Gradient systems of harmonic polynomials

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Abstract

We characterize all local phase-portraits of the finite and infinite singular points of the gradient systems defined by the real harmonic polynomials in two variables.

We classify all the non-equivalent topological phase portraits of the gradient systems in the Poincaré disc defined by harmominc polynomials of degree less than five.

Keywords: harmonic polynomial, lgradient system, phase portrait.
2010 MSC: 34C07, 37C27.

1. Introduction and statement of the main results

There are several papers studying the dynamics of the differential equations in the plane $\mathbb{R}^2$ which come from holomorphic functions in one complex variable, when we separate them in their real and imaginary components, see for instance the articles [1, 6, 8, 9, 10, 13, 14]. Also there are some papers studying differential equations in $\mathbb{R}^4$ coming from functions of one quaternion variable, see [2, 7, 15] and the papers quoted there.

On the other hand the gradient differential equations in the plane $\mathbb{R}^2$ defined by the gradient of a smooth function of two real variables have been studied by several authors, see for example [3, 4, 11].

The objective of this paper is to study the gradient differential equations in the plane $\mathbb{R}^2$ which come from harmonic polynomials of two real variables.

Let $\mathbb{P}_n(\mathbb{R}^2)$ be the set of *harmonic polynomials* of degree $n \geq 1$ in the real variables

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x and y, i.e. $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$ if and only if

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

For every polynomial $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$ we consider its associated *gradient system*

$$\dot{x} = \frac{\partial P}{\partial x}(x, y), \qquad \dot{y} = \frac{\partial P}{\partial y}(x, y). \tag{1}$$

As usual the dot denotes the derivative with respect to the time t.

Our first result describes the local phase portrait of system (1) around its singular points.

Theorem 1. *Let $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$ and $p \in \mathbb{R}^2$ be a singular point of the gradient system* (1)*. Then the local phase portrait at the point p is union of $2(\alpha + 1)$ hyperbolic sectors with $\alpha \in \{1, \ldots, n-1\}$. So the (topological) index of p is $-\alpha$.*

We can now focus on global aspects of the dynamic of system (1). Thus in our second result we estimate the number of singular points of system (1), and the study the relationship between the index of these points with the degree of the harmonic polynomial $P(x, y)$.

Theorem 2. *Let $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$.*

(a) The number N of the finite singular points of the gradient system (1) *satisfies $1 \leq N \leq n - 1$.*

(b) The sum of the indices of all finite singular points of the gradient system (1) *is $1 - n$.*

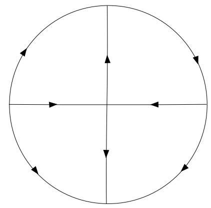
The third result provides the number of singular points at infinity of system (1) depending on the degree of the harmonic polynomial defining the gradient system.

Theorem 3. *Let $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$. Then gradient system* (1) *has $2n$ infinite singular points, and all of them are hyperbolic nodes.*

Finally the next result provides all phase portraits in the Poincaré disc of the gradient systems associated with the harmonic polynomials of degree less than 5. See chapter 5 of [5] for the definition and properties of the Poincaré compactification of a polynomial differential system in the Poincaré disc.

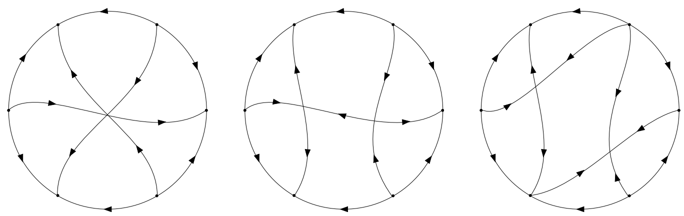
Theorem 4. *The following statements hold.*

(a) The phase portrait in the Poincaré disc of the gradient systems defined by a quadratic harmonic polynomial is topologically equivalent to the one of Figure 1.



(a)

Figure 1: Global phase portrait of a gradient system associated with a harmonic polynomial of degree 2.



(a) (b) (c)

Figure 2: Global phase portraits of a gradient system associated with a harmonic polynomial of degree 3.

