CENTERS OF PLANAR GENERALIZED ABEL EQUATIONS

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Abstract. We deal with the differential equation

$$\dot{r} = \frac{dr}{d\theta} = a(\theta) r^n + b(\theta) r^m,$$

where (r, θ) are the polar coordinates in the plane $\mathbb{R}^2$, m and n are integers such that $m > n \geq 2$, and a, b are C^1 functions. Note that when $n = 2$ and $m = 3$ we have an Abel differential equation. For this class of generalized Abel equations we characterize a new family of centers.

1. Introduction and statement of the results

Consider the generalized Abel equation

$$\dot{r} = \frac{dr}{d\theta} = a(\theta) r^n + b(\theta) r^m, \tag{1}$$

defined in the plane $(r, \theta) \in [0, +\infty) \times \mathbb{S}^1$ in polar coordinates where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Here m and n are integers such that $m > n \geq 2$, $\theta \in [-\pi, \pi]$ and a, b are C^1-functions.

The origin of the plane is a *center* for the differential equation (1) if there is a neighborhood of it where all the solutions are periodic except the equilibrium point at the origin.

When $n = 2$ and $m = 3$ the differential equation (1) is a particular family of Abel equations. In fact the Abel equations are of the form

$$\dot{r} = a(\theta) + b(\theta) r + c(\theta) r^2 + d(\theta) r^3,$$

and they appeared by the first time in the works of Niels Henryk Abel, see [7]. Today there are more than 1400 papers in MathSciNet with the name "Abel equation" in their tittle, see for instance the papers [1, 2, 3, 5, 6, 8] for results on centers in the Abel equations and the references quoted therein.

The main objective of this work is to provide a new family of centers in the generalized Abel equation (1). Thus, our main result is the following.

Theorem 1. *If $a(\theta)$ and $b(\theta)$ are C^1 odd functions and $m \geq 2n-1$, then the origin $r = 0$ of the differential equation* (1) *is a center.*

The proof of Theorem 1 is given in the next section. We note that Theorem 1 in the particular case $n = 2$ and $m = 3$, i.e. when equation (1) is an Abel equation, already was obtained in [4] by Araujo, Lemos and Alves.

2010 *Mathematics Subject Classification.* 34C25, 34A34.
Key words and phrases. Centers, generalized Abel equations.

2. Proof of Theorem 1

In order to prove Theorem 1 we introduce some auxiliary results that will be used in its proof.

Proposition 2. *The origin* $r = 0$ *of equation* (1) *is a center if and only if*

$$\int_{-\pi}^{\pi} a(\theta)\, d\theta = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} b(\theta) r(\theta;\rho)^{m-n}\, d\theta = 0, \tag{2}$$

for $|\rho| < \rho_0$ *with* ρ_0 *sufficiently small where* $r(\theta;\rho)$ *is the solution of equation* (1) *such that* $r(-\pi;\rho) = \rho$.

Proof. First we prove sufficiency. Note that dividing equation (1) by r^n and integrating it, we obtain

$$-\frac{1}{n-1} r^{1-n}(\theta;\rho) = \int_{-\pi}^{\theta} a(s)\, ds + \int_{-\pi}^{\theta} b(s) r(s;\rho)^{m-n}\, ds - \frac{1}{(n-1)\rho^{n-1}},$$

where ρ is the initial condition. So we have

$$r^{n-1}(\theta;\rho) = \frac{\rho^{n-1}}{1-(n-1)\rho^{n-1}[\int_{-\pi}^{\theta} a(s)\, ds + \int_{-\pi}^{\theta} b(s) r(s;\rho)^{m-n}\, ds]},$$

which yields

$$r(\theta;\rho) = \frac{\rho}{\left(1-(n-1)\rho^{n-1}[\int_{-\pi}^{\theta} a(s)\, ds + \int_{-\pi}^{\theta} b(s) r(s;\rho)^{m-n}\, ds]\right)^{\frac{1}{n-1}}}. \tag{3}$$

Taking $\theta = \pi$ in the previous equation we have

$$r(\pi;\rho) = \frac{\rho}{\left(1-(n-1)\rho^{n-1}[\int_{-\pi}^{\pi} a(s)\, ds + \int_{-\pi}^{\pi} b(s) r(s;\rho)^{m-n}\, ds]\right)^{\frac{1}{n-1}}}.$$

If (2) holds, then $x(\pi;\rho) = \rho$, and the sufficiency in the theorem follows.

