Meromorphic integrability of the Hamiltonian systems with homogeneous potentials of degree −4 *

Jaume Llibre[a] and Yuzhou Tian[b,*]

[a]Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail: jllibre@mat.uab.cat

[b]School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai 519082, P.R. China

E-mail: tianyzh3@mail2.sysu.edu.cn

Abstract We characterize the meromorphic Liouville integrability of the Hamiltonian systems with Hamiltonian $H=\left(p_1^2+p_2^2\right)/2+V\left(q_1,q_2\right)$, being $V\left(q_1,q_2\right)$ a homogeneous potential of degree -4, with the exception of the potential $V_8=1/\left(q_1^4+6\mu q_1^2q_2^2+q_2^4\right)$ when $\mu\in\{-5/3,-2/3\}$. For this potential we only can prove that it has no polynomial first integral.

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1 Introduction and main results

Hamiltonian systems play an important role in the theory of the dynamical systems due to the fact that these systems occur frequently in mathematical physics, particularly in mechanics, engineering and other fields. In order to describe global information on the Hamiltonian systems is good to find sufficient number of first integrals. The fact that a Hamiltonian system has some additional first integral independent with its Hamiltonian is a rare phenomenon which lead to a difficult problem, how to determine whether a given Hamiltonian system has additional independent first integrals.

In this work we consider the Hamiltonian systems of two degrees of freedom

$$\dot{q}_i=p_i,\quad \dot{p}_i=-\frac{\partial V}{\partial q_i},\quad i=1,2, \tag{1.1}$$

with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{2}p_i+V\left(\mathbf{q}\right), \tag{1.2}$$

where $V\left(\mathbf{q}\right)=V\left(q_1,q_2\right)\in\mathbb{C}\left[q_1,q_2\right]$ is a homogeneous polynomial potential of degree $k\in\mathbb{Z}$.

Let $A=A\left(\mathbf{q},\mathbf{p}\right)$ and $B=B\left(\mathbf{q},\mathbf{p}\right)$ be two functions with $\mathbf{q}=\left(q_1,q_2\right)$ and $\mathbf{p}=\left(p_1,p_2\right)$. Their *Poisson bracket* is defined as

$$\{A,B\}=\sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{1.3}$$

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