(b) The phase portrait in the Poincaré disc of the gradient system defined by a cubic harmonic polynomial is topologically equivalent to one of the three phase portraits of Figure 2.

(c) The phase portrait in the Poincaré disc of the gradient system defined by a quartic harmonic polynomial is topologically equivalent to one of the nine phase portraits of Figure 3.

Theorems 1, 2, 3 and 4 are proved in section 3.

2. Preliminary results

In this section we introduce the basic definitions, notations and results which shall be used to analyze of the local portraits of the finite and infinite singular points of system (1).

Let $\mathbb{H}_n(\mathbb{R}^2)$ be the set of harmonic homogeneous polynomials of degree $n \geq 1$ in the variables x and y. Now consider the following polynomials

$$u_n(x,y) = \mathbf{Re}((x+iy)^n) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k},$$

$$v_n(x,y) = \mathbf{Im}((x+iy)^n) = \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1},$$

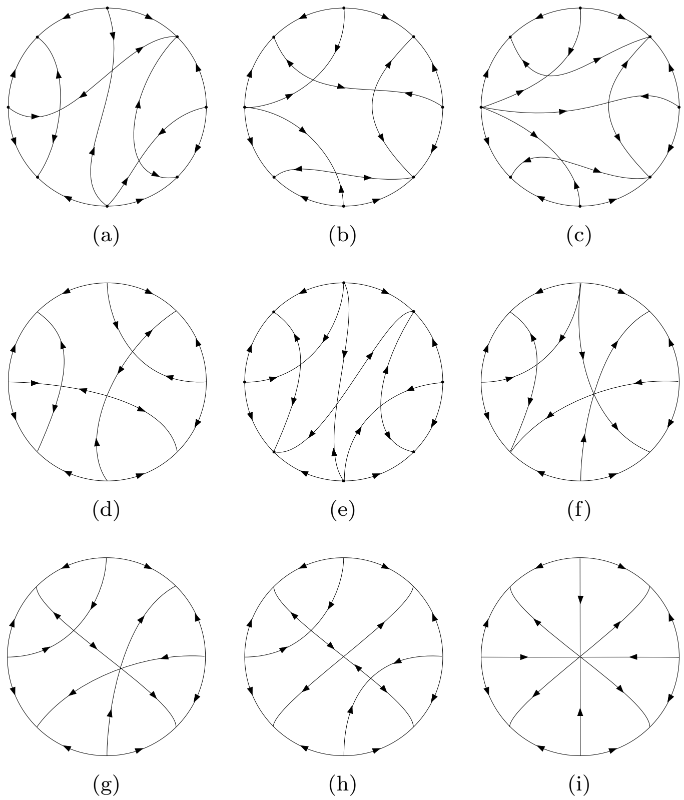


Figure 3: Global phase portraits of a gradient system associated with a harmonic polynomial of degree 4.

where n is a positive integer and i is the imaginary unity.

The next proposition show that the set $\{u_n(x), v_n(x)\}$ span $\mathbb{H}_n(\mathbb{R}^2)$ and it is proved in [16].

Proposition 5. *For $n \geq 1$, the set of polynomials $\{u_n(x,y), v_n(x,y)\}$ is a basis of $\mathbb{H}_n(\mathbb{R}^2)$.*

The following lemmas summarize some properties of functions $u_n(x,y)$ and $v_n(x,y)$.

Lemma 6. *For $n \geq 1$ the following properties hold.*

(a) $yu_n(x,y) + xv_n(x,y) = v_{n+1}(x,y)$;

(b) $xu_n(x,y) - yv_n(x,y) = u_{n+1}(x,y)$;

(c) $\dfrac{\partial}{\partial x}u_n(x,y) = nu_{n-1}(x,y)$;

(d) $\dfrac{\partial}{\partial y}u_n(x,y) = -nv_{n-1}(x,y)$;

(e) $\dfrac{\partial}{\partial x} v_n(x,y) = n v_{n-1}(x,y)$;

(f) $\dfrac{\partial}{\partial y} v_n(x,y) = n u_{n-1}(x,y)$.

