

PERIODS OF HOLOMORPHIC MAPS ON COMPACT RIEMANN SURFACES AND PRODUCT OF SPHERES

JAUME LLIBRE¹ AND VÍCTOR F. SIRVENT²

Dedicated to O.M. Sharkovsky in his 82nd birthday.

ABSTRACT. In this article we consider non-constant holomorphic maps on Riemann surfaces and product of Riemann spheres, we give conditions on the maps in order that they have arbitrary large prime numbers as periods. We use Lefschetz fixed point theory and in particular we compute the Lefschetz numbers of period m for large m 's.

1. INTRODUCTION

In the theory of the dynamical systems and mainly in the study of the iteration of self-maps on a topological manifold X , the periodic orbits play an important role. More precisely, let $f : X \rightarrow X$ be a continuous map, a point $x \in X$ is *periodic* of *period* $k \in \mathbb{N}$ if $f^k(x) = x$ and $f^j(x) \neq x$ for $j = 1, \dots, k - 1$. The set $\{x, f(x), \dots, f^{k-1}(x)\}$ is the *periodic orbit* of the periodic point x of period k . If $k = 1$ then the periodic point x is called a *fixed point*. We shall denote by $Per(f)$ the set of periods of the map f . A natural question is to ask for all the possible periods that the map f can exhibit. In some situations the knowledge of some possible periods of the map gives understanding of some global properties of the dynamics of the map, as in the case for continuous self-maps on the interval. If a continuous map on the interval has a periodic orbit of period three, then the map has orbits of all possible periods (*cf.* [8, 12]).

The differential topological methods are very useful for understanding the periodic structure of continuous self-maps on manifolds on dimensions greater than 1, because the topology of the manifold plays an important role, in particular we use the Lefschetz fixed point theory.

Date: September 6, 2019.

2010 Mathematics Subject Classification. 37C25, 37C30, 37E15.

Key words and phrases. Lefschetz numbers, holomorphic maps, periodic point, product of spheres, torus.

In this article we study the periodic structure of non-constant holomorphic maps on Riemann surfaces and product of Riemann spheres, in particular we give conditions on homology such that the maps have arbitrary large prime numbers as periods; the corresponding results are Theorems 4, 5 and Corollary 8. In section 2 we consider the self-maps on Riemann surfaces and in section 3 the maps on product of Riemann spheres.

Let X be an n -dimensional topological manifold and f a continuous self-map on X . The map f induces a homomorphism on the k -th rational homology group of X for $0 \leq k \leq n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over \mathbb{Q} and f_{*k} is a linear map whose matrix has integer entries.

The *Lefschetz number* of f is defined as

$$(1) \quad L(f) := \sum_{k=0}^n (-1)^k \text{trace}(f_{*k}).$$

The *Lefschetz Fixed Point Theorem* states that if $L(f) \neq 0$ then f has a fixed point (cf. [3] or [9]).

The *Lefschetz numbers of period m* , introduced in [4] and [10], are defined by

$$(2) \quad \ell(f^m) := \sum_{r|m} \mu(r) L(f^{m/r}),$$

where the sum is taken over all divisors r of m and μ is the Möbius function defined by

$$\mu(m) := \begin{cases} 1 & \text{if } m = 1; \\ 0 & \text{if there is a } k \text{ such that } k^2 \text{ divides } m; \\ (-1)^s & \text{if } m = p_1 \cdots p_s \text{ with } p_i \text{ distinct primes.} \end{cases}$$

By the Möbius inversion formula

$$L(f^m) = \sum_{r|m} \ell(f^r).$$

Observe that $L(f^m)$ and $\ell(f^m)$ are integer numbers for all m .

The *Lefschetz zeta function* of f is defined as

$$\zeta_f(t) = \exp \left(\sum_{m \geq 1} \frac{L(f^m)}{m} t^m \right).$$

This function keeps the information of the Lefschetz number for all the iterates of f , so this function gives information about the set of periods

of f . This function is rational, moreover its expression is

$$(3) \quad \zeta_f(t) = \prod_{k=0}^n \det(Id_{*k} - tf_{*k})^{(-1)^{k+1}},$$

where $n = \dim M$, $n_k = \dim H_k(M, \mathbb{Q})$, $Id := Id_{*k}$ is the identity map on $H_k(M, \mathbb{Q})$, and by convention $\det(Id_{*k} - tf_{*k}) = 1$ if $n_k = 0$ (cf. [6]).

