HILBERT'S 16TH PROBLEM. WHEN DIFFERENTIAL SYSTEMS MEET VARIATIONAL METHODS

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Abstract. We provide an upper bound for the number of limit cycles that planar polynomial differential systems of a given degree may have. The bound turns out to be a polynomial of degree four in the degree of the system. The strategy brings together variational and dynamical system techniques by transforming the task of counting limit cycles into counting critical points for a certain smooth, non-negative functional, through Morse inequalities, for which limit cycles are global minimizers. We thus solve the second part of Hilbert's 16th problem providing a uniform upper bound for the number of limit cycles which only depends on the degree of the polynomial differential system.

Contents

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1. Introduction

We deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = x' = P(x,y), \qquad \frac{dy}{dt} = y' = Q(x,y), \tag{1.1}$$

where the maximum degree of the polynomials P and Q is n. This n is called the *degree* of the polynomial differential system (1.1).

We recall that a *limit cycle* of the differential system (1.1) is a periodic orbit of this system isolated in the set of all periodic orbits of system (1.1). As far as we known the notion of limit cycle appeared in the year 1885 in the work of Poincaré [42]. Moreover, he proved that a polynomial differential equation (1.1) without saddle connections has finitely many limit cycles, see [41].

In the Second International Congress of Mathematicians, celebrated in Paris in 1900, Hilbert [24] proposed a list of 23 relevant problems to be solved during the XX century. The 16–th problem of this list reads:

Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6th order, I have satisfied myself–by a complicated process, it is true–that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be *of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4th order in three dimensional space can really have.*

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential systems. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limits) for a differential system of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the nth degree in x and y. Written homogeneously, this is

$$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0,$$

where X, Y, and Z are rational integral homogeneous functions of the nth degree in x, y, z, and the latter are to be determined as functions of the parameter t.

Clearly the 16–th Hilbert problem is formulated in two parts. The first part is about the mutual disposition of the maximal number of separate branches of an algebraic curve, and its extension to nonsingular real algebraic varieties. The second part is dedicated to limit cycles of polynomial differential systems in $\mathbb{R}^2$, where Hilbert asked for the maximal number and relative position of limit cycles of polynomial differential systems (1.1). Usually the first part of the 16–th Hilbert problem is studied in real algebraic geometry, while the second part is considered in the theory of differential systems. Hilbert also pointed out that there might exist possible connections between these two parts. Some of these connections are described in the survey about the 16–th Hilbert problem written by Jibin Li, see [30].

From now on, when we talk about the 16–th Hilbert problem, we always mean the second part of the 16–th Hilbert problem.

In 1988 Noel Lloyd [36] observed with respect to the 16–th Hilbert problem that *the striking aspect is that the hypothesis is algebraic, while the conclusion is topological.*

Arnold in 1977 and 1983 (see [1] and [2], respectively) stated the *weakened, infinitesimal or tangential* 16*–th Hilbert problem* which we do not consider here, but there are some surveys for this modified problem. See for instance the survey of Ilyashenko [28] on the 16–th Hilbert problem, the already mentioned survey of Jibin Li, the book of Colin Christopher and Chengzhi Li [14], the survey due to Kaloshin [29], the one of Yakovenko [50], or more recently the work of Binyamini, Novikov and Yakovenko [7].

According to Smale [45], except for the Riemann hypothesis, the second part of the 16–th Hilbert problem seems to be the most elusive of Hilbert's problems. Smale states the following version of the second half of 16–th Hilbert problem with respect to the number of limit cycles:

Consider the polynomial differential system (1.1) *in* $\mathbb{R}^2$. *Is there a bound* K *on the number of limit cycles of the form* $K \leq n^q$ *where* n *is the maximum of the degrees of* P *and* Q, *and* q *is a universal constant*?

The topological configurations or possible distribution of limit cycles mentioned as position by Hilbert has also attracted the attention of many authors. Coleman in his work [16] on the 16–Hilbert problem said: *For* $n > 2$ *the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes, or eyes enclosing more than a single critical point, can exist.* Here "eye" means a nest of limit cycles. We shall see later that some of the questions on the possible topological configurations of limit cycles realized by polynomial differential systems have been solved.

1.1. **On the number of limit cycles.** Dulac [18] claimed in 1923 that any polynomial differential system (1.1) always has finitely many limit cycles. Ilyashenko [26] found an error in Dulac's paper in 1985. Later, two long works have appeared, independently, providing proofs of Dulac's assertion: one due to Écalle [19] in 1992, and the other to Ilyashenko [27] in 1991. As Smale mentioned in [45], these two books have yet to be thoroughly digested by the mathematical community. We remark that in no case the results of Écalle and Ilyashenko prove that there exists a uniform upper bound for the maximum number of limit cycles of all the polynomial differential systems of a given degree.

Bamon [3] proved in 1986 that any polynomial differential system of degree 2 has finitely many limit cycles. His result uses previous results of Ilyashenko.

Here a *homoclinic* or *heteroclinic loop* is formed by $k = 1$ or $k > 1$ saddles (eventually some saddles can be repeated), and k different separatrices connecting these saddles and forming a loop (eventually some points of this loop can be identified in a repeated saddle) in such a way that at least in one of the two sides of the loop a Poincaré return map is defined. Let $\mu_i < 0 < \lambda_i$ be the eigenvalues of these saddles. If

$$\prod_{i=1}^{k} \frac{\lambda_i}{\mu_i} \neq 1,$$

then the loop is called *simple*. From the work of Poincaré [42] (see Theorem XVII), it follows that if a polynomial differential system (1.1) has all its saddle connections forming a simple homoclinic or heteroclinic loop, then the system also has finitely many limit cycles, see the nice work of Sotomayor [46] for more details. There are many other results providing upper bounds for the maximum number of limit cycles which can accumulate or bifurcate from different kinds of homoclinic and heteroclinic loops.

In 1957 Petrovskii and Landis [39] claimed that the polynomial differential systems of degree $n = 2$ have at most 3 limit cycles. Soon (in 1959) a gap was found in the arguments of Petrovskii and Landis, see [40]. Later, Lan Sun Chen and Ming Shu Wang [11] in 1979, and Songling Shi [44] in 1982, provided the first polynomial differential systems of degree 2 having 4 limit cycles, and up to now 4 is the maximum number of limit cycles known for a polynomial differential system of degree 2.

Lower bounds for the maximum number of limit cycles that a polynomial differential system of degree n can have have been given mainly by Christopher and Lloyd [15], Jibin Li [30], and more recently by Maoan Han and Jibin Li [23].

There are also some relevant results about the 16th Hilbert problem restricted to algebraic limit cycles, see Appendix 1.

1.2. **Statement of main results.** The following theorem provides an upper bound for the maximum number of limit cycles that a polynomial differential system of degree n can have. So it provides an answer to the second part of the 16-th Hilbert problem, and an answer to the stronger version of it stated by Smale.

Theorem 1. *An upper bound for the maximum number $H(n)$ of limit cycles that a polynomial differential system of degree $n > 1$ can have is*

$$H(n) \leq \frac{5}{2}n^4 - \frac{13}{2}n^3 + 6n^2 \quad \textit{if } n \textit{ is even, and}$$
$$H(n) \leq \frac{5}{2}n^4 - \frac{13}{2}n^3 + 5n^2 \quad \textit{if } n \textit{ is odd.}$$

The number $H(n)$ is usually called the *Hilbert number* for the polynomial differential systems of degree n. This upper bound for $H(n)$ yields a universal exponent $q = 4$ for the stronger version due to Smale.

A more detailed version of Theorem 1, in the generic case in which all the components of the algebraic curve

$$P_x + Q_y = 0$$

are homeomorphic to a straight line or to a circle, and the number of contact points of the vector field with the previous curve is finite, is the following one. We recall that a point of the divergence curve is of *contact* if the vector field is either tangent to the curve at this point, or it is a singular point of the vector field.

Theorem 2. *Consider a polynomial differential system* (1.1) *of degree $n > 1$. Assume that*

(i) *all of the connected components of the curve $P_x + Q_y = 0$, are homeomorphic to a straight line or to an oval (i.e. the algebraic curve $P_x + Q_y = 0$ has no singular points);*
(ii) *N is the finite number of real solutions of the polynomial system*

$$\begin{aligned} P(P_{xx} + Q_{yx}) + Q(P_{xy} + Q_{yy}) &= 0, \\ P_x + Q_y &= 0, \end{aligned} \tag{1.2}$$

(i.e. the contact points of the vector field (P, Q) with the curve $P_x + Q_y = 0$);
(iii) *M is the number of connected components of the curve $P_x + Q_y = 0$.*

Then an upper bound for the number of limit cycles $H(n)$ that the differential system (1.1) *may have is*

$$H(n) \leq n^2(M + N). \tag{1.3}$$

The paper is dedicated to proving Theorems 1 and 2. In particular, it is of the utmost importance to understand the role played by the following two pieces of information:

(I) the divergence curve

$$\text{Div} = \text{Div}(x, y) \equiv P_x(x, y) + Q_y(x, y) = 0$$

and

(II) the contact points of the vector field (P, Q) with the curve $\text{Div} = 0$ of system (1.2).

All of our efforts are concentrated in proving Theorem 2, together with a short but clear discussion about its validity in a non-generic situation in Section 12. Section 2 is devoted to proving Theorem 1 based on Theorem 2, and its version for a non-generic situation, as just indicated. We also explore the particular situations of quadratic and Lienard differential systems.

Before getting into a serious discussion for a rigorous proof of Theorem 2, it is instructive to have an overall description of the principle and main steps that will guide us, as they are quite different from the typical techniques utilized in polynomial differential systems. This is clearly stated in Section 3. We have also written two final appendices with additional information on Hilbert's 16th problem.