Proof. We have that

$$\begin{aligned}(x+iy)^{n+1} &= (x+ix)(x+iy)^n \\ &= (x+iy)(u_n(x+iy)+iv_n(x,y)) \\ &= xu_n(x,y)-yv_n(x,y)+i(yu_n(x,y)+xv_n(x,y)).\end{aligned}$$

So $v_{n+1}(x,y) = yu_n(x,y)+xv_n(x,y)$ and $u_{n+1}(x,y) = xu_n(x,y)-yv_n(x,y)$. This proves statements (a) and (b).

Now we compute

$$\begin{aligned}\frac{\partial}{\partial x}u_n(x,y) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k}\right), \\ &= \sum_{k=0}^{[(n-1)/2]}(-1)^k(n-2k)\binom{n}{2k}x^{n-2k-1}y^{2k}, \\ &= n\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n-1}{2k}x^{n-2k-1}y^{2k}, \\ &= nu_{n-1}(x,y).\end{aligned}$$

So statement (c) is proved. In a similar way can be proved statements (d), (e) and (f). $\square$

Lemma 7. *If $P(x,y)\in\mathbb{P}_n(\mathbb{R}^2)$, then there are $\alpha_i,\beta_i\in\mathbb{R}$ for $i=0,\ldots,n$ such that*

$$P_n(x,y) = \alpha_0+\beta_0+\sum_{i=1}^{n}\alpha_i u_i(x,y)+\beta_i v_i(x,y),$$

where $|\alpha_n|+|\beta_n|\neq 0$.

Proof. It follows immediately from Proposition 5. $\square$

Lemma 8. *If $P(x,y)\in\mathbb{P}_n(\mathbb{R}^2)$, then*

$$G(x,y) = \int_0^y \frac{\partial P}{\partial x}(0,t)dt - \int_0^x \frac{\partial P}{\partial y}(t,y)dt$$

is a first integral of the gradient system (1).

Proof. Since that $P(x,y)$ is a harmonic polynomial of degree $n \geq 1$, it follows that $G(x,y)$ is also a polynomial of degree n such that

$$\frac{\partial G}{\partial x}(x,y) = -\frac{\partial P}{\partial y}(x,y),$$

and

$$\begin{aligned}\frac{\partial G}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left(-\int_0^x \frac{\partial P}{\partial y}(t,y)dt + \int_0^y \frac{\partial P}{\partial x}(0,t)dt\right)\\ &= -\int_0^x \frac{\partial^2 P}{\partial y^2}(t,y)dt + \frac{\partial P}{\partial x}(0,y)\\ &= \int_0^x \frac{\partial^2 P}{\partial x^2}(t,y)dt + \frac{\partial P}{\partial x}(0,y) = \frac{\partial P}{\partial x}(x,y).\end{aligned}$$

So we have

$$\left(\frac{\partial G}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial P}{\partial y}\right)(x,y) = 0.$$

Therefore $G(x,y)$ is a first integral of the gradient system (1). □

We associate to each harmonic polynomial a complex function associated as follows. Given the polynomial $P(x,y) \in \mathbb{P}_n(\mathbb{R}^2)$ we define the complex polynomial

$$R_P(z,\bar{z}) = \frac{\partial P}{\partial x}(x,y) - i\frac{\partial P}{\partial y}(x,y),$$

where $z = x + iy$.

In what follows we establish some properties of the complex function $R_P(z,\bar{z})$.

Lemma 9. *Let $P(x,y) \in \mathbb{P}_n(\mathbb{R}^2)$.*

(a) $\dfrac{\partial R_P}{\partial \bar{z}}(z,\bar{z}) = 0.$

(b) $R_P(z,\bar{z})$ is a polynomial of degree $n-1$ in the variables z and $\bar{z}$.

(c) A point $(x,y) \in \mathbb{R}^2$ is a singular point of the gradient system (1) if and only if the complex number $z = x + iy$ is a zero of the polynomial $R_P(z,\bar{z})$.