Using formal computations we can write the Lefschetz zeta function as a formal infinite product that involves the $\ell(f^d)$, for details see [1]:

$$(4) \quad \zeta_f(t) = \exp\left(\sum_{m \geq 1} \frac{L(f^m)}{m} t^m\right) = \prod_{d \geq 1} (1 - t^d)^{-\frac{\ell(f^d)}{d}}.$$

The following result characterizes when the product in the identity (4) has a finite number of nontrivial factors, i.e. when $\ell(f^d) \neq 0$ for only finitely many m .

Theorem 1. *Let X be an n -dimensional topological manifold and f a continuous self-map on X . The zeros and poles of the Lefschetz zeta function of f are roots of unity if and only if $\ell(f^m) \neq 0$ for only finitely many m .*

Proof. “The only if part”. Let m_1, \dots, m_k be the only values of m such that $\ell(f^m) \neq 0$. According to (4),

$$\zeta_f(t) = (1 - t^{m_1})^{-\ell(f^{m_1})/m_1} \dots (1 - t^{m_k})^{-\ell(f^{m_k})/m_k}.$$

Therefore the roots and poles of $\zeta_f(t)$ are roots of unity.

“The if part”. Since $\zeta_f(t)$ is a rational function it has only a finite number of poles and zeros, i.e. let $\omega_1, \dots, \omega_l$ be such zeros and poles. So they are of the form $\omega_j = e^{2\pi i r_j / k_j}$, for some positive integer k_j and $0 \leq r_j \leq k_j - 1$. According to Gauss’s lemma the minimal polynomial of ω_j over the integers is the k_j -th cyclotomic polynomial, i.e. $\Phi_{k_j}(t)$. Hence $\Phi_{k_1}, \dots, \Phi_{k_l}$ are factors either in the numerator or denominator of $\zeta_f(t)$. By basic properties of the cyclotomic polynomials (cf. [7]):

$$\Phi_{k_j}(t) = \prod_{d|k_j} (1 - t^d)^{\mu(k_j/d)}.$$

Therefore there exist non-zero integers c_i , with $1 \leq i \leq s$, for some s , such that

$$\zeta_f(t) = \prod_{i=1}^s (1 - t^{d_i})^{c_i}$$

where d_i is a divisor of the least common multiple of $k_1 \dots k_l$. \square

The relationship between the numbers $\ell(f^m)$ and the periodic structure for holomorphic maps was given in [5] in the following result.

Theorem 2 (Theorem A of [5]). *Let M be a compact complex manifold and $f : M \rightarrow M$ be a nonconstant holomorphic map. Then, there exists $M > 0$ such that for all $p \in \mathbb{N}$ prime and $p > M$, $\ell(f^p) \neq 0$ if and only if $p \in \text{Per}(f)$.*

2. COMPACT RIEMANN SURFACES OF GENUS g

Let $X = M_g$ be a compact surface of genus g endowed with a complex structure and $f : M_g \rightarrow M_g$ be a non-constant holomorphic map. The homology groups of X with rational coefficients are:

$$H_0(M_g, \mathbb{Q}) = H_2(M_g, \mathbb{Q}) = \mathbb{Q}, \quad H_1(M_g, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{2g}.$$

The identity (3) allows to write the Lefschetz zeta function of f as

$$(5) \quad \zeta_f(t) = \frac{p(t)}{(1-t)(1-Dt)},$$

where D is the degree of f and $p(t) = \det(\text{Id} - tf_{*1})$, which is a polynomial of degree at most $2g$.

Proposition 3. *Let $f : M_g \rightarrow M_g$ be a continuous map. Then*

- (a) *If $p(t)$ is not a product of cyclotomic polynomials, then $\ell(f^m) \neq 0$ for infinitely many m .*
- (b) *If $|D| \neq 0, 1$ and D^{-1} is not an eigenvalue of f_{*1} of multiplicity 1, then $\ell(f^m) \neq 0$ for infinitely many m .*

Proof. For continuous self-maps on M_g an expression of its Lefschetz zeta function is given in (5), which is written in irreducible form, i.e. $1-t$ and $1-Dt$ are not factors of $p(t)$. If $p(t)$ is not a product of cyclotomic polynomials then $\zeta_f(t)$ has zeros which are not roots of unity, according to Theorem 1, there are infinitely many m 's such that $\ell(f^m) \neq 0$.