2. Hilbert's 16th problem in the generic case

Suppose that the polynomials P and Q are coprime, i.e. equilibria are isolated. If the polynomials P and Q have a common factor, it can be removed by doing a change in the independent variable of the polynomial differential system, and the upper bound on the number of limit cycles of the new polynomial system obtained is also an upper bound for the number of limit cycles of the initial polynomial system having P and Q a common factor.

It is well-known that under small perturbations in the coefficients of P and Q the components of the algebraic curve $\text{Div} = 0$ are homeomorphic either to a straight line or to an oval, and that the number of contact points of the vector field (P, Q) with the curve $\text{Div} = 0$ is finite. We refer to this previous case as the generic one. Given that the bound for the generic case is uniform on the degree n of the system, we will show in Section 12 that such an upper bound for the number of limit cycles of a polynomial differential system of degree n in the generic case extends to a non–generic polynomial differential system of degree n. So here we focus on proving Theorem 1 in the generic case. To do so, we recall two classical theorems:

Theorem 3 (Bezout Theorem)**.** *Let $R(x, y)$ and $S(x, y)$ be two polynomials with coefficients in $\mathbb{R}$. If both polynomials do not share a non-trivial common factor, then the algebraic system of equations*

$$R(x, y) = S(x, y) = 0$$

has at most degree(R)degree(S) solutions.

For a proof of Theorem 3 see for instance [20].

Theorem 4 (Harnack Theorem)**.** *The maximum number of connected components of an algebraic curve of degree k is*

(a) $1 + (k - 1)(k - 2)/2$ *if k is even,*
(b) $(k - 1)(k - 2)/2$ *if k is odd.*

For a proof of Theorem 4 see for instance [22].

Proof of Theorem 1 in the generic case assuming Theorem 2. We need to find an upper bound for the number N of the solutions that system (1.2) may have when P and Q are polynomials of at most degree n. By Bezout's theorem we have that $N \leq 2(n-1)^2$, because in the generic case we discard the possibility that the two equations of system (1.2) have a non–trivial common factor.

By Theorem 4 the number M of components of $\mathrm{Div} = 0$ satisfies $M \leq \frac{1}{2}(n-2)(n-3)+1$, if n is even, and $M \leq \frac{1}{2}(n-2)(n-3)$, if n is odd.

The final number in the statement of the theorem is then a direct consequence of Theorem 2, i.e.

$$\begin{aligned} n^2(N+M) \leq&\ n^2\left(\frac{1}{2}(n-2)(n-3)+1+2(n-1)^2\right) \\ =&\ \frac{5}{2}n^4 - \frac{13}{2}n^3 + 6n^2, \end{aligned}$$

if n is even, while

$$\begin{aligned} n^2(N+M) \leq&\ n^2\left(\frac{1}{2}(n-2)(n-3)+2(n-1)^2\right) \\ =&\ \frac{5}{2}n^4 - \frac{13}{2}n^3 + 5n^2, \end{aligned}$$

if n is odd. This yields the numbers in the statement of Theorem 1 in the generic case. □

Two corollaries of Theorem 2 are the following ones.

Corollary 5. *If the divergence of a quadratic polynomial differential system* (1.1) *is constant or zero, then it has no limit cycles. Otherwise if the straight line* $\mathrm{Div} = 0$ *of a quadratic polynomial differential system* (1.1) *has:*

(a) *two contact points, then it cannot have more than* 12 *limit cycles.*
(b) *one contact point, then it cannot have more than* 8 *limit cycles.*
(c) *no contact points, then it has no limit cycles.*

Proof. The set $\mathrm{Div} = 0$ for a quadratic polynomial differential system is either empty, or a straight line, or the whole plane. If it is empty, i.e. if Div is a non-zero constant, then the system has no limit cycles by Bendixon criterium (see for instance Theorem 7.10 of [17]). If it is the whole plane, the system is Hamiltonian and so it has no limit cycles. We assume that $\mathrm{Div} = 0$ is a straight line. So using (1.3), we have $n = 2$, $M = 1$, $N \in \{0, 1, 2\}$.

If $N = 2$, then $n^2(M + N) = 4(1 + 2) = 12$.
If $N = 1$, we obtain $n^2(M + N) = 4(1 + 1) = 8$.
If $N = 0$, then there are no contact points and the limit cycles cannot intersect the straight line $\mathrm{Div} = 0$. Again by Bendixon criterium the system has no limit cycles. □

Corollary 6. *We consider the Liénard polynomial differential systems*

$$\dot{x} = P(x,y) = y - f(x), \quad \dot{y} = Q(x,y) = g(x), \tag{2.1}$$

where p is the degree of f, and q is the degree of g. So $n = \max\{p, q\}$. A system (2.1) *cannot have more than $2\max\{p, q\}^2(p-1)$ limit cycles.*

Proof. It is well known that a system (2.1) has at most $p-1$ connected components for the curve $\text{Div} = f'(x) = 0$ corresponds to the critical values of the polynomial f. Note that each component is a vertical straight line in the (x, y)–plane. System (1.2) becomes

$$(y - f(x))f''(x) = 0, \qquad f'(x) = 0,$$

for system (2.1). If $f''(x) = f'(x) = 0$ has a solution x_0, the vertical straight line $x = x_0$ is formed by contact points, so it is invariant and we do need to take it into account, because limit cycles cannot intersect such straight line. Therefore a connected component of the curve $\text{Div} = 0$ has one single contact point $(x_0, f(x_0))$ for each zero x_0 of the polynomial $f'(x)$ such that $f''(x_0) \neq 0$. Using again (1.3), we have $n = \max\{p, q\}$, and $M = N \leq p - 1$. Hence

$$n^2(M + N) = 2\max\{p, q\}^2(p-1).$$

This completes the proof. □

We now turn to treat rigorous proofs of all the ingredients that we shall use for proving Theorem 2, and its version for a non-generic situation.

3. Overview

Consider the polynomial planar differential system (1.1). We will be working with the following functional associated with it in a natural way

$$E_0(x, y) = \int_0^1 \frac{1}{2}(P(x, y)y' - Q(x, y)x')^2 \, dt. \tag{3.1}$$

The functional E_0 is some kind of measure of how far a closed path

$$(x, y) = (x(t), y(t)), \quad x(0) = x(1), y(0) = y(1),$$

parameterized in $[0, 1]$, is from being a close path formed by orbits of system (1.1). It is clear that E_0 is smooth ($\mathcal{C}^\infty$), $E_0 \geq 0$, and $E_0 = 0$ for every limit cycle of the system (1.1) (re)parameterized in the interval $[0, 1]$ for normalization. Note that indeed $E_0(x, y) = 0$ for a closed path (x, y) formed by orbits of system (1.1), because

$$P(x, y)y' - Q(x, y)x' = 0.$$

There are several other possibilities for which $E_0(x, y)$ vanishes:

(i) (x, y) could be a constant path;
(ii) (x, y) could be a periodic orbit surrounding a center;
(iii) (x, y) could be a limit cycle run counterclockwise or clockwise, or even run several times in either orientation; or could be a reparameterization of a limit cycle, even with a different starting point for $t = 0$.
(iv) (x, y) could be formed by a singular point and a homoclinic orbit.
(v) Other possibilities.

The point is to realize that limit cycles of system (1.1) are definitely zeroes of the functional E_0. Thus, our aim will be accomplished if we can find an upper bound, depending on the degree of the system (1.1), of the zeroes of E_0.

Where may our bound for limit cycles come from? What might our driving idea be? The fundamental thought is to try to bound the number of global minima of E_0 (or of a suitable perturbation of it) through the number of its critical paths other than absolute minimizers, that is in the minima where $E_0 = 0$. This will be done through Morse inequalities. The matter is to organize the critical points of a functional, that needs to enjoy a number of important properties, in different classes in such a way that certain numerical identities and inequalities with the number of critical points in each class hold. Fundamental concepts like non-degenerate critical point, (Morse) index of such critical points, the Palais-Smale condition, etc, need to be discussed to better understand and appreciate the rigorous statement about Morse inequalities.

Note that the one-dimensional version of Morse inequalities is essentially the classical Rolle's theorem. Some readers may be familiar with the mountain-pass lemma which is a quite successful tool in non-linear PDEs, see for instance [8]. Morse inequalities are like a big, global mountain pass lemma. We state here the version of it that can be checked in [5]. We have found this version particularly helpful for our purposes in this work.

Theorem 7. *Let $E : \mathbf{H} \to \mathbb{R}$ be a $\mathcal{C}^2$-functional defined over a Hilbert space $\mathbf{H}$, which is bounded from below, coercive, enjoying the Palais-Smale property, and having a finite number of critical points, all of which are non-degenerate and of a finite index. Put M_k for the (finite) number of critical points of E for each fixed index k. Then*

$$M_0 \geq 1, \quad M_1 - M_0 \geq -1, \quad M_2 - M_1 + M_0 \geq 1, \quad \dots, \quad \sum_{k=0}^{\infty} (-1)^k M_k = 1. \tag{3.2}$$

Notice that under the assumed hypotheses, all sums involved in this statement are finite sums.

We therefore need a suitable functional E, defined over an appropriate Hilbert space $\mathbf{H}$, that needs to comply with a series of properties if it is to be eligible for Morse inequalities to be applied. This will be our first important step.

We will be using below the notation

$$\Sigma = \sum_{k=0}^{\infty} (-1)^k M_k, \quad \Sigma(S) = \sum_{k=0}^{\infty} (-1)^k M_k(S), \tag{3.3}$$

for a given subset $S \subset \mathbf{H}$, where $M_k(S)$ indicate the number of critical points of E of index k contained in S.

