the Euler-Jacobi formula to $C(x,y)=L_{15}^{+-}L_{28}^{++}L_{37}$ we get that p_4 and p_6 must have the same sign and applying the Euler-Jacobi formula to $C(x,y)=L_{15}^{+-}L_{28}^{++}L_{46}$ we get that p_3 and p_7 must have the same sign, and applying the Euler-Jacobi formula to $L_1^+L_5^-L_{46}$ taking into account that $p_2=p_2^+$ and $p_8=p_8^+$ we get that $p_3=p_3^-$ and $p_7=p_7^-$. Then $p_4=p_4^-$ and $p_5=p_5^-$ and we get the configuration is $(3+;+,2-,+,-)$.

The quadratic–quartic system (1) with

$$
\begin{aligned}
P(x,y) = &\ x^2-y^2-1,\\
Q(x,y) = &\ -x^4+\frac{9x^3y}{11}+x^2y^2-x^2y-\frac{26xy^3}{33}+\frac{8y^3}{11}+y^2+1,
\end{aligned}
$$

has the singular points

$(-1,0)$, $(-1,0)$, $(2,\sqrt{3})$, $(2,-\sqrt{3})$, $(3,2\sqrt{2})$, $(3,-2\sqrt{2})$, $(4,\sqrt{15})$, $(4,-\sqrt{15})$,

in the configuration $(3+;5)$.

Clearly configurations of the form $(2+;*)$ cannot occur because the eight singular points would be on a straight line, and by Lemma 2 this straight line will be full of singular points, a contradiction. Moreover, configurations of the form $(1+;*)$ have no meaning. This concludes the proof of statement (b) of Theorem 1.

2.3. **Proof of statement (c) of Theorem 1.** We consider now the case in which $\sum_{a\in A} i_X(a)=0$. In this case we have 4 points with positive index and 4 points with negative one. We separate this study into different cases. Note that configurations $(7;1)$, $(6;2)$ and $(5;3)$ are not possible because any convex hull of seven, six or five points on a conic has at most four points in the boundary of the convex hull except for the ellipse, but in the case of the ellipse cannot be points in the 1–level. Furthermore, by the explanation in the proof of statement (b) we get that the configurations $(3;4;1)$, $(3;3;2)$, $(2;*)$ and $(1;*)$ are not possible. In short, the unique possible configurations are (8), $(4;4)$, $(4;3;1)$ and $(3;5)$. We will study them separately.

Configuration (8): Assume that the subscripts of the points in A are in such a way that $p_1,p_2,p_3,p_4,p_5,p_6,p_7$ and p_8 are ordered in $\partial\hat{A}$ in counterclockwise sense. Also we consider the subscripts in $\mathbb{Z}/8\mathbb{Z}$. Take the cubic

$$C_i(x,y)=L_{i,i+1}(x,y)L_{i+2,i+3}(x,y)L_{i+4,i+5}.$$

Then the Euler Jacobi formula applied to C_i yields

$$\frac{C_i(p_{i+6})}{J(p_{i+6})}+\frac{C_i(p_{i+7})}{J(p_{i+7})}=0,$$

so $J(p_{i+6})J(p_{i+7})<0$ for all i. Hence the indices of P_j and p_{j+1} are different and the configuration of A must be $(8)=(+,-,+,-,+,-,+,-)$.

The quadratic–quartic system (1) with

$$\begin{aligned} P(x,y)=&\ x^2+y^2-1,\\ Q(x,y)=&\ -2x^4+x^3y+x^2y^2-3xy^3+3y^4-x^3-x^2y-xy^2-y^3+x^2\\ &+xy-4y^2+x+y+1, \end{aligned}$$

has the singular points

$$(0,\pm1),\quad (\pm1,0),\quad \Big(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\Big)$$

in the configuration $(+,-,+,-,+,-,+,-)$.