Proof. The real part $u(x,y)$ and the imaginary part $v(x,y)$ of $R_P(z,\bar{z})$ are

$$u(x,y) = \frac{\partial P}{\partial x}(x,y) \qquad \text{and} \qquad v(x,y) = -\frac{\partial P}{\partial y}(x,y),$$

respectively. Therefore the partial derivatives of $u(x,y)$ and $v(x,y)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial^2 P}{\partial x^2}(x,y), \; \frac{\partial u}{\partial y} = \frac{\partial^2 P}{\partial y \partial x}(x,y), \; \frac{\partial v}{\partial x} = -\frac{\partial^2 P}{\partial x \partial y}(x,y), \; \frac{\partial v}{\partial y} = -\frac{\partial^2 P}{\partial y^2}(x,y).$$

Since $P(x, y)$ is a harmonic polynomial, these partial derivatives have the following relationships:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

So $R_P(z, \bar{z})$ satisfies the Cauchy–Riemann equations, and consequently

$$\frac{\partial R_P}{\partial \bar{z}}(z, \bar{z}) = 0$$

This proved statement (a).

Now we determine the polynomial $R_P(z, \bar{z})$ in function of the coefficients of $P(x, y)$. Using Lemma 7 we have that

$$P(x, y) = \alpha_0 + \beta_0 + \sum_{k=1}^{n} \alpha_k u_k(x, y) + \beta_k v_k(x, y), \quad \text{with} \quad |\alpha_n| + |\beta_n| \neq 0.$$

Using Lemma 6 we get

$$\begin{aligned} \frac{\partial P}{\partial x}(x, y) &= \sum_{k=1}^{n} k(\alpha_k u_{k-1}(x, y) + \beta_k v_{k-1}(x, y)), \\ \frac{\partial P}{\partial y}(x, y) &= \sum_{k=1}^{n} k(-\alpha_k v_{k-1}(x, y) + \beta_k u_{k-1}(x, y)). \end{aligned} \tag{2}$$

Therefore

$$\begin{aligned} R_P(z, \bar{z}) = & \sum_{k=1}^{n} k(\alpha_k u_{k-1}(x, y) + \beta_k v_{k-1}(x, y)) - i \sum_{k=1}^{n} k(-\alpha_k v_{k-1}(x, y) + \beta_k u_{k-1}(x, y)) \\ = & \sum_{k=1}^{n} k(\alpha_k - i\beta_k)(\mathbf{Re}(z^{k-1}) + i\mathbf{Im}(z^{k-1})) \\ = & \sum_{k=1}^{n} k(\alpha_k - i\beta_k) z^{k-1}. \end{aligned}$$

As $\alpha_n, \beta_n \in \mathbb{R}$ are such that $|\alpha_n| + |\beta_n| \neq 0$ it follows that $R_P(z, \bar{z})$ is a polynomial in z of degree $n - 1$. This proves statement (b). Statement (c) follows directly from the definition of $R_P(z, \bar{z})$. $\square$

Using statement (c) of Lemma 9 we define the *multiplicity* of the singular point $(x, y) \in \mathbb{R}^2$ of the gradient system (1) as the multiplicity of the complex number $z = x + iy$ as a zero of the polynomial $R_P(z, \bar{z})$.

Remark 10. *If (x, y) is a singular point of the gradient system* (1) *defined by $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$, then its multiplicity is at most $n - 1$.*

3. Proofs of the theorems

Proof of Theorem 1. Let $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$. Without loss of generality we can assume that $(0, 0)$ is a singular point of the gradient system (1) of multiplicity α with $\alpha \leq n-1$, i.e., $z = 0 + i0$ is a root of multiplicity α of the polynomial $R_P(z, \bar{z})$. Therefore, using Lemma 9 we can rewrite,

$$R_P(z, \bar{z}) = (a_\alpha + ib_\alpha)z^\alpha + z^\alpha Q(z)$$

where $(a_\alpha + ib_\alpha) \in \mathbb{C} \setminus \{0\}$, and $Q(z)$ is a polynomial such that $Q(0) = 0$ and degree$(Q(z)) \geq 1$. So the real part $u(x, y)$ and the imaginary part $v(x, y)$ of $R_P(z, \bar{z})$ are