Morse inequalities are quite flexible. In particular, we would like to highlight the following interesting properties:

(1) Additivity. If E complies with the assumptions in the above theorem, and a and b are non-critical values of E with $a < b$, then

$$\Sigma(\{E \leq a\}) + \Sigma(\{a < E \leq b\}) = \Sigma(\{E \leq b\}),$$

where each Σ is given by (3.3) for the corresponding critical points in the given subset.

(2) Morse inequality in valleys. If a is a non-critical value, Morse inequalities are valid in each connected component of $\{E \le a\}$, informally called valleys of E, and so

$$\Sigma = 1 \text{ in each component of } \{E \le a\}.$$

(3) As a consequence of the previous items, and since it is unlikely that one could determine precisely the indexes of specific critical points, if a and b are non-critical with $a < b$, and $\{E \le b\}$ is connected,

$$\sharp\{E \le a\} \le 1 + \sum_{k=0}^{\infty} M_k, \quad M_k = M_k(\{a < E \le b\}),$$

where $\sharp\{E \le a\}$ stands for the number of connected components of $\{E \le a\}$. If we realize that the sum

$$\sum_{k=0}^{\infty} M_k(\{a < E \le b\})$$

is actually the total number $\mathcal{C}(\{a < E \le b\})$ of critical points of E in the set $\{a < E \le b\}$, the previous inequality becomes

$$\sharp\{E \le a\} \le 1 + \mathcal{C}(\{a < E \le b\}). \tag{3.4}$$

It is in this form (3.4) that we would like to make use of Morse inequalities. Note that all sums are finite under hypotheses implied in Theorem 7. Our intention is to identify every single limit cycle of our differential system (1.1) with a component of $\{E \le a\}$, for a suitable perturbation E of E_0 in (3.1), and some appropriately chosen a, while finding an upper bound $C(n)$, depending exclusively on the degree n, for the right-hand side of (3.4) for a suitable value b

$$\mathcal{C}(\{a < E \le b\}) \le C(n).$$

If we succeed in carrying out this task, and given that there is also a special component of the set $\{E \le a\}$ determined by constant paths, we will have our bound

$$H(n) \le C(n). \tag{3.5}$$

At any rate, according to Theorem 7, there is a number of crucial properties that E must comply with before we can even make use of inequality (3.4). In particular, the zero set of E (if it is to be non-negative) must be finite, but this is not true for E_0 in (3.1) as we have stated several times earlier. As a matter of fact, zeros of E_0 are not even isolated as indicated earlier. A number of important changes and steps are to be covered to reach our objective (3.5).

3.1. **Some changes and new difficulties.**

Our functional E_0 in (3.1) misses all of the requirements in Theorem 7. In the natural space $H^1([0,1];\mathbb{R}^2)$ of absolutely continuous paths with square-integrable, (weak) derivatives, where E_0 is defined, it is not even coercive. The initial solution we propose to this difficulty is canonical and consists in passing to a smaller Hilbert space of more regular paths, namely, $H^2([0,1];\mathbb{R}^2)$, whose paths have components with a weak derivatives up to order two which are square-integrable, and to modify our first version of the functional E_0 to the perturbation

$$E_\varepsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u}\|^2_{H^2([0,1];\mathbb{R}^2)}, \quad \mathbf{u} = (x, y).$$

We will be more explicit about these spaces in Section 4: inner products, norms, coercivity, basic properties. For each fixed, positive ε, this functional is coercive in $H^2([0,1];\mathbb{R}^2)$. We can make it comply with all of the requirements in Theorem 7, if we perturb it further to

$$E_\epsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u}\|^2_{H^2} + \langle \mathbf{v}_\epsilon, \mathbf{u}\rangle + \frac{1}{2\varepsilon}\|\mathbf{v}_\varepsilon\|^2, \tag{3.6}$$

where the path $\mathbf{v}_\varepsilon$ is to be chosen appropriately in order that E_ϵ precisely satisfies all of the necessary requirements of Theorem 7.

On the other hand, the space of admissible paths that we would like to consider need to be restricted as well. They must be 1-periodic

$$\mathbf{u}(0) = \mathbf{u}(1), \quad \mathbf{u}'(0) = \mathbf{u}'(1), \tag{3.7}$$

and have rotation index +1 to avoid redundant and useless multiplicity. Our ambient space $\mathbf{H}$ will thus be the subspace of $H^2([0,1];\mathbb{R}^2)$ complying with (3.7), and generated by those paths with rotation index +1. We will denote by $H^2_{O,+1}([0,1];\mathbb{R}^2)$ this Hilbert space, which is a subspace (with the same inner product) of $H^2([0,1];\mathbb{R}^2)$.

These unavoidable changes give rise to new difficulties. To begin with, a new small positive parameter ε enters into the discussion, and so everything depends on ε except the ambient space: the functional, the critical paths, the Morse indexes and the numbers M_k, etc. Absolute minimizers of E_0, in particular limit cycles of system (1.1), may no longer be critical paths for E_ε though somehow, for small ε, we expect them not to be far from being so. Moreover, functional E_ε is a singular perturbation of E_0. It is well-known that these problems are delicate, and require fine arguments in proofs.

3.2. **Steps to be covered.** We will be working with the functional E_ε in (3.6) regarded over the space

$$\mathbf{H} = H^2_{O,+1}([0,1];\mathbb{R}^2),$$

where ε can be chosen as small as may be convenient. We plan to apply Theorem 7 to this situation. Our first main step is then the following.

- Show that a path $\mathbf{v}_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$ can be chosen so that Theorem 7 can be applied to E_ε given by (3.6). This is the content of Sections 5 and 6.

It is important, according to the brief discussion after Theorem 7, to clarify the use we pretend to make of Morse inequalities, and how to organize the counting procedure. As indicated, we will accomplish this by considering the restriction and validity of Morse inequalities to sublevel sets of E_ε of the form $\{E_\varepsilon \le a\}$, where a could depend on ε.

- Discuss how Morse inequalities extend to subsets of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ of the form

  $$\{E_\varepsilon \le a_\varepsilon\}, \quad \{a_\varepsilon < E_\varepsilon \le b_\varepsilon\},$$

  for non-critical values (of E_ε) $a_\varepsilon < b_\varepsilon$. Show how these versions of Morse inequalities can be utilized to find that

$$\sharp\{E_\varepsilon \le a_\varepsilon\} \le 1 + \mathcal{C}\{a_\varepsilon < E_\varepsilon \le b_\varepsilon\}, \tag{3.8}$$

where the left-hand side is the number of connected components of $\{E_\varepsilon \leq a_\varepsilon\}$, and the right-hand side is the full set of critical paths contained in $\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}$. This discussion can be found in Section 4.1.

The next important step would be:

- Each limit cycle of (1.1) identifies, in a one-to-one fashion, a connected component of the sublevel set $\{E_\varepsilon \leq a_\varepsilon\}$ where $a_\varepsilon > 0$ is a suitable, non-critical value. Hence, each limit cycle counts as unity in the left-hand side of (3.8). Section 7 includes this objective.

Each such component is informally called a valley of E_ε, as we have already indicated earlier, and it may contain smaller valleys inside. There is an additional special component identified with constant paths. It can be ignored at the expense of transforming (3.8) into

$$H(n) \leq \mathcal{C}\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}. \tag{3.9}$$

The core of our estimate is the following.

- For a_ε and b_ε well chosen, (3.9) is valid, and one can show an upper bound

$$\mathcal{C}\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\} \leq C(n) \tag{3.10}$$

  uniformly for every ε sufficiently small, where $C(n)$ only depends on the degree of our differential polynomial system (1.1).

Once we have covered successfully all of our steps above, we would have our upper bound

$$H(n) \leq C(n). \tag{3.11}$$

3.3. **Main steps of the upper bound** (3.10)**.** We divide this last step, the upper bound (3.10), in various phases. For the counting procedure to be valid, it is necessary to restrict attention to a generic situation in which the connected components of the algebraic curve

$$P_x + Q_y = 0$$

are homeomorphic to either a straight line or an oval.

- Argue how small perturbations of the components $P(x, y)$ and $Q(x, y)$ in (1.1), without changing their degree, produce a similar differential system which is generic in the sense just indicated. Show that the upper bound (3.11) extends unchanged for every polynomial differential system of the same degree, provided it is true for such generic differential systems. This argument is provided in Section 12.

As a consequence of this fact, we can restrict attention to such generic situations to find the upper bound (3.10) depending only on the degree n.

Since we aim at counting all of critical paths in a set of the form $\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}$, we need to understand the defining features of critical paths. This requires to:

- Examine carefully the Euler-Lagrange system of optimality associated with E_ε over our ambient space $H^2_{O,+1}([0,1];\mathbb{R}^2)$. A full analysis of this can be checked in Section 8.

The counting procedure itself proceeds in two main stages:

- Classify and count all possible different asymptotic behaviors of branches of critical paths of E_ε as ε tends to zero (Section 9). The sum $N+M$ in the statement of Theorem 2 provides an upper bound for the number of such asymptotic behaviors.
- For each possible behavior in the previous item, determine how many branches can possibly share the same asymptotic limit. We show that there cannot be more than n^2 of such branches where n is the degree of system (1.1) (Section 10).

Section 11 pretends to summarize all of our previous conclusions to facilitate the counting procedure itself. It also explores how to select the values of a_ε and b_ε so that all of our arguments above are correct.

4. Some analytical preliminaries

We briefly state here some basic notions about spaces of functions with weak derivatives having suitable integrability properties, as well as recalling concepts like the coercivity of a functional. It may be convenient to do so for some interested readers not familiar with these concepts. We refer to [9] for a main, accesible source in this regard, and much more related information.

