POLYNOMIAL, RATIONAL AND ANALYTIC FIRST INTEGRALS FOR AN ASYMMETRIC FAMILY OF 3-DIMENSIONAL LOTKA-VOLTERRA SYSTEMS

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Abstract. We extend the study of the integrability of the classical model of competition between three species studied by May and Leonard (SIAM J. Appl. Math. **29** (1975), 243–256), to a more realistic asymmetric model. Our results provide all polynomial, rational and analytic first integrals of this extended model. We also classify all the invariant algebraic surfaces of these asymmetric models.

1. Introduction and statement of the main results

Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and sciences in general. For a 3–dimensional system the knowledge of two first integrals whose gradients are linearly independent in $\mathbb{R}^3$ except perhaps in a zero Lebesgue measure set, determine completely its phase portrait due to the fact that the intersections of the invariant levels of these two first integrals determine the trajectories of the system. On the other hand, the knowledge of only one first integral do not determine completely the phase portrait but reduces the study of the dynamics of the system in one dimension (i.e. from dimension 3 to dimension 2). So the study of the existence of first integrals is an important subject in the qualitative theory of differential equations. Many different methods have been used for studying the existence of first integrals of non–linear differential systems based on: Noether symmetries [6], the Darboux theory of integrability [9, 19], the Lie symmetries [1, 25], the Painlevé analysis [3], the use of Lax pairs [13], the direct method [10, 11], the linear compatibility analysis method [26], the Carlemann embedding procedure [7, 2], the quasimonomial formalism [4], etc.

In this paper we use the Darboux theory of integrability to study the existence of first integrals for the following asymmetric model (which is the asymmetrization of the initial model used by May and Leonard [20] for studying the competition among three species). This asymmetric model is

$$\begin{aligned} \dot{X} &= X(1 - X - a_1 Y - b_1 Z), \\ \dot{Y} &= Y(1 - b_2 X - Y - a_2 Z), \\ \dot{Z} &= Z(1 - a_3 X - b_3 Y - Z), \end{aligned} \tag{1}$$

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