If $D \neq 0$ and D^{-1} is not an eigenvalue of f_{*1} of multiplicity 1, then D^{-1} is a root or pole of $\zeta_f(t)$, because $p(t) = \det(\text{Id} - tf_{*1})$. So if $D \neq -1, 1$ since D is an integer, we have that $\zeta_f(t)$ has a root or pole which is not a root of unity. Therefore $\ell(f^m) \neq 0$ for infinitely many m 's. \square

From Proposition 3 and Theorem 2 it yields the following result.

Theorem 4. *Let M_g be an orientable surface of genus g endowed with a complex structure and $f : M_g \rightarrow M_g$ be a nonconstant holomorphic map such that its Lefschetz zeta function is given in (5).*

- (a) *If $p(t)$ is not a product of cyclotomic polynomials.*
- (b) *If $|D| \neq 0, 1$ and D^{-1} is not an eigenvalue of f_{*1} of multiplicity 1.*

Then there exists an integer $N > 0$ such that for all prime $p > N$, p is a period of the map f .

Theorem 5. *Let M_g be an orientable surface of genus g endowed with a complex structure and $f : M_g \rightarrow M_g$ be a nonconstant holomorphic map. Let $\lambda_1, \dots, \lambda_{2g}$ be the eigenvalues of f_{*1} and D the degree of f . If any of these conditions hold*

- (a) $|D| > \max\{1, |\lambda_1|, \dots, |\lambda_{2g}|\}$.
- (b) *There exists $|\lambda_j| > \max\{1, |D|, |\lambda_i| : i \neq j\}$.*
- (c) *There exists i_1, \dots, i_k such that $\lambda_{i_1} = \dots = \lambda_{i_k}$ and*

$$|\lambda_{i_1}| > \max\{1, |D|, |\lambda_j| : j \neq i_1, \dots, i_k\}.$$

- (d) *There exists i_1, \dots, i_k such that*

$$|\lambda_{i_1}| = \dots = |\lambda_{i_k}| > \max\{1, |D|, |\lambda_j| : j \neq i_1, \dots, i_k\}.$$

Then there exists an integer $N > 0$ such that for all prime $p > N$, p is a period of the map f .

Proof. From the definition of the Lefschetz numbers (1) and for all $m \in \mathbb{N}$ we have

$$L(f^m) = 1 - (\lambda_1^m + \dots + \lambda_{2g}^m) + D^m.$$

For p prime, according to (2)

$$\begin{aligned} \ell(f^p) &= L(f^p) - L(f) \\ &= (1 - (\lambda_1^p + \dots + \lambda_{2g}^p) + D^p) - (1 - (\lambda_1 + \dots + \lambda_{2g}) + D) \\ &= (D^p - D) - (\lambda_1^p - \lambda_1) - \dots - (\lambda_{2g}^p - \lambda_{2g}). \end{aligned}$$

If conditions (a) or (b) are held, it can be easily checked that $\ell(f^p) \neq 0$ for sufficiently large prime p . Hence, by Theorem 2 these prime numbers belong to $\text{Per}(f)$.

If $\lambda_{i_1} = \dots = \lambda_{i_k}$, the number $\ell(f^p)$ can be written as

$$\ell(f^p) = -k(\lambda_{i_1}^p - \lambda_{i_1}) + (D^p - D) - \sum_{j \neq i_1, \dots, i_k} (\lambda_j^p - \lambda_j).$$

Hence

$$\begin{aligned} |\ell(f^p)| &\geq k|\lambda_{i_1}^p - \lambda_{i_1}| - |D^p - D| - \sum_{j \neq i_1, \dots, i_k} |\lambda_j^p - \lambda_j| \\ &= |\lambda_{i_1}^p - \lambda_{i_1}| \left(k - \frac{|D^p - D|}{|\lambda_{i_1}^p - \lambda_{i_1}|} - \sum_{j \neq i_1, \dots, i_k} \frac{|\lambda_j^p - \lambda_j|}{|\lambda_{i_1}^p - \lambda_{i_1}|} \right). \end{aligned}$$