The underlying natural Hilbert space for E_ε is

$$\mathbf{H} = \{(x,y) : [0,1] \to \mathbb{R}^2 : \int_0^1 [x^2 + y^2 + (x')^2 + (y')^2 + (x'')^2 + (y'')^2]\, dt < \infty\}.$$

This is nothing but the classical Sobolev space $H^2([0,1];\mathbb{R}^2)$ of paths with square-integrable weak derivatives up to order two. The inner product in this space is

$$\langle (x_1,y_1),(x_2,y_2)\rangle = \int_0^1 (x_1x_2 + y_1y_2 + x_1'x_2' + y_1'y_2' + x_1''x_2'' + y_1''y_2'')\, dt,$$

and the associated norm

$$\|(x,y)\|^2 = \int_0^1 [x^2 + y^2 + (x')^2 + (y')^2 + (x'')^2 + (y'')^2]\, dt.$$

Norms and inner products occurring henceforth are meant to be these. Paths in $\mathbf{H}$ have continuous first derivatives. Since limit cycles are $\mathcal{C}^\infty$, they belong to this space.

Coercivity for a general functional E defined in a Hilbert space $\mathbf{H}$ means that

$$E(\mathbf{u}) \to +\infty \quad \text{as} \quad \|\mathbf{u}\| \to \infty \text{ with } \mathbf{u} \in \mathbf{H}.$$

If a base functional E_0 defined in $\mathbf{H}$ is non-negative, the perturbation

$$E_\varepsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u} - \mathbf{u}_\varepsilon\|^2$$

automatically becomes coercive for every positive ε, and every fixed element $\mathbf{u}_\varepsilon$.

To summarize our analytical framework, we will concentrate on the functional

$$E_\varepsilon : \mathbf{H} \equiv H^2_{O,+1}([0,1];\mathbb{R}^2) \to \mathbb{R}^+$$

where this space $\mathbf{H}$ is the subspace of $H^2_O([0,1];\mathbb{R}^2)$ generated by paths having rotation index $+1$, and

$$H^2_O([0,1];\mathbb{R}^2) = \{\mathbf{u} \in H^2([0,1];\mathbb{R}^2) : \mathbf{u}(0) = \mathbf{u}(1), \mathbf{u}'(0) = \mathbf{u}'(1)\},$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_0^1 (\mathbf{u}(t)\cdot\mathbf{v}(t) + \mathbf{u}'(t)\cdot\mathbf{v}'(t) + \mathbf{u}''(t)\cdot\mathbf{v}''(t))\,dt,$$

$$\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2_{H^2([0,1];\mathbb{R}^2)} = \int_0^1 \left(|\mathbf{u}''(t)|^2 + |\mathbf{u}'(t)|^2 + |\mathbf{u}(t)|^2\right) dt,$$

$$E_0(\mathbf{u}) = \frac{1}{2}\int_0^1 (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')^2\,dt,$$

$$E_\varepsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v}_\varepsilon\rangle + \frac{1}{2\varepsilon}\|\mathbf{v}_\varepsilon\|^2 = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u} + \frac{1}{\varepsilon}\mathbf{v}_\varepsilon\|^2,$$

and

$$\mathbf{u} = (x,y), \quad \mathbf{F}(\mathbf{u}) = (P(x,y), Q(x,y)),$$
$$\mathbf{F}^\perp(\mathbf{u}) = (-Q(\mathbf{u}), P(\mathbf{u})), \quad \mathbf{v}_\varepsilon = (X_\varepsilon, Y_\varepsilon).$$

A fundamental fact for us to bear in mind is that convergence in $H^2([0,1];\mathbb{R}^2)$ implies uniform convergence of first derivatives ([9]).

4.1. **Morse inequalities.** The discussion in this subsection is well-known to experts in Morse theory. Since most likely many interested readers will not be familiar with this material, we have made an special effort in explaining facts in the most transparent and intuitive way that we have found with precise references to formal sources. For proofs of main results that we are about to state, we refer to our two principal references [5], [10]. Simple proofs of some specific consequences of more general results that we will be employing are indicated.

The statement of Morse inequalities, which is our main basic tool, involves the notion of Morse index for a non-degenerate critical point of a smooth functional. Suppose

$$E : \mathbf{H} \to \mathbb{R}$$

is a non-negative, coercive, $\mathcal{C}^2$-functional defined over a Hilbert space $\mathbf{H}$. Let $\mathbf{u} \in \mathbf{H}$ be a critical point of E, i.e. $E'(\mathbf{u}) = \mathbf{0}$. A real number $c \in \mathbb{R}$ is a critical value of E if there is a critical point $\mathbf{u}$, $E'(\mathbf{u}) = \mathbf{0}$, such that $c = E(\mathbf{u})$.

Definition 8. *When the number of negative eigenvalues of $E''(\mathbf{u})$ is finite, such number is called the (Morse) index of $\mathbf{u}$.*

Morse inequalities can be found in several places, for instance, Corollary (6.5.10) of [5] or Theorem 4.3 of Chapter 1 in [10]. A main, indispensable condition for these inequalities to hold is the *Palais-Smale condition* (see Section 5.2). For a general, smooth $\mathcal{C}^1$- functional

$$E : \mathbf{H} \to \mathbb{R},$$

this important compactness property reads:

> If for a sequence $\{\mathbf{u}_j\}$ we have that $E(\mathbf{u}_j) \leq K$ for all j and a fixed positive constant K, and $E'(\mathbf{u}_j) \to \mathbf{0}$ as $j \to \infty$, then a certain subsequence of $\{\mathbf{u}_j\}$ converges (strongly) in $\mathbf{H}$.

If E is coercive, we can replace the boundedness of E along the sequence $\{\mathbf{u}_j\}$ by the uniform boundedness of $\{\mathbf{u}_j\}$ in $\mathbf{H}$. We recall that Theorem 7 is the version of Morse inequalities in [5] (Corollary (6.5.10) as indicated above).

We will show several special situations in which this fundamental result can be used. The first one focuses on the validity of the same inequalities for "valleys" of E.

Definition 9. *Let*

$$E : \mathbf{H} \to \mathbb{R}$$

be a $\mathcal{C}^2$-functional. A bounded, connected and with $\mathcal{C}^1$-boundary subset $H \subset \mathbf{H}$ is a valley for E, if

$$\langle E'(\mathbf{u}), \nu(\mathbf{u}) \rangle > 0$$

for every $\mathbf{u}$ in the boundary ∂H of H, where $\nu(\mathbf{u})$ is the outer normal vector to H.

Note that if a is a non-critical value of E, then every connected component of the level set $\{E \leq a\}$ is a valley. This observation makes the following definition suitable for our purposes. Note that

$$\{E \leq a\} = \{\mathbf{u} \in \mathbf{H} : E(\mathbf{u}) \leq a\}.$$

Definition 10. *A functional*

$$E : \mathbf{H} \to \mathbb{R}$$

is called a Morse functional if it is $\mathcal{C}^2$-, non-negative, coercive, enjoys the Palais-Smale property, and has a finite number of critical points over each level set $\{E \leq a\}$ for each non-critical value a, all of which are non-degenerate and with a finite index.

Morse theory is concerned about Morse functionals. Indeed, Corollary (6.5.11) in [5] reads as follows.

Proposition 11. *Let*

$$E : \mathbf{H} \to \mathbb{R}$$

be a Morse functional. Let H be a valley of E (in particular a connected component of the level set $\{E \leq a\}$ for a non-critical value a). Then Morse inequalities (3.2) *are valid restricted to critical points of E in H.*

This result shows that Morse inequalities are valid restricted to every valley of E. As just indicated, Proposition 11 is proved in [5] for balls, but the proof is exactly the same for arbitrary bounded, connected domains H, as the sentence before the statement of this corollary in [5] asserts.

Another special situation where Morse inequalities can be used refers to its extension to infinite dimensional manifolds modeled over Hilbert spaces. Such generalization is treated in a formal way in both of our basic references [5] and [10]. All of the main concepts necessary to state and prove them in this more general context (including the Palais-Smale condition) are extended in a natural way. Theorem 7 is also valid in the context of such a manifold, though the statement of Morse inequalities in this case involves Betti numbers of the manifold, and it becomes more technical. Fortunately, we do not need to examine these more complex situations.

There are many more general results on the validity of Morse inequalities restricted to subsets of $\mathbf{H}$ other than the one in Proposition 11 for valleys. See some of these in Section 6.1 in page 55 of [10].

5. The functional E_ε. Finiteness of the number of critical closed paths over finite level sets

We discuss again, now in greater detail, the analytical scenario in which we will be working. As indicated above, our first objective is to specify the underlying ambient space, and the form and properties of the perturbation E_ε of our basic functional E_0 given by

$$E_0(\mathbf{u}) = \frac{1}{2}\int_0^1 (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')^2\,dt,$$

in such a way that E_ε turns out to be a Morse functional (Definition 10 below), and so it will be eligible for applying all results in Section 4.1 below. This essentially amounts to checking assumptions in Theorem 7 stated in Section 3.

The natural initial space for our analysis is the Hilbert space $H^2_O([0,1];\mathbb{R}^2)$ of 1-periodic paths with square integrable (weak) derivatives up to order two. However, it is evident that there is a useless multiplicity for each element of this space as regards its image set in $\mathbb{R}^2$. Each path in $H^2_O([0,1];\mathbb{R}^2)$ admits infinitely many reparameterizations: many of them can be deformed continuously from each other without changing the image set of the path; but there are others which cannot be deformed continuously into each other without changing the image set. Think about the possibility of running each path in $H^2_O([0,1];\mathbb{R}^2)$ several times, either counter- or clockwise. It will be very convenient for us to avoid such fruitless multiplicity. After all, for our counting procedure of limit cycles, we would like to identify each one of them through a single element of our ambient space. Our intention is to specify a certain subspace of $H^2_O([0,1];\mathbb{R}^2)$ which will, essentially, contain a unique representative of each limit cycle.