Assume now that equation (1) has a center on $r = 0$. We first note that any solution of equation (1), $r(\theta;\rho)$ can be expanded in power series in ρ for $|\rho| < \rho_0$ with ρ_0 sufficiently small and $\theta \in [-\pi, \pi]$ in the form

$$r(\theta;\rho) = r_0(\theta) + r_1(\theta)\rho + r_2(\theta)\rho^2 + \dots \tag{4}$$

Clearly $r_0(\theta) = 0$ because $r(\theta;0) = 0$. Substituting (4) into equation (1) we get $\dot r_1(\theta) = 0$ because $n \geq 2$, and since $r(\pi;\rho) = \rho$ we must have $r_1(\theta) = 1$. Hence we have

$$r(\theta;\rho) = \rho + r_2(\theta)\rho^2 + \dots \tag{5}$$

Note that a sufficient and necessary condition for $r = 0$ to be a center of (1) is that

$$r_i(\pi;\rho) = 0 \quad \text{for } i = 2, 3, \dots.$$

Substituting (4) into (1) and computing the coefficients in ρ^j for $j \geq 2$ we get that

$$\dot r_i(\theta;\rho) = 0 \quad \text{for } i = 2, \dots, n-1.$$

Since $r_i(\pi;\rho) = 0$ we must have $r_i(\theta;\rho) = 0$ for $i = 2, \dots, n-1$. On the other hand

$$\dot r_n(\theta;\rho) = a(\theta) \quad \text{that is} \quad r_n(\theta;\rho) = \int_{-\pi}^{\theta} a(\theta)\, d\theta.$$

Since $r_n(\pi;\rho)=0$ we obtain that $\int_{-\pi}^{\pi} a(s)\,ds=0$. But then, from (3) we readily get that $\int_{-\pi}^{\pi} b(s)r(s;\rho)^{m-n}\,ds=0$, which proves the necessity. This concludes the proof of the proposition. □

Note that by equation (3) a solution of equation (1) is equivalent to a solution of the integral equation

$$r(\theta;\rho)=\frac{\rho}{\big(1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)r(s;\rho)^{m-n})\,ds\big)^{\frac{1}{n-1}}}, \tag{6}$$

for $\theta\in[-\pi,\pi]$, where $r(-\pi;\rho)=\rho$. We denote

$$y(\theta)=r^{n-1}(\theta;\rho) \quad \text{and} \quad \ell=\frac{m-n}{n-1}. \tag{7}$$

Then equation (6) becomes

$$y(\theta)=\frac{\rho^{n-1}}{(1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^{\ell})\,ds}. \tag{8}$$

We define the operator $T\colon B_M\to C([-\pi,\pi])$ by

$$T(y)(\theta)=\frac{\rho^{n-1}}{1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^{\ell})\,ds},$$

for $\theta\in[-\pi,\pi]$, where

$$B_M=\{y\in E: \|y\|_\infty\le M\},$$

being E the closed subspace of $C([-\pi,\pi])$ defined by

$$E=\{y\in C([-\pi,\pi]):\ y \text{ is an even function}\}.$$

Note that if the operator T has a unique fixed point, i.e., a unique $y\in B_M$ such that $T(y)(\theta)=y$ then (8) has an even solution. As usual $\|r\|_\infty=\max_{\theta\in[-\pi,\pi]}|r(\theta)|$.

We take the notation

$$J_\theta(y)=\frac{\rho^{n-1}}{1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^{\ell})\,ds}.$$

We define

$$A=\max_{\theta\in[-\pi,\pi]}|a(\theta)|,\quad B=\max_{\theta\in[-\pi,\pi]}|b(\theta)|.$$

Proposition 3. *For*

$$0\le\rho<\min\left\{\frac{1}{\big(4\pi(n-1)(A+BM^{\ell})\big)^{1/(n-1)}},\Big(\frac{M}{2}\Big)^{1/(n-1)}\right\}, \tag{9}$$

the operator $T\colon B_M\to C([-\pi,\pi])$ is continuous and compact. Moreover $T(B_M)\subset B_M$.

Proof. For each $y_1,y_2\in C([-\pi,\pi])$, we have

$$|T(y_1)(\theta)-T(y_2)(\theta)|=\big|(n-1)J_\theta(y_1)J_\theta(y_2)\big|\left|\int_{-\pi}^{\theta}b(s)(y_1(s)^{\ell}-y_2(s)^{\ell})\,ds\right|,$$

for $\theta \in [-\pi, \pi]$. Note that since

$$0 \le \rho \le \frac{1}{\big(4\pi(n-1)(A+BM^\ell)\big)^{1/(n-1)}},$$

we have

$$|(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^\ell)\,ds| \le 2\pi(n-1)\rho^{n-1}(A+BM^\ell) < \frac{1}{2}.$$

Therefore

$$1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^\ell)\,ds > 1/2, \tag{10}$$

which yields $J_\theta(y) \le 2\rho^{n-1}$. Moreover, since $m \ge 2n-1$ we have that $\ell \ge 1$ and by the Mean Value theorem we get

$$|y_1(s)^\ell - y_2(s)^\ell| \le \ell M^{\ell-1}\|y_1-y_2\|_\infty.$$

Hence

$$\begin{aligned}|T(y_1)(\theta)-T(y_2)(\theta)| &\le 4(n-1)\rho^{2n-2}\int_{-\pi}^{\theta}\big|b(s)(y_1(s)^\ell-y_2(s)^\ell)\big|\,ds\\ &\le 8\pi(n-1)\rho^{2n-2}B\ell M^{\ell-1}\|y_1-y_2\|_\infty,\end{aligned}$$

for each $\theta \in [-\pi,\pi]$. So,

$$\|T(y_1)-T(y_2)\|_\infty \le 8\pi(n-1)\rho^{2n-2}B\ell M^{\ell-1}\|y_1-y_2\|_\infty$$

for all $y_1, y_2 \in C([-\pi,\pi])$. Hence the operator T is continuous.