$$\begin{aligned}
u(x, y) &= a_\alpha \mathbf{Re}(z^\alpha) - b_\alpha \mathbf{Im}(z^\alpha) + \mathbf{Re}(z^\alpha)\mathbf{Re}(Q(z)) - \mathbf{Im}(z^\alpha)\mathbf{Im}(Q(z)), \\
v(x, y) &= a_\alpha \mathbf{Im}(z^\alpha) + b_\alpha \mathbf{Re}(z^\alpha) + \mathbf{Im}(z^\alpha)\mathbf{Re}(Q(z)) + \mathbf{Re}(z^\alpha)\mathbf{Im}(Q(z)).
\end{aligned}$$

From the definition of multiplicity of a singular point it follows that

$$\begin{aligned}
\frac{\partial P}{\partial x}(x, y) = \quad & a_\alpha \mathbf{Re}((x + iy)^\alpha) - b_\alpha \mathbf{Im}((x + iy)^\alpha) + \mathbf{Re}((x + iy)^\alpha)\mathbf{Re}(Q((x + iy))) \\
& - \mathbf{Im}((x + iy)^\alpha)\mathbf{Im}(Q((x + iy))), \\
\frac{\partial P}{\partial y}(x, y) = \quad & -a_\alpha \mathbf{Im}((x + iy)^\alpha) - b_\alpha \mathbf{Re}((x + iy)^\alpha) - \mathbf{Im}((x + iy)^\alpha)\mathbf{Re}(Q((x + iy))) \\
& - \mathbf{Re}((x + iy)^\alpha)\mathbf{Im}(Q((x + iy))).
\end{aligned}$$

In order to identify the local phase portrait of $(0, 0)$ we apply the polar blow up $(x, y) = (r \cos\theta, r \sin\theta)$ and get the following system, which after the rescaling of the time $ds = r^{1-\alpha}dt$, it can be written as

$$\begin{aligned}
\dot{r} &= r\left[a_\alpha \cos((\alpha + 1)\theta) - b_\alpha \sin((\alpha + 1)\theta) + \mathbf{Re}(e^{i(\alpha+1)\theta}Q(re^{i\theta}))\right], \\
\dot{\theta} &= -a_\alpha \sin((\alpha + 1)\theta) - b_\alpha \cos((\alpha + 1)\theta) - \mathbf{Im}(e^{i(\alpha+1)\theta}Q(re^{i\theta})),
\end{aligned}$$

where the dot denotes the derivative with respect to s. This system has $2(\alpha+1)$ singular points with $r = 0$. In particular, if $(r, \theta) = (0, \theta_j)$ is one of them, then θ_j is a zero of the function $\varphi(\theta) = -a_\alpha \sin((\alpha + 1)\theta) - b_\alpha \cos((\alpha + 1)\theta)$, and its eigenvalues are $\varphi'(\theta_j)$ and $-\varphi'(\theta_j)/(\alpha + 1)$. Therefore $(0, \theta_j)$ is a hyperbolic saddle. Going back through the changes of variables we get that the local phase portrait at the singular point $(0, 0)$ is given by $2(\alpha + 1)$ hyperbolic sectors, and by the Poincaré formula of the index (see for instance section 6.7 of [5]), the index of this singular point is the $-\alpha$. Hence the theorem is proved. $\square$

We can rewrite Theorem 1 in terms of the multiplicity of the singular point.

