WEIERSTRASS INTEGRABILITY OF COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. We characterize the complex differential equations of the form

$$\frac{dy}{dx} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x),$$

where $a_j(x)$ are meromorphic functions in the variable x for $j = 0, \ldots, n$ that admit either a Weierstrass first integral or a Weierstrass inverse integrating factor.

1. Introduction and statement of the main results

Let x and y be complex variables. In this paper we study the differential equations of the form

$$\frac{dy}{dx} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) \quad \text{with} \quad a_n(0) \not\equiv 0, \tag{1}$$

where $a_j(x)$ are meromorphic functions of x for $j = 0, \ldots, n$. In particular, the differential equation (1) contains the well-known *Abel differential equations* when $n = 3$, the *Riccati differential equations* when $n = 2$, and the *linear differential equations* when $n = 1$.

In what follows instead of working with the differential equation (1) we shall work with the equivalent differential system

$$\dot{x} = 1, \quad \dot{y} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) \quad \text{with} \quad a_n(0) \not\equiv 0, \tag{2}$$

in $\mathbb{C}^2$, where the dot denotes derivative with respect to the time t, real or complex.

The objective of this paper is to study the integrability of the differential equations (1) restricted to a special kind of first integrals. For such systems the notion of integrability is based on the existence of a first integral, and we want to characterize when the differential equations (1) have either a Weierstrass first integral or a Weierstrass inverse integrating factor.

When one studies the integrability of a differential system, the first class of functions to look for first integral is the polynomial functions. Then, one can go a step further and try to look for analytic first integrals. Usually, this is a very hard task and instead of this, one studies the first integrals that can be described by formal series. The use of formal series in the study of differential equations is a classical tool (see for instance [4], where the author used formal series to prove the Dulac's conjecture). Here, guided by the fact that the equations in (1) are polynomial in the variable y, we study the first integrals that are polynomials in the variable y and formal series in the variable x, called Weierstrass first integrals.

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