Configuration $(4;4)$: Denote by $p_1,\dots,p_4$ the points in $A\cap\partial\hat{A}$ ordered in counterclockwise sense, and by $p_5,\dots,p_8$ the points in $A\cap\partial\hat{A}_1$ also ordered in counterclockwise sense. Applying the Euler-Jacobi formula to $C(x,y)=$

$L_{12}L_{34}L_{5,6}$, $C(x,y) = L_{12}L_{34}L_{67}$, and $C(x,y) = L_{12}L_{34}L_{78}$ we get that the configuration is $(4;+,-,+,-)$. Moreover we have that in ∂A there are two points with positive index and two points with negative one. We note that the configuration $(4;4)$ only can be realized in a hyperbola or in a conic formed by two straight lines intersecting in a point. Then applying three times the Euler-Jacobi formula to a cubic formed by the product of three straight lines, two of them defined by two non-contiguous sides of the quadrilateral defined by the boundary of $\partial\hat{A}_1$ and the third straight line defined by one side of the quadrilateral defined by the boundary of $\partial\hat{A}$, we obtain that the configuration $(2+,2-;+,-,+,-)$ is not possible. In summary the unique possible configuration is $(+,-,+,-;+,-,+,-)$.

The quadratic–quartic system (1) with

$$\begin{aligned} P(x,y) = & \quad x^2 - y^2 - 1, \\ Q(x,y) = & \quad -x^4 + \frac{23x^2y^2}{12} - x^2y - y^4 + y^3 + y^2 + y - 1, \end{aligned}$$

has the eight singular points $(\pm 2, \pm\sqrt{3})$, $(\pm 3, \pm 2\sqrt{2})$, in the configuration $(+,-,+,-;+,-,+,-)$.

Configuration $(4;3;1)$: The arguments used in the proof of case $K=3$ of statement (b) of Theorem 1 applied to the configuration $(4;3;1)$ show that such a configuration only can be realized if the conic $P(x,y)=0$ is a hyperbola. So in the following arguments we take into account that the eight singular points of system (1) are in a hyperbola. Denote by $p_1,\dots,p_4$ the points in $A\cap\partial\hat{A}$ ordered in counterclockwise sense and by p_5,p_6,p_7 the points in the 1st-level of A and p_8 in the 2nd-level of A. Applying the Euler-Jacobi formula, iteratively, to $C(x,y) = L_{12}L_{34}L_{56}$, $C(x,y) = L_{12}L_{34}L_{67}$ and $C(x,y) = L_{12}L_{34}L_{57}$ we get that, without loss of generality, $p_8 = p_8^+$ and $p_5 = p_5^-$, $p_6 = p_6^-$ and $p_7 = p_7^-$. So, we have three points with index $+$ and one point with index $-$ in $\partial\hat{A}$. By the previous consideration we can assume that $\partial\hat{A}$ is a quadrilateral. Take $L_{k,k+1}^{++}$ where p_k^+ are two contiguous points in $\partial\hat{A}$. Applying the Euler-Jacobi formula to $C(x,y) = L_{5,8}^{-+}L_{7,6}^{--}L_{k,k+1}^{++}$ we reach a contradiction because the remaining points in $\partial\hat{A}$ are in different sides of $L_{k,k+1}^{++}$ and have different signs. So, the configuration $(4;3;1)$ is not possible.

Configuration $(3;5)$: As in the proof of the case $K=3$ of statement (b) of Theorem 1 all the singular points lie in a hyperbola with one point in one branch of the hyperbola and the other seven points in the other branch of the hyperbola. Denote by p_1 the point in one branch of the hyperbola and by $p_2,p_3,p_4,p_5,p_6,p_7,p_8$ the remaining points which are in the other branch of the hyperbola and ordered. Applying the Euler-Jacobi formula to $C(x,y) = L_{28}L_{37}L_{46}$ we get that p_1 and p_5 have different signs so we can assume without loss of generality that $p_1 = p_1^+$ and $p_5 = p_5^-$. Now applying again the Euler-Jacobi formula to $C(x,y) = L_{15}^{+-} L_{28} L_{37}$ we get that p_4 and p_6 must have the same sign, applying the Euler-Jacobi formula to $C(x,y) = L_{15}^{+-} L_{28} L_{46}$ we get that p_3 and p_7 must have the same sign, and finally applying the Euler-Jacobi formula to $C(x,y) = L_{15}^{+-} L_{37} L_{46}$ we get that p_2 and p_8 must have the same sign, but then the number of positive and negative indices would be odd which is not possible. In short, such a configuration is not possible. This proves statement (c) of Theorem 1 and concludes the proof of the theorem.

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