Remark 11. *Let $P(x, y) \in \mathbb{P}_n(\mathbb{R}^2)$ and $p \in \mathbb{R}^2$ be a singular point of system* (1) *of multiplicity α. Then the local phase portrait at the point p is union of $2(\alpha+1)$ hyperbolic sectors, and its index is $-\alpha$.*

Proof of Theorem 2. We consider $P(x,y) \in \mathbb{P}_n(\mathbb{R}^2)$. By Lemma 9 $R_P(z,\bar{z})$ is a polynomial of degree $n-1$, and $\partial R_P(z,\bar{z})/\partial \bar{z} = 0$. By the Fundamental Theorem of Algebra $R_P(z,\bar{z})$ has $n-1$ complex zeros counting their multiplicities. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of $R_P(z,\bar{z})$, with multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_k$ respectively, then

$$R_P(z,\bar{z}) = c(z-z_1)^{\alpha_1}(z-z_2)^{\alpha_2}\ldots(z-z_k)^{\alpha_k}$$

where $c \in \mathbf{C} \setminus \{0\}$ and $\alpha_1 + \ldots + \alpha_k = n-1$. Now from statement ($c$) of Lemma 9 we get that the gradient system (1) has the singular points $(\mathbf{Re}(z_l), \mathbf{Im}(z_l)) \in \mathbb{R}^2$ for $l = 1, \ldots, k$. The multiplicities of these singular points are α_l for $l = 1, \ldots, k$. Therefore from the Remark (11) we get that the sum of the indices of all singular points is $\sum_{l=1}^{k} \alpha_l = 1-n$. This completes the proof of the statements of the theorem. □

Proof of Theorem 3. From (2) the gradient system (1) defined by the harmonic polynomial $P(x,y) \in \mathbb{P}_n(\mathbb{R}^2)$ is

$$\begin{aligned}
\dot{x} &= \frac{\partial P}{\partial x}(x,y) = \sum_{k=1}^{n} k(\alpha_k u_{k-1}(x,y) + \beta_k v_{k-1}(x,y)),\\
\dot{y} &= \frac{\partial P}{\partial y}(x,y) = \sum_{k=1}^{n} k(-\alpha_k v_{k-1}(x,y) + \beta_k u_{k-1}(x,y)),
\end{aligned}$$

with

$$|\alpha_n| + |\beta_n| \neq 0. \tag{3}$$

From the Poincaré compactification the infinite singular points in the Poincaré disc of the gradient system (1) are determined by the real solutions of the system

$$y(\alpha_n u_{n-1}(x,y) + \beta_n v_{n-1}(x,y)) - x(-\alpha_n v_{n-1}(x,y) + \beta_n u_{n-1}(x,y)) = 0, \quad x^2 + y^2 = 1.$$

Then, by statements (a) and (b) of Lemma 6, taking $(x,y) = (\cos\theta, \sin\theta)$ and $z = e^{i\theta}$ we have that the solutions of the previous system are the zeros of the function

$$\begin{aligned}
G(\theta) = \;& \alpha_n v_n(\cos\theta, \sin\theta) - \beta_n u_n(\cos\theta, \sin\theta)\\
= \;& \alpha_n \mathbf{Im}(z^n) - be_n \mathbf{Re}(z^n)\\
= \;& \alpha_n \sin(n\theta) - \beta_n \cos(n\theta).
\end{aligned}$$

This function has $2n$ simple zeros in $[0, 2\pi)$. So the gradient system (1) has $2n$ infinite singular points. Now we shall determine the local phase portraits at these infinite singular points studying the compactification of the gradient system in the local charts U_1 and U_2, using the notation of chapter 5 of [5].

The expression of the gradient system in the local chart U_1 is

$$
\begin{aligned}
\dot{u} = & \ -\sum_{i=1}^{n} kv^{n-k}\big((\alpha_k u - \beta_k)u_{k-1}(1,u) + (\alpha_k + \beta_k u)v_{k-1}(1,u)\big) \\
= & \ -\sum_{i=1}^{n} kv^{n-k}\big(\alpha_k v_k(1,u) - \beta_k u_k(1,u)\big), \qquad (4) \\
\dot{v} = & \ -\sum_{i=1}^{n} kv^{n+1-k}\big(\alpha_k u_{k-1}(1,u) + \beta_k v_{k-1}(1,u)\big),
\end{aligned}
$$

where in the expression of $\dot{u}$ we have used statements (a) and (b) of Lemma 6. If $(u_0, 0)$ is an infinite singular point of this system, then u_0 is a real root of the polynomial

$$F(u) = n(\alpha_n v_n(1,u) - \beta_n u_n(1,u)). \qquad (5)$$

We note that θ_0 satisfies $G(\theta_0) = 0$ if and only if $\tan(\theta_0)$ satisfies $F(\tan(\theta_0)) = 0$.