If $|\lambda_{i_1}| > \max\{1, |D|, |\lambda_j| : j \neq i_1, \dots, i_k\}$, the ratios

$$\frac{|D^p - D|}{|\lambda_{i_1}^p - \lambda_{i_1}|}, \quad \frac{|\lambda_j^p - \lambda_j|}{|\lambda_{i_1}^p - \lambda_{i_1}|}$$

are arbitrary small for sufficiently large p . Hence $|\ell(f^p)| > 0$ for large prime numbers p . So by Theorem 2, these numbers p belong to $Per(f)$. This proves statement (c).

If $|\lambda_{i_1}| = \dots = |\lambda_{i_k}| > 1$ then

$$\begin{aligned} |\ell(f^p)| &\geq (|\lambda_{i_1}^p - \lambda_{i_1}| + \dots + |\lambda_{i_k}^p - \lambda_{i_k}|) - |D^p - D| - \sum_{j \neq i_1, \dots, i_k} |\lambda_j^p - \lambda_j| \\ &= |\lambda_{i_1}|^{p-1}((\lambda_{i_1} - 1) + \dots + (\lambda_{i_k} - 1)) - |D^p - D| - \sum_{j \neq i_1, \dots, i_k} |\lambda_j^p - \lambda_j| \\ &\geq C_1 |\lambda_{i_1}|^p - |D^p - D| - \sum_{j \neq i_1, \dots, i_k} |\lambda_j^p - \lambda_j|, \end{aligned}$$

for some constant $C_1 > 0$.

Hence

$$|\ell(f^p)| \geq |\lambda_{i_1}|^p \left(C_1 - \frac{|D^p - D|}{|\lambda_{i_1}|^p} - \sum_{j \neq i_1, \dots, i_k} \frac{|\lambda_j^p - \lambda_j|}{|\lambda_{i_1}|^p} \right).$$

If $|\lambda_{i_1}| > \max\{1, |D|, |\lambda_j| : j \neq i_1, \dots, i_k\}$ then the ratios

$$\frac{|D^p - D|}{|\lambda_{i_1}|^p}, \quad \frac{|\lambda_j^p - \lambda_j|}{|\lambda_{i_1}|^p}.$$

are small for arbitrary large p . Therefore we can conclude $\ell(f^p) \neq 0$, for large p . This proves statement (d). \square

We note that the minimal periods of holomorphic maps on surfaces has been studied in [11].

3. PRODUCT OF RIEMANN SPHERES

In this section we consider the product of Riemann spheres, which have a natural complex structure. We define $X(n) := \underbrace{\mathbb{S}^2 \times \cdots \times \mathbb{S}^2}_{n\text{-times}}$.

Theorem 6. *Let $f : X(n) \rightarrow X(n)$ be a continuous map. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of f_{*2} .*

- (a) *If $|\lambda_i| > 1$ for $1 \leq i \leq n$, then $\ell(f^p) \neq 0$ for all prime number p .*
- (b) *If $\lambda_i = -1$ for some i then $\ell(f^p) = 0$, for all odd prime numbers.*
- (c) *If $\lambda_i = -1$ for some i and $\lambda_j \neq 0$ for $1 \leq j \leq n$ then $\ell(f^2) \neq 0$.*

Before proving Theorem 6 we present the following result that determines the value of the Lefschetz numbers for the maps on $X(n)$.

Theorem 7 ([2]). *Let $f : X(n) \rightarrow X(n)$ be a continuous map. Then, for all $m > 0$,*

$$L(f^m) = \det(\text{Id} + f_{*2}^m),$$

where Id is the identity map on \mathbb{Q}^n . In particular,

$$L(f^m) = (-1)^n C_{f_{*2}^m}(-1),$$

where $C_h(t) := \det(t \text{Id} - h)$ denotes the characteristic polynomial of h .

Proof of Theorem 6. Due to Theorem 7, we have

$$L(f^m) = (-1)^n (1 + \lambda_1^m) \cdots (1 + \lambda_n^m).$$

If p is a prime number then

$$(6) \quad \ell(f^p) = L(f^p) - L(f)$$

$$(7) \quad = (-1)^n ((1 + \lambda_1^p) \cdots (1 + \lambda_n^p) - (1 + \lambda_1) \cdots (1 + \lambda_n)).$$

If $\lambda_i > 1$ for $1 \leq i \leq n$ then $\lambda_i^p > \lambda_i$ so $(-1)^n \ell(f^p) > 0$ for every prime p .

