INTEGRABILITY IN THE LIOUVILLE SENSE FOR COMPLEX HAMILTONIAN SYSTEMS

JAUME LLIBRE[1], RAFAEL RAMÍREZ[2] AND NATALIA SADOVSKAIA[3]

ABSTRACT. The aim of this paper is to study the integrability in the Liouville sense of the complex Hamiltonian vector field

$$\Gamma_H = \sum_{j=1}^{M} \left(\frac{\partial H}{\partial P_j} \frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j} \frac{\partial}{\partial P_j} \right),$$

where $H = H(P_1, \ldots, P_M, z_1, \ldots, z_M)$ is a holomorphic function in an open subset of $\mathbb{C}^{2M}$. In particular we characterize the Liouvillian integrability of Γ_H with Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{M} P_j^2 + \sum_{k,j=1,\, k<j}^{M} a_{kj}(z_k - z_j) a_{jk}(z_j - z_k).$$

1. Introduction and statement of the main results

During the last decade the study of complex Hamiltonian systems has become of considerable theoretical interest. In particular until now in Physics the main interest in the complex Hamiltonian systems was focused into studying Hamiltonian systems in two dimensional real phase space ((x, p)–plane) described by a Hamiltonian $H(x, p)$ (see for instance [7]) passing this system to complex notation. For this purpose several methods of complexification are used. One type of complexification is obtained defining a z-plane (see for instance [11]) with $z = p + i\omega_0 x$ and $\bar{z} = p - i\omega_0 x$, where ω_0 is a real parameter. This type of complexification is discussed in several text-books on quantum mechanics.

Another type of complexification consists in extending each real variable x and p in the Hamiltonian $H(x, p)$ to convenient complex variables, i.e. extending the real two dimensional phase space (x-p plane) to a complex space with four real degrees of freedom. This complexification can be realized taking $x = x_1 + i\, p_2$ and $p = p_1 + i\, x_2$. This complexification is used to develop a new complex mechanics for which the classical and quantum mechanics appear as special subcases.

The complex Hamiltonian system with Hamiltonian $H(z, p)$ where $z = x_1 + ix_2$ and $p = p_1 + i\, p_2$ was studied in the paper [6], where the authors considered the

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