The linear part of system (4) at the infinite singular point $(u_0, 0)$ has the two eigenvalues

$$\lambda_1 = n(\alpha_n v_n'(1,u_0) - \beta_n u_n'(1,u_0)) = n^2(\alpha_n u_{n-1}(1,u_0) + \beta_n v_{n-1}(1,u_0)) = n^2\zeta,$$

and $\lambda_2 = n\zeta$, where we have used statements (d) and (f) of Lemma 6. It clear that $\lambda_1\lambda_2 \geq 0$. We claim that $\lambda_1\lambda_2 > 0$, so all the infinite singular points will be hyperbolic nodes, except perhaps the origin of the local chart U_2 in case that it be an infinite singular point.

Now we prove the claim. Assume that $\lambda_1\lambda_2 = 0$. Then $\alpha_n u_{n-1}(1,u_0) + \beta_n v_{n-1}(1,u_0) = 0$ and from (5), it follows that

$$\begin{pmatrix} -v_n(1,u_0) & u_n(1,u_0) \\ u_{n-1}(1,u_0) & v_{n-1}(1,u_0) \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (6)$$

The determinant of the matrix associated to this linear system in the variables α_n and β_n is

$$
\begin{aligned}
& -v_n(1,u_0)v_{n-1}(1,u_0) - u_n(1,u_0)u_{n-1}(1,u_0) = \\
& -(u_0 u_{n-1}(1,u_0) + v_{n-1}(1,u_0))v_{n-1}(1,u_0) - (u_{n-1}(1,u_0) - u_0 v_{n-1}(1,u_0))u_{n-1}(1,u_0) = \\
& -v_{n-1}^2(1,u_0) - u_{n-1}^2(1,u_0) = \\
& -\left(\mathbf{Im}((1+iu_0)^{n-1})\right)^2 - \left(\mathbf{Re}((1+iu_0)^{n-1})\right)^2 < 0,
\end{aligned}
$$

where again we have used statements (a) and (b) of Lemma 6. Since the determinant of the linear system (5) is non–zero, we get that $(\alpha_n, \beta_n) = (0, 0)$, which is in contradiction with (3). Hence the claim is proved.

Working in a similar way as we did in local chart U_1 we obtain that the gradient system (1) in the local chart U_2 is

$$\dot{u} = \sum_{i=1}^{n} kv^{n-k}(\alpha_k v_k(u,1) - \beta_k u_k(u,1)),$$
$$\dot{v} = \sum_{i=1}^{n} kv^{n+1-k}(\alpha_k v_{k-1}(u,1) - \beta_k u_{k-1}(u,1)).$$

Using the same arguments of the chart U_1 we can prove that, if the origin of the local chart U_2 is an infinite singular point then it is a hyperbolic node. So the gradient system (1) has exactly $2n$ singular points at infinity, and all of them are hyperbolic nodes. □

Proof of Theorem 4. Suppose that $P(x, y)$ is a harmonic polynomial of degree 2, then the gradient system (1) is linear. Applying Theorems 1 and 2 we have that the system admits a unique finite singularity which is of saddle type. From Theorem 3 it follows that this system has four infinite singularities, which are hyperbolic nodes. Therefore we get that the global phase portrait in the Poincaré disc for this system is topologically equivalent to the phase portrait of Figure 1. This proves statement (a).

Assume now that $P(x, y)$ is a harmonic polynomial of degree 3. From Theorems 1, 2 and 3 we get all the information about the local phase portraits of the finite and infinite singular points for the grandient system (1). In particular one of the following cases holds:

Either the gradient system (1) has two finite singular points, each one having four hyperbolic sectors, and six infinite singular points all of them hyperbolic nodes.

Or the gradient system (1) has a unique finite singularity, having six hyperbolic sectors, and six infinite singular points all of them hyperbolic nodes.