If $\lambda_i < -1$ for $1 \leq i \leq n$ then $\lambda_i^p < \lambda_i$ for every prime $p > 2$. If n is even then $\ell(f^p) > 0$ and if n is odd $\ell(f^p) < 0$. Similarly if $p = 2$ then $(-1)^n \ell(f^p) > 0$ for all n . This proves statement (a).

If $\lambda_i = -1$ for some i , then it follows from (7) that $\ell(f^p) = 0$ for p an odd prime. Moreover if $\lambda_j \neq 0$ for all $1 \leq j \leq n$ then, from (7) we have $\ell(f^2) \neq 0$. This proves statements (b) and (c), respectively. \square

The following corollary is a consequence of Theorem 6 and Theorem 2.

Corollary 8. *Let $f : X(n) \rightarrow X(n)$ be a non-constant holomorphic map. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of f_{*2} . If $|\lambda_i| > 1$, for $1 \leq i \leq n$ then there exists an integer N such that all prime numbers $p > N$ are periods of f .*

Theorem 9. *Let $f : X(n) \rightarrow X(n)$ be a non-constant holomorphic map. If f_{*2} has an eigenvalue bigger than one in modulus and that it does not have root of unity as eigenvalues. Then there exists N such that all prime numbers $p > N$ are periods of f .*

Theorem 9 is a particular case of the following more general result proved in [13].

Theorem 10. *Let $f : X(n) \rightarrow X(n)$ be a continuous map. We assume that f_{*2} has an eigenvalue bigger than one in modulus and that it does not have root of unity as eigenvalues. Then there exists N such that $\ell(f^m) \neq 0$ for $m \geq N$.*

ACKNOWLEDGEMENTS

The first author is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

REFERENCES

- [1] I. K. BABENKO AND S. A. BOGATYI, The behaviour of the index of periodic points under iterations of a mapping, *Math. USSR Izv.*, **38** (1992), 1–26.
- [2] P. BERRIZBEITIA, M.J. GONZÁLEZ, AND V.F. SIRVENT, On the Lefschetz zeta function and the minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms on products of ℓ -spheres, *Topology Appl.* **235** (2018) 428–444.
- [3] R.F. BROWN, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
- [4] A. DOLD, Fixed point indices of iterated maps, *Invent. math.* **74** (1983), 419–435.
- [5] N. FAGELLA AND J. LLIBRE, Periodic points of holomorphic maps via Lefschetz numbers, *Trans. Amer. Math. Soc.* **352** (2000), 4711–4730.
- [6] J. FRANKS, Period doubling and the Lefschetz formula, *Trans. Amer. Math. Soc.* **287** (1985), 275–283.
- [7] S. LANG, *Algebra*, Addison-Wesley, Boston, MA, 1971.

- [8] T. Y. LI AND J. YORKE, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.
- [9] S. LEFSCHETZ, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.
- [10] J. LLIBRE, Lefschetz numbers for periodic points, *Contemporary Math.*, vol 152 (1993), 215–227, Amer. Math. Soc., Providence.
- [11] J. LLIBRE AND W. MARZANTOWICZ, *Periods of holomorphic maps on surfaces*, *Funtiones et Approximatio* **40.2** (2009), 309–326.
- [12] O.M. SHARKOVSKY, Co-existence of cycles of a continuous mapping of the line into itself (Russian), *Ukrain. Mat. Zh.* **16** (1964), 61–71.
- [13] V.F. SIRVENT A note on the periodic structure of transversal maps on the torus and the product of spheres, *Pre-print* (submitted).

¹ DEPARTAMENT DE MATEMÀTIQUES. UNIVERSITAT AUTÒNOMA DE BARCELONA, BELLATERRA, 08193 BARCELONA, CATALUNYA, SPAIN

Email address: jllibre@mat.uab.cat

² DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD CATÓLICA DEL NORTE, ANTOFAGASTA, CHILE

Email address: victor.sirvent@ucn.cl