Let $\mathcal{H}_{+1}$ stand for the subset of $H^2_O([0,1];\mathbb{R}^2)$ incorporating all paths which are one-to-one (no self-intersections), and have rotation index $+1$. Recall that, by the classical theorem of H. Hopf [25], the rotation index of such a smooth curve (in particular paths in $H^2_O([0,1];\mathbb{R}^2)$ are) is well-defined and either $+1$ or -1, depending on whether they are oriented counter- or clock-wise.

Definition 12. *We will designate by*

$$\mathbf{H} \equiv H^2_{O,+1}([0,1];\mathbb{R}^2) \tag{5.1}$$

the Hilbert subspace of $H^2_O([0,1];\mathbb{R}^2)$ *generated by* $\mathcal{H}_{+1}$.

Paths in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ are limits (in the norm of the Hilbert space $H^2([0,1];\mathbb{R}^2)$, and uniformly in $\mathcal{C}^1$) of paths in $\mathcal{H}_{+1}$. In particular, elements of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ need not be one-to-one (constant paths belong to $H^2_{O,+1}([0,1];\mathbb{R}^2)$, for instance). On the other hand, paths for which the rotation index is well-defined and is different from $+1$ do not belong to $H^2_{O,+1}([0,1];\mathbb{R}^2)$. There is still the ambiguity of preserving the image set of a path in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ with infinitely many reparameterizations, but these will not pose a particular difficulty as they can be deformed continuously into each other without changing an essential ingredient of our functional as we will see below.

The next important result clearly specifies the form of our perturbations E_ε and its main properties as concerns the possibility of using Morse inequalities. Its proof is the goal of this and the next sections.

Theorem 13. *For every positive ε, there is a $\mathcal{C}^\infty$-, one-to-one path $\mathbf{v}_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$ such that $\|\mathbf{v}_\varepsilon\| \leq K\varepsilon$, with K independent of ε, positive and sufficiently small (in particular, $\|\mathbf{v}_\varepsilon\| \to 0$ as $\varepsilon \searrow 0$), and the perturbed functional*

$$E_\varepsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v}_\varepsilon \rangle + \frac{1}{2\varepsilon}\|\mathbf{v}_\varepsilon\|^2 \tag{5.2}$$

is non-negative, coercive, $\mathcal{C}^2$, complies with the Palais-Smale condition, and has a finite number (possibly depending on ε and α) of non-degenerate critical closed paths in every finite level set of the form $\{E_\varepsilon \leq \alpha\}$ for arbitrary non-critical value α.

Theorem 13 will be proved later, after Lemma 19 below.

Note that the terms added to E_0 are such that

$$\frac{\varepsilon}{2}\|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v}_\varepsilon \rangle + \frac{1}{2\varepsilon}\|\mathbf{v}_\varepsilon\|^2 = \frac{\varepsilon}{2}\|\mathbf{u} + \frac{1}{\varepsilon}\mathbf{v}_\varepsilon\|^2.$$

For proving this main result, we need some preliminary abstract definitions and facts, which we state next for the sake of completeness, most of which can be found in the book of Berger [5], among others.

Suppose that

$$E : \mathbf{H} \to \mathbb{R}$$

is a smooth $\mathcal{C}^2$-functional defined in a Hilbert space $\mathbf{H}$. We shall use the following definitions.

(i) A critical point $\mathbf{x} \in \mathbf{H}$ of E is called *non-degenerate* is the self-adjoint operator

$$E''(\mathbf{x}) : \mathbf{H} \to \mathbf{H}$$

is invertible. Otherwise, $\mathbf{x}$ is said to be *degenerate.*

(ii) An element $\mathbf{x} \in \mathbf{H}$ is called a *regular* point for a non-linear $\mathcal{C}^1$-operator

$$\mathbf{F} : \mathbf{H} \to \mathbf{H}$$

if the linear operator

$$\mathbf{F}'(\mathbf{x}) : \mathbf{H} \to \mathbf{H}$$

is surjective. Otherwise, $\mathbf{x}$ is called a *singular* point for $\mathbf{F}$. When $\mathbf{F}$ is the derivative of a $\mathcal{C}^2$-functional $E : \mathbf{H} \to \mathbb{R}$, then a critical point $\mathbf{x}$ for E is degenerate (respectively, non-degenerate) if it is singular (respectively, regular) for $\mathbf{F} = E'$. Note that in this case $\mathbf{F}' = E''$ is a self-adjoint operator, and so it is surjective if and only if it is bijective, see Section 2.7 in [9], for instance. The image $\mathbf{F}(\mathbf{x})$ of a singular point $\mathbf{x}$ is called a singular value of $\mathbf{F}$.

(iii) A mapping

$$\mathbf{F} : \mathbf{H} \to \mathbf{H}$$

is a *non-linear Fredholm operator* if its Fréchet derivative

$$\mathbf{F}'(\mathbf{x}) : \mathbf{H} \to \mathbf{H}$$

is a linear Fredholm map for each $\mathbf{x} \in \mathbf{H}$. The *index* of $\mathbf{F}$ is defined to be the difference of the dimensions of the kernel and the cokernel of $\mathbf{F}'(\mathbf{x})$. This index is independent of $\mathbf{x}$.

(iv) The functional E is a *Fredholm functional* if

$$E' : \mathbf{H} \to \mathbf{H}$$

is a Fredholm mapping, i.e. if

$$E''(\mathbf{x}) : \mathbf{H} \to \mathbf{H}$$

is a linear Fredholm map for each $\mathbf{x} \in \mathbf{H}$.

We state several interesting facts (page 100 in [5]).

Proposition 14. *The following statements hold.*

a) *Any diffeomorphism between Banach spaces is a Fredholm map of index zero.*
b) *If $\mathbf{F}$ is a Fredholm map, and $\mathbf{G}$ is a compact operator, then the sum $\mathbf{F} + \mathbf{G}$ is also Fredholm with the same index as $\mathbf{F}$.*

We recall two additional classic results. The first one is the Inverse Function Theorem (page 113, [5]) for Banach spaces.

Theorem 15. *Let $\mathbb{F}$ be a $\mathcal{C}^1$-mapping defined in a neighborhood of some point $\overline{\mathbf{x}}$ of a Banach space $\mathbf{X}$, with range in a Banach space $\mathbf{Y}$. If $\mathbb{F}'(\overline{\mathbf{x}})$ is a linear homeomorphism of $\mathbf{X}$ onto $\mathbf{Y}$, then $\mathbb{F}$ is a local homeomorphism of a neighborhood $\mathbf{U}(\overline{\mathbf{x}})$ of $\overline{\mathbf{x}}$ to a neighborhood of $\mathbb{F}(\overline{\mathbf{x}})$.*

The second one is a version of Sard's theorem for infinite-dimensional spaces (page 125 of [5]).

Theorem 16. *Let $\mathbf{F}$ be a $\mathcal{C}^q$-Fredholm mapping of a separable Banach space $\mathbf{X}$ into a separable Banach space $\mathbf{Y}$. If $q > \max(\text{index } \mathbf{F}, 0)$, the set of singular values of $\mathbf{F}$ are nowhere dense (its closure has empty interior) in $\mathbf{Y}$.*

The proof of Theorem 13 will make use of Proposition 14, and Theorems 15 and 16. The use of these general results requires the compactness of E_0' as a main ingredient. Moreover, we will need to show the Palais-Smale condition for E_ε. We treat these two issues in the next two subsections.

5.1. **Compactness of E_0'.** For our functional E_0, it is easy to find an expression for

$$\langle E_0'(\mathbf{u}), \mathbf{U} \rangle, \quad \mathbf{u}, \mathbf{U} \in H^2_{O,+1}([0,1]; \mathbb{R}^2).$$

Indeed, by definition we have

$$\langle E_0'(\mathbf{u}), \mathbf{U} \rangle = \frac{d}{d\tau} E_0(\mathbf{u} + \tau \mathbf{U}) \bigg|_{\tau=0}. \tag{5.3}$$

Since

$$E_0(\mathbf{u} + \tau \mathbf{U}) = \frac{1}{2} \int_0^1 [\mathbf{F}^\perp(\mathbf{u} + \tau \mathbf{U}) \cdot (\mathbf{u}' + \tau \mathbf{U}')]^2 \, dt,$$

then from (5.3) we have

$$\langle E_0'(\mathbf{u}), \mathbf{U} \rangle = \int_0^1 (\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}')[(D\mathbf{F}^\perp(\mathbf{u})\mathbf{U}) \cdot \mathbf{u}' + \mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{U}'] \, dt. \tag{5.4}$$

We are, therefore, seeking an element

$$\mathbf{v}(=E_0'(\mathbf{u}))\in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

such that

$$\mathbf{v}\cdot\mathbf{U}=\langle E_0'(\mathbf{u}),\mathbf{U}\rangle,$$

that is to say

$$\int_0^1(\mathbf{v}\cdot\mathbf{U}+\mathbf{v}'\cdot\mathbf{U}'+\mathbf{v}''\cdot\mathbf{U}'')\,dt=\int_0^1(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')[(D\mathbf{F}^\perp(\mathbf{u})\mathbf{U})\cdot\mathbf{u}'+\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{U}']\,dt \tag{5.5}$$

for every

$$\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

There is a unique such $\mathbf{v}$, which turns out to be the minimizer (with respect to

$$\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2))$$

of the augmented functional

$$\int_0^1[\frac{1}{2}|\mathbf{U}''|^2+\frac{1}{2}|\mathbf{U}'|^2+\frac{1}{2}|\mathbf{U}|^2-(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')[(D\mathbf{F}^\perp(\mathbf{u})\mathbf{U})\cdot\mathbf{u}'+\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{U}']]\,dt. \tag{5.6}$$

The existence of a unique minimizer for this problem, which is quadratic, is a direct consequence of the classical Lax-Milgram Theorem (see Corollary 5.8 of [9] for instance).