The finite singular points of our gradient systems are formed by four or six hyperbolic sectors their topological index are -1 or -2, respectively. So these systems cannot have periodic orbits because the sum of the indices in the bounded region limited by a periodic orbit must be one, see for instance Theorem 6.8.1 of [12]. Moreover our gradient systems cannot have graphics because the unique possible graphics must involve the finite singular points having hyperbolic sectors, and consequently such graphics either are formed by a singular point and a homoclinic orbit to it, or by two singular points and two heteroclinic orbits connected them. But such graphics again in the bounded region limited by them must satisfy that the sum of the indices in that region must be one, and this is not possible. Therefore, by the Poincaré-Bendixson Theorem (see for instance Theorem 1.30 of [5]) the α– and ω–limit sets of any orbit always are a finite or an infinite singular point. So the phase portrait of a gradient system defined by a harmonic polynomial of degree 3 is topologically equivalent to some of the phase portraits of Figure 2. This proves statement (b).

Finally assume now that $P(x, y)$ is a harmonic polynomial of degree 4. From Theorems 1, 2 and 3 one of the following cases hold:

Either the gradient system (1) has three finite singular points, each one having four hyperbolic sectors, and eight infinite singular points all of them hyperbolic nodes.

Or the gradient system (1) has two finite singular points, one having four hyperbolic sectors and the other having six of those sectors, and eight infinite singular points all of them hyperbolic nodes.

Or the gradient system (1) has a unique finite singularity, having eight hyperbolic sectors, and eight infinite singular points all of them hyperbolic nodes.

Consequently the phase portraits of a gradient system defined by a harmonic polynomial of degree 4 must be topologically equivalent to some of the phase portraits of Figure 2. This proves statement (c). This completes the proof of the theorem. □

Acknowledgements

The first author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

References

References

[1] L. Brickman and E. S. Thomas, *Conformal equivalence of analytic flows*, J. Differential Equations **25** (1977), 310–324.

[2] Z.F. Cai and K.I. Kou, *Solving quaternion ordinary differential equations with two-sided coefficients*, Qual. Theory Dyn. Syst. **17** (2018), 441–462.

[3] C.C. Chicone, *Quadratic gradients on the plane are generically Morse–Smale*, J. Differential Equations **33** (1979), 159–166.

[4] C. Chicone and P. Ehrlich, *Gradient–like and integrable vector fields on* $\mathbb{R}^2$, Manuscripta Math. **49** (1984), 141–164.

[5] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer, New York, 2006.

[6] A. Garijo, A. Gasull and X. Jarque, *Normal forms for singularities of one dimensional holomorphic vector fields*, Elect. J. Differential Equations **2004** (2004), No. 122, pp. 1–7.

[7] A. Gasull, J. Llibre and X. Zhang, *One-dimensional quaternion homogeneous polynomial differential equations*, J. Math. Phys. **50** (2009), no. 8, 082705, 17 pp.

[8] O. Hájek, *Notes on meromorphic dynamical systems I*, Czech. Math. J. **91** (1966), 14–27.

[9] O. Hájek, *Notes on meromorphic dynamical systems II*, Czech. Math. J. **91** (1966), 28–35.

[10] D.J. Needham and A. C. King, *On meromorphic complex differential equations*, Dynamics and stability systems **9** (1994), 99–122.

[11] S. Smale, *On gradient dynamical systems*, Ann. of Math. **74** (1961), 199–206.

[12] S.Strogatz, *Nonlinear dynamics and chaos*, Addison–Wesley, 1994.

[13] R. Sverdlove, *Vector fields defined by complex functions*, J. Differential Equations **34** (1978), 427–439.

[14] L. Teyssier, *Examples of non-conjugated holomorphic vector fields and foliations*, J. Differential Equations, 205 (2004) pp. 390-407.

[15] X. Zhang, *Global structure of quaternion polynomial differential equations*, Comm. Math. Phys. **303** (2011), 301–316.

[16] S. Axler S., P. Bourdon , W. Ramey , *Harmonic Function Theory*, Graduate Texts in Mathematics, Springer, New York, 2001