For proving that the operator T is compact, we shall see that T is bounded and equicontinuous. Now we prove that it is bounded. Indeed, by (10) and the condition in ρ in (9) we have

$$|T(y)(\theta:-\pi)| \le 2\rho^{n-1}, \quad \text{for all } y \in B_M \text{ and } \theta \in [-\pi,\pi]$$

and so

$$\|T(y)\|_\infty \le 2\rho^{n-1} \quad \text{for all } y \in B_M, \tag{11}$$

proving that the operator T is bounded.

Now we show that T is equicontinuous. For each $\theta_1, \theta_2 \in [-\pi,\pi]$ (that without loss of generality we can assume that $\theta_2 > \theta_1$), and any $y \in B_M$, we have

$$\begin{aligned}|T(y)(\theta_1)-T(y)(\theta_2)| &= \big|(n-1)J_{\theta_1}(y)J_{\theta_2}(y)\big|\times\\ &\quad\left|\int_{-\pi}^{\theta_1}(a(s)+b(s)y(s)^\ell)\,ds-\int_{-\pi}^{\theta_2}(a(s)+b(s)y(s)^\ell)\,ds\right|\\ &\le 4(n-1)\rho^{2n-2}\int_{\theta_1}^{\theta_2}|a(s)+b(s)y(s)^\ell|\,ds\\ &\le 4(n-1)\rho^{2n-2}(A+BM^\ell)|\theta_2-\theta_1|.\end{aligned}$$

Therefore $T(B_M)$ is an equicontinuous subset of $C([-\pi,\pi])$. By Ascoli-Arzela Theorem (see for instance [9]) we have that $T\colon B_M \to C([-\pi,\pi])$ is compact.

Since the functions $a(\theta)$ and $b(\theta)$ are odd and by assumptions $y(\theta)$ is an even function (and so also $y(\theta)^\ell$ is an even function), we have that $a(s)+b(s)y(s)^\ell$ is odd. Taking into account that the integral of an odd function is an even function we conclude that $\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^{\ell})\,ds$ is an even function in θ. Hence, for each $y \in E$ we have that $T(y)(\theta) = T(y)(-\theta)$ for all $\theta \in [-\pi, \pi]$. Therefore, $T(y) \in E$ for every $y \in E$. Moreover, by (9) and (11) we have that

$$\|T(y)\|_\infty \leq 2\rho^{n-1} < M \quad \text{for all } y \in B_M.$$

So $T\colon B_M \to B_M$ is well defined. This concludes the proof of the proposition. □

Proposition 4. *Under the assumptions of Theorem 1 there are infinitely many closed even solutions $r(\theta;\rho)$ of system* (1) *for ρ satisfying* (9).

Proof. It follows from Proposition 3 that the operator $T\colon B_M \to B_M$ is well defined, continuous and compact. By the Schauder fixed point Theorem, see [9], the operator T has a fixed point y satisfying

$$T(y)(\theta) = y(\theta) = \frac{\rho}{1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)y(s)^{\ell})\,ds}$$

and $y(-\pi) = \rho^{n-1}$ for each ρ satisfying (9). From (7) there exists $r(\theta;\rho)$ such that

$$r(\theta;\rho) = \frac{\rho}{\left(1-(n-1)\rho^{n-1}\int_{-\pi}^{\theta}(a(s)+b(s)r(s;\rho)^{m-n})\,ds\right)^{\frac{1}{n-1}}}$$

and $r(-\pi;\rho) = \rho$. Note that $r(-\theta;\rho) = r(\theta;\rho)$ and so the solution is closed and even. In short, there are many closed even solutions of system (1) near the origin. □

Proof of Theorem 1. To prove Theorem 1 we first show that if $\bar{r}(\theta;\rho)$ is a solution of equation (1) that satisfies $\bar{r}(-\pi;\rho) = \rho$ with ρ satisfying (9), then $\bar{r}(\theta;\rho)$ is closed and even. Indeed, by Proposition 4 there is $r(\theta;\rho)$ a closed even solution of system (1) such that $r(-\pi;\rho) = \rho$, and by the uniqueness of solutions of an ordinary differential equation, we obtain that $\bar{r}(\theta;\rho) = r(\theta;\rho)$. Hence if a and b are odd functions in the variable θ, then each solution of equation (1) with initial condition ρ satisfying (9) is a closed even solution. Hence, for any ρ satisfying (9) we have

$$\int_{-\pi}^{\pi} a(s)\,ds = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} b(s)r(s;\rho)^{m-n}\,ds = 0.$$

Therefore it follows from Proposition 2 that $r = 0$ is a center for equation (1). □

Acknowledgements

The first author is partially supported by a FEDER-MINECO grant MTM2016-77278-P, a MINECO grant MTM2013-40998-P, and an AGAUR grant number 2014SGR-568. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

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