Therefore the equation for

$$\mathbf{v}=E_0'(\mathbf{u})\in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

will be the associated Euler-Lagrange system for the functional (5.6) as it is given by this last theorem

$$[\mathbf{v}'']''-[\mathbf{v}'+(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')\mathbf{F}^\perp(\mathbf{u})]'+\mathbf{v}+(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')\mathbf{u}'^TD\mathbf{F}^\perp(\mathbf{u})=\mathbf{0}\text{ in }(0,1). \tag{5.7}$$

Its weak formulation is exactly (5.5).

Lemma 17. *Let $\{\mathbf{u}_j\}$ be a uniformly bounded sequence in $H^2_{O,+1}([0,1];\mathbb{R}^2)$, and $\{\mathbf{v}_j\}$ the sequence of derivatives*

$$\mathbf{v}_j=E_0'(\mathbf{u}_j)\in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

which are solutions of (5.7) for $\mathbf{u}=\mathbf{u}_j$. Then $\{\mathbf{v}_j\}$ is relatively compact in $H^2_{O,+1}([0,1];\mathbb{R}^2)$.

Proof. For the sake of brevity, set

$$\mathbf{G}_j\equiv(\mathbf{F}^\perp(\mathbf{u}_j)\cdot\mathbf{u}_j')\mathbf{F}^\perp(\mathbf{u}_j),\quad \mathbf{H}_j\equiv(\mathbf{F}^\perp(\mathbf{u}_j)\cdot\mathbf{u}_j')\mathbf{u}_j'^TD\mathbf{F}^\perp(\mathbf{u}_j).$$

If $\{\mathbf{u}_j\}$ is uniformly bounded in $H^2_{O,+1}([0,1];\mathbb{R}^2)$, we know that a certain subsequence (not relabelled) of $\{\mathbf{u}_j\}$ converges weakly to some $\mathbf{u}$ in $H^2([0,1];\mathbb{R}^2)$. Set

$$\mathbf{G}\equiv(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')\mathbf{F}^\perp(\mathbf{u}),\quad \mathbf{H}\equiv(\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')\mathbf{u}'^TD\mathbf{F}^\perp(\mathbf{u}).$$

If we put

$$\mathbf{v}_j=E_0'(\mathbf{u}_j),\quad \mathbf{v}=E_0'(\mathbf{u}),$$

then (5.5) implies

$$\int_0^1[\mathbf{v}_j''\cdot\mathbf{U}''+(\mathbf{v}_j'+\mathbf{G}_j)\cdot\mathbf{U}'+(\mathbf{v}_j+\mathbf{H}_j)\cdot\mathbf{U}]\,dt=0,$$

$$\int_0^1[\mathbf{v}''\cdot\mathbf{U}''+(\mathbf{v}'+\mathbf{G})\cdot\mathbf{U}'+(\mathbf{v}+\mathbf{H})\cdot\mathbf{U}]\,dt=0,$$

for every

$$\mathbf{U} \in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

By substracting one from the other

$$\int_0^1 [(\mathbf{v}_j'' - \mathbf{v}'') \cdot \mathbf{V}'' + (\mathbf{v}_j' - \mathbf{v}' + \mathbf{G}_j - \mathbf{G}) \cdot \mathbf{V}' + (\mathbf{v}_j - \mathbf{v} + \mathbf{H}_j - \mathbf{H}) \cdot \mathbf{V}]\, dt = 0$$

for every

$$\mathbf{U} \in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

We can take $\mathbf{U} = \mathbf{v}_j - \mathbf{v}$ to find that

$$\int_0^1 [|\mathbf{v}_j'' - \mathbf{v}''|^2 + (\mathbf{v}_j' - \mathbf{v}' + \mathbf{G}_j - \mathbf{G}) \cdot (\mathbf{v}_j' - \mathbf{v}') + (\mathbf{v}_j - \mathbf{v} + \mathbf{H}_j - \mathbf{H}) \cdot (\mathbf{v}_j - \mathbf{v})]\, dt = 0.$$

This equality can be reorganized as

$$\|\mathbf{v}_j - \mathbf{v}\|^2_{H^2([0,1];\mathbb{R}^2)} = -\int_0^1 [(\mathbf{G}_j - \mathbf{G}) \cdot (\mathbf{v}_j' - \mathbf{v}') + (\mathbf{H}_j - \mathbf{H}) \cdot (\mathbf{v}_j - \mathbf{v})]\, dt,$$

Hence, by the standard Hölder inequality for integrals, we can also have

$$\|\mathbf{v}_j - \mathbf{v}\|_{H^2([0,1];\mathbb{R}^2)} \leq \|\mathbf{G}_j - \mathbf{G}\|_{L^2([0,1];\mathbb{R}^2)} + \|\mathbf{H}_j - \mathbf{H}\|_{L^2([0,1];\mathbb{R}^2)}. \tag{5.8}$$

Since the weak convergence of $\mathbf{u}_j$ to $\mathbf{u}$ in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ implies weak convergence up to second derivatives, by the classical Rellich–Kondrachov Theorem, which implies that the injection $W^{2,p} \subset W^{1,p}$ is always compact (see Theorem 9.16 in [9] for the case with first derivatives $W^{1,p} \subset L^p$), we conclude the convergences

$$\mathbf{G}_j \to \mathbf{G}, \quad \mathbf{H}_j \to \mathbf{H}$$

strongly in $L^2([0,1];\mathbb{R}^2)$. The proof is then a direct consequence of (5.8). □

As a direct consequence of Lemma 17, we have:

Corollary 18. *The map*

$$E_0' : H^2_{O,+1}([0,1];\mathbb{R}^2) \to H^2_{O,+1}([0,1];\mathbb{R}^2)$$

is compact.

This compactness property is the only reason why the functional E_0 has to be perturbed by a norm involving up to second derivatives. If we had just perturbed E_0 up to first derivatives, we would not have the strong convergence of the vector fields $\mathbf{G}_j$ and $\mathbf{H}_j$ to $\mathbf{G}$ and $\mathbf{H}$, respectively, in the proof of Lemma 17.

5.2. **Palais-Smale property for E_ε.** We need to show that our perturbation E_ε in (5.2) complies with the Palais-Smale property: if $\{\mathbf{u}_j\}$ is a sequence in $\mathbf{H}$ such that

$$E_\varepsilon(\mathbf{u}_j) \text{ is bounded}, \quad E_\varepsilon'(\mathbf{u}_j) \to \mathbf{0},$$

then some subsequence of $\{\mathbf{u}_j\}$ converges in $\mathbf{H}$. Note that, since E_ε is coercive in $\mathbf{H}$ given in (5.1), we can replace the boundedness of E_ε along the sequence $\{\mathbf{u}_j\}$ by the uniform boundedness of $\{\mathbf{u}_j\}$ in $\mathbf{H}$.

Lemma 19. *For each positive ε fixed, the functional E_ε is bounded from below, coercive, and enjoys the Palais-Smale property.*

Proof. Note first that the identity

$$\frac{\varepsilon}{2}\|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v}_\varepsilon\rangle = \frac{\varepsilon}{2}\|\mathbf{u} + \frac{1}{\varepsilon}\mathbf{v}_\varepsilon\|^2 - \frac{1}{2\varepsilon}\|\mathbf{v}_\varepsilon\|^2$$

together with the fact that $E_0 \geq 0$, implies the coercivity of each E_ε on $\mathbf{u}$. On the other hand,

$$E'_\varepsilon = E'_0 + \varepsilon\mathbf{1} + \mathbf{v}_\varepsilon, \tag{5.9}$$

where $\mathbf{1} : \mathbf{H} \to \mathbf{H}$ is the identity operator.

Suppose $\{\mathbf{u}_j\}$ is uniformly bounded. Since E'_0 is compact (Corollary 18), there is a subsequence $\mathbf{u}_j$ (not relabelled) such that

$$E'_0(\mathbf{u}_j) \to \overline{\mathbf{u}}, \quad \overline{\mathbf{u}} \in \mathbf{H}.$$

Under the Palais-Smale conditions, if $E'_\varepsilon(\mathbf{u}_j) \to 0$, due to (5.9), we have

$$\varepsilon\mathbf{u}_j = E'_\varepsilon(\mathbf{u}_j) - E'_0(\mathbf{u}_j) - \mathbf{v}_\varepsilon \to -\overline{\mathbf{u}} - \mathbf{v}_\varepsilon \text{ as } j \to \infty.$$

Hence $\{\mathbf{u}_j\}$ converges strongly in $\mathbf{H}$. This is exactly the Palais-Smale property for each E_ε. □

5.3. **Main proof.** We are now in a position to prove Theorem 13. Recall that, except for an harmless additive constant,

$$E_\varepsilon(\mathbf{u}) = \int_0^1 \Big[\frac{1}{2}(\mathbf{F}^\perp(\mathbf{u}(t)) \cdot \mathbf{u}'(t))^2 + \frac{\varepsilon}{2}(|\mathbf{u}''(t)|^2 + |\mathbf{u}'(t)|^2 + |\mathbf{u}(t)|^2)$$
$$+(\mathbf{u}(t) \cdot \mathbf{v}_\varepsilon(t) + \mathbf{u}'(t) \cdot \mathbf{v}'_\varepsilon(t) + \mathbf{u}''(t) \cdot \mathbf{v}''_\varepsilon(t))\Big]\, dt.$$

As a matter of fact, the proof of Theorem 13 is essentially described in abstract terms in page 355 of [5]. For the sake of completeness we prove it using all the preliminary results in this section.

Proof of Theorem 13. Consider the functional

$$\tilde{E}_\varepsilon : H^2_{O,+1}([0,1]; \mathbb{R}^2) \to \mathbb{R}$$

given by

$$\tilde{E}_\varepsilon(\mathbf{u}) = E_0(\mathbf{u}) + \frac{\varepsilon}{2}\|\mathbf{u}\|^2.$$

Its derivative

$$\mathbb{F}(\mathbf{u}) = \tilde{E}'_\varepsilon(\mathbf{u}) = \varepsilon\mathbf{u} + E'_0(\mathbf{u})$$

is the sum of a diffeomorphism, $\varepsilon\mathbf{1}$, and a compact operator, E'_0. By Proposition 14, this derivative is a Fredholm operator of index zero. By Theorem 16, the set of critical values of the derivative $\mathbb{F}$, that is

$$\{\mathbb{F}(\mathbf{u}) \in H^2_{O,+1}([0,1]; \mathbb{R}^2) : \mathbb{F}'(\mathbf{u}) = \mathbf{0}\},$$

is no where dense, and consequently, we can choose an element

$$\mathbf{v}_\varepsilon \in H^2_{O,+1}([0,1]; \mathbb{R}^2),$$

with the properties claimed in the statement of the theorem, so that every solution $\mathbf{u}$ of the equation

$$\mathbb{F}(\mathbf{u}) + \mathbf{v}_\varepsilon = \mathbf{0}$$

is not a singular point for $\mathbb{F}$, i.e.

$$\mathbb{F}'(\mathbf{u}) = \tilde{E}''_0(\mathbf{u})$$

is bijective, and so $\mathbf{u}$ is non-degenerate. This argument implies indeed that the critical closed paths of E_ε are non-degenerate, once $\mathbf{v}_\varepsilon$ has been chosen in this way and has been added to $\tilde{E}_\varepsilon$ because

$$E''_\varepsilon(\mathbf{u}) = \tilde{E}''_\varepsilon(\mathbf{u}).$$

The Inverse Function Theorem 15 implies directly that non-degenerate critical closed paths of a $\mathcal{C}^2$-functional, E_ε, are isolated.

Finally, we argue why the number of critical paths in sets of the form $\{E_\varepsilon \leq \alpha\}$ is finite. Indeed, if we let α be a positive real number and assume that there is an infinite number $\{\mathbf{u}_j\}$ of critical closed paths with

$$E_\varepsilon(\mathbf{u}_j) \leq \alpha, \quad E'_\varepsilon(\mathbf{u}_j) = \mathbf{0},$$

the Palais-Smale condition for E_ε would ensure the existence of a suitable subsequence converging to some $\overline{\mathbf{u}}$ which would be a critical, non-isolated path. This is a contradiction with the previous statement about the fact that the critical closed paths are isolated, and so the number of such critical closed paths has to be finite. This completes the proof of Theorem 13 □

6. Finiteness of Morse indexes for critical points of E_ε

Another main issue to apply Morse inequalities demands to have finite Morse indexes for every critical, non-degenerate path of E_ε. To prove this is the main goal of this section.

We need to focus on the Hessian $E''_\varepsilon(\mathbf{u})$ at a critical path $\mathbf{u}$, $E'_\varepsilon(\mathbf{u}) = \mathbf{0}$, and show that it has a finite number of negative eigenvalues that could depend of $\mathbf{u}$ and on ε. E_ε has a linear part, which drops out in the Hessian, a quadratic part coming from the norm in $H^2([0,1];\mathbb{R}^2)$, and a non-linear, lower-order term. Indeed

$$E''_\varepsilon(\mathbf{u}) = E''_0(\mathbf{u}) + \varepsilon\mathbf{1}. \tag{6.1}$$

Once again, the main fact in which we can support our proof is the compactness of the linear operator

$$E''_0(\mathbf{u}) : H^2_{O,+1}([0,1];\mathbb{R}^2) \to H^2_{O,+1}([0,1];\mathbb{R}^2)$$

for each fixed $\mathbf{u}$.

If, in general, we have a certain smooth, $\mathcal{C}^2$-functional $E : \mathbf{H} \to \mathbb{R}$ over a Hilbert space $\mathbf{H}$ with derivative $E' : \mathbf{H} \to \mathbf{H}$, there are various ways to deal with the second derivative, but probably the best suited for our purposes is to consider the derivative

$$\langle E''(\mathbf{u}), (\mathbf{U}, \overline{\mathbf{U}})\rangle = \left.\frac{d}{d\delta}\right|_{\delta=0} \langle E'(\mathbf{u} + \delta\mathbf{U}), \overline{\mathbf{U}}\rangle,$$

where both vector fields $\mathbf{U}$ and $\overline{\mathbf{U}}$ belong to $\mathbf{H}$. In our situation, and in view of (5.4), we have

$$\begin{aligned}\langle E_0''(\mathbf{u}), (\mathbf{U}, \overline{\mathbf{U}})\rangle = & \left.\frac{d}{d\delta}\right|_{\delta=0} \int_0^1 (\mathbf{F}^\perp(\mathbf{u}+\delta\mathbf{U})\cdot(\mathbf{u}'+\delta\mathbf{U}'))[\nabla\mathbf{F}^\perp(\mathbf{u}+\delta\mathbf{U})\overline{\mathbf{U}}\cdot(\mathbf{u}'+\delta\mathbf{U}') \\ & \qquad +\mathbf{F}^\perp(\mathbf{u}+\delta\mathbf{U})\cdot\overline{\mathbf{U}}'] \, dt \\ = & \int_0^1 (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')[\nabla^2\mathbf{F}^\perp(\mathbf{u}):(\mathbf{U},\overline{\mathbf{U}},\mathbf{u}')+\nabla\mathbf{F}^\perp(\mathbf{u}):(\overline{\mathbf{U}},\mathbf{U}') \\ & \qquad +\nabla\mathbf{F}^\perp(\mathbf{u}):(\mathbf{U},\overline{\mathbf{U}}')] \, dt \\ & + \int_0^1 [\nabla\mathbf{F}^\perp(\mathbf{u})\mathbf{U}\cdot\mathbf{u}'+\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{U}'][\nabla\mathbf{F}^\perp(\mathbf{u})\overline{\mathbf{U}}\cdot\mathbf{u}'+\mathbf{F}^\perp(\mathbf{u})\cdot\overline{\mathbf{U}}'] \, dt\end{aligned}$$

Through this long formula, we can understand, for such a $\mathbf{u}$ given and fixed, the linear operator

$$E_0''(\mathbf{u}) : H^2_{O,+1}([0,1];\mathbb{R}^2) \to H^2_{O,+1}([0,1];\mathbb{R}^2).$$

Let

$$\mathbf{U} \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

be given. The image

$$\mathbf{V} = E_0''(\mathbf{u})\mathbf{U} \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

of $\mathbf{U}$ under the linear map $E_0''(\mathbf{u})$ is determined through the identity

$$\langle E_0''(\mathbf{u}), (\mathbf{U}, \overline{\mathbf{U}})\rangle = \langle E_0''(\mathbf{u})\mathbf{U}, \overline{\mathbf{U}}\rangle = \langle \mathbf{V}, \overline{\mathbf{U}}\rangle \tag{6.2}$$

for all

$$\overline{\mathbf{U}} \in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

The element $\mathbf{V}$ defined through (6.2) is the solution of a standard quadratic variational problem for which the weak form of its optimality condition is precisely (6.2). This form is especially suited to show the compactness we are after. Set

$$\begin{aligned}\overline{\mathbf{F}} = & (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{u}')\nabla^2\mathbf{F}^\perp(\mathbf{u}):\mathbf{u}' + \nabla\mathbf{F}^\perp(\mathbf{u})\mathbf{u}'\otimes\nabla\mathbf{F}^\perp(\mathbf{u})\mathbf{u}', \\ \overline{\mathbf{G}} = & \nabla\mathbf{F}^\perp(\mathbf{u}) + \mathbf{F}^\perp(\mathbf{u})\otimes\nabla\mathbf{F}^\perp(\mathbf{u})\mathbf{u}', \\ \overline{\mathbf{H}} = & \nabla\mathbf{F}^\perp(\mathbf{u}) + \nabla\mathbf{F}^\perp(\mathbf{u})\mathbf{u}'\otimes\mathbf{F}^\perp(\mathbf{u}), \\ \overline{\mathbf{J}} = & \mathbf{F}^\perp(\mathbf{u})\otimes\mathbf{F}^\perp(\mathbf{u}).\end{aligned}$$

This choice is dictated so that

$$\langle E_0''(\mathbf{u}), (\mathbf{U}, \overline{\mathbf{U}})\rangle = \int_0^1 [\overline{\mathbf{F}}(\mathbf{U},\overline{\mathbf{U}}) + \overline{\mathbf{G}}(\mathbf{U}',\overline{\mathbf{U}}) + \overline{\mathbf{H}}(\mathbf{U},\overline{\mathbf{U}}') + \overline{\mathbf{J}}(\mathbf{U}',\overline{\mathbf{U}}')] \, dt.$$

Exactly as in Lemma 17, one can show the following.

Lemma 20. *For fixed, given*

$$\mathbf{u} \in H^2_{O,+1}([0,1];\mathbb{R}^2),$$

the operator

$$\mathbf{U} \mapsto \mathbf{V} = E_0''(\mathbf{u})\mathbf{U}$$

is self-adjoint and compact.

Proof. Assume $\{\mathbf{U}_j\}$ is bounded in $H^2_{O,+1}([0,1];\mathbb{R}^2)$. In particular, and for reasons already pointed out earlier, $\mathbf{U}_j' \to \mathbf{U}'$ uniformly for some

$$\mathbf{U} \in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

Let $\mathbf{V}_j$ and $\mathbf{V}$ determined through (6.2), respectively. Then

$$\|\mathbf{V}_j - \mathbf{V}\|^2 = \langle E_0''(\mathbf{u})(\mathbf{U}_j - \mathbf{U}), \mathbf{V}_j - \mathbf{V}\rangle,$$

and

$$\|\mathbf{V}_j - \mathbf{V}\| \le \|E_0''(\mathbf{u})(\mathbf{U}_j - \mathbf{U})\|.$$

The key point is to realize, in the formulas above, that in $E_0''(\mathbf{u})(\mathbf{U}_j - \mathbf{U})$ only up to first derivatives of the differences $\mathbf{U}_j - \mathbf{U}$ occur, and these converge strongly to zero. Hence

$$\|\mathbf{V}_j - \mathbf{V}\| \to 0.$$

□

Theorem 21. *For each ε positive, every critical point of E_ε is non-degenerate, and has a finite Morse index.*

Proof. The proof relies on the standard fact that eigenvalues of a linear, self-adjoint, compact operator in a Banach space, like $E_0''(\mathbf{u})$, always has a sequence of (real) eigenvalues converging to zero (see, for instance, Chapter 6 of [9]). According to (6.1), eigenvalues of $E_\varepsilon''(\mathbf{u})$ are eigenvalues of $E_0''(\mathbf{u})$ plus ε, and so there cannot be an infinite number of negative eigenvalues. □

7. Identification of limit cycles with valleys of E_ε, and relevant critical closed paths

It is elementary to realize that each limit cycle (regardless of how it is paremeterized in $[0,1]$) identifies one valley of the level set $\{E_\varepsilon \le a_\varepsilon\}$ for a_ε, non-critical, suitably chosen possibly depending on ε, and uniformly away from 0. The main issue we would like to address, and which is fundamental for our strategy, is to show that two distinct limit cycles cannot lie in the same connected component of $\{E_\varepsilon \le a_\varepsilon\}$ if a_ε is non-critical and appropriately selected. Keep in mind the possibility, for each limit cycle, of changing the starting point, or of reparameterizing the curve. The possibility of describing limit cycles by reparameterization going around several times counter- or clock-wise has been discarded in our space $H^2_{O,+1}([0,1];\mathbb{R}^2)$.

Each path

$$\mathbf{u} \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

whose image set

$$\{\mathbf{u}(t) : t \in [0,1]\} \subset \mathbb{R}^2$$

is a certain limit cycle will give rise to a certain valley of E_ε, when ε is sufficiently small. What we need to ensure is that distinct limit cycles cannot stay in the same valley for values a_ε well chosen.

Theorem 22. *Let $\mathbf{u}$ and $\mathbf{v}$ be two elements of our space $H^2_{O,+1}([0,1];\mathbb{R}^2)$ representing two different limit cycles of our differential system. For well-chosen a_ε non-critical, and uniformly away from zero, they cannot belong to the same connected component of $\{E_\varepsilon \le a_\varepsilon\}$.*

Proof. The idea of the proof is the following. Arguing by contradiction, if we assume that $\mathbf{u}$ and $\mathbf{v}$ are in the same connected component of the set $\{E_\varepsilon \le a_\varepsilon\}$, with $a_\varepsilon \to 0$, then there exists a continuous family of closed paths

$$\sigma_\varepsilon(s,t) : [0,1]\times[0,1] \to \mathbb{R}^2, \quad \sigma_\varepsilon(0,\cdot) = \mathbf{u}, \sigma_\varepsilon(1,\cdot) = \mathbf{v}, \sigma_\varepsilon(s,\cdot) \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

with $E_\varepsilon(\sigma_\varepsilon(s,\cdot)) \le a_\varepsilon$, so that each path $\sigma_\varepsilon(s,\cdot)$ does not need to be a closed solution of system (1.1). Working with this family, we will prove that there is a limit family $\sigma(s,\cdot)$ of closed paths as $\varepsilon \to 0$ that it is formed by closed orbits of system (1.1), i.e. $E_0(\sigma(s,\cdot)) = 0$, but this will be a contradiction because $\mathbf{u}$ and $\mathbf{v}$ are isolated limit cycles.

We proceed in two steps. Suppose, first that the sequence

$$\sigma_\varepsilon(s,\cdot) : [0,1] \to H^2_{O,+1}([0,1];\mathbb{R}^2) \tag{7.1}$$

is uniformly bounded in the space

$$L^\infty([0,1]; H^2_{O,+1}([0,1];\mathbb{R}^2)). \tag{7.2}$$

Through a reparametrization of each curve $\sigma_\varepsilon(\cdot,t)$ through a uniformly bounded multiple of arc-length, for instance, we can assume that a suitable sequence (not relabeled) converges weakly in the same space, but strongly in

$$L^\infty([0,1]; H^1([0,1];\mathbb{R}^2))$$

(notice the change in the target space, see Corollary 9.13 in [9] for more details) to some continuous path

$$\sigma : [0,1] \to H^1([0,1];\mathbb{R}^2).$$

Since, for every $s \in [0,1]$,

$$0 \le E_0(\sigma_\varepsilon(s,\cdot)) \le E_\varepsilon(\sigma_\varepsilon(s,\cdot)) \le a_\varepsilon \to 0, \tag{7.3}$$

we conclude that $E_0(\sigma(s,\cdot)) = 0$, i.e. σ is a continuous deformation from $\mathbf{u}$ to $\mathbf{v}$ through orbits of our differential system (1.1). This is impossible because $\mathbf{u}$ and $\mathbf{v}$ are isolated limit cycles.

Assume next that the sequence in (7.1) is not uniformly bounded in the space (7.2). For every fixed $r > 0$, again by reparameterization if necessary, there is some s_0, depending on r and increasing with r, such that working in the subinterval $[0,s_0]$ instead of the full interval $[0,1]$, we are in a situation where the conclusion of the first step can be applied to find a continuous limit path

$$\sigma : [0,s_0] \to H^1([0,1];\mathbb{R}^2), \quad L^\infty([0,s_0]; H^1([0,1];\mathbb{R}^2)),$$

such that, as in (7.3),

$$E_0(\sigma(s,\cdot)) = 0, \quad s \in [0,s_0].$$

we can assume that

$$\|\sigma_\varepsilon([0,s_0]) - \mathbf{u}\|_{H^2_{O,+1}([0,1];\mathbb{R}^2)} \le r,$$
$$\|\sigma_\varepsilon(s_0) - \mathbf{u}\|_{H^2_{O,+1}([0,1];\mathbb{R}^2)} = r, \tag{7.4}$$

for some fixed (independent of ε) $s_0 \in [0,1]$. This implies that the sequence $\{\sigma_\varepsilon\}$ is a uniformly bounded set in the space

$$L^\infty([0,s_0]; H^2_{O,+1}([0,1];\mathbb{R}^2)).$$

As before, a suitable sequence (not relabeled) converges weakly in the same space, but strongly in

$$L^\infty([0, s_0]; H^1([0, 1]; \mathbb{R}^2))$$

to some continuous path

$$\sigma : [0, s_0] \to H^1([0, 1]; \mathbb{R}^2).$$

In particular $\sigma(0) = \mathbf{u}$, and from (7.3) we have

$$E_0(\sigma(s, \cdot)) = 0 \text{ for every } s \in [0, s_0],$$

with a non-empty set

$$\{\sigma(s, \cdot) : s \in [0, s_0]\} \setminus \{\mathbf{u}\},$$

because (7.4) implies

$$\|\sigma(s_0) - \mathbf{u}\|_{H^1([0,1];\mathbb{R}^N)} = r.$$

This implies, because $\mathbf{u}$ is an isolated close path formed by orbits of system (1.1), that $\sigma(s, \cdot)$ is a reparameterization of $\mathbf{u}$, in our space $H^2_{O,+1}([0, 1]; \mathbb{R}^2)$, for every $s \in [0, s_0]$. But since r can be taken arbitrarily large, we would be able to find reparameterizations of $\mathbf{u}$, a bounded path in $\mathbb{R}^2$, with size in $H^2_{O,+1}([0, 1]; \mathbb{R}^2)$ arbitrarily large, in particular with second derivatives arbitrarily large. This is not possible. Note that if second derivatives were uniformly bounded, by integration, first derivatives and paths themselves would be bounded and we would fall into the first step already treated. □

From now on we focus on understanding critical closed paths of E_ε for arbitrary small ε for which E_ε is away from zero, to see if Morse inequalities restricted to these may lead to our desired uniform bound.

8. The equation for critical closed paths

The object of this section is to derive and study the differential equations which must satisfy critical closed paths of E_ε in $H^2_{O,+1}([0, 1]; \mathbb{R}^2)$. The proof uses standard ideas in Calculus of Variations, but we include them for the sake of completeness.

Let

$$F(t, \mathbf{u}, \mathbf{z}, \mathbf{Z}) : [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

be a ($\mathcal{C}^\infty$-) function with respect to $(t, \mathbf{u}, \mathbf{z}, \mathbf{Z})$, with partial derivatives $F_\mathbf{u}$, $F_\mathbf{z}$, $F_\mathbf{Z}$. Assume that

$$F_\mathbf{u}(t, \mathbf{v}, \mathbf{v}', \mathbf{v}'')$$

and

$$F_\mathbf{z}(t, \mathbf{v}, \mathbf{v}', \mathbf{v}'') - \int_0^t F_\mathbf{u}(t, \mathbf{v}, \mathbf{v}', \mathbf{v}'')\, ds$$

belong to $L^1((0, 1); \mathbb{R}^2)$ for every feasible

$$\mathbf{v} \in H^2_{O,+1}([0, 1]; \mathbb{R}^2).$$

Theorem 23. *Suppose that the functional*

$$E(\mathbf{u}) = \int_0^1 F(t, \mathbf{u}(t), \mathbf{u}'(t), \mathbf{u}''(t))\, dt$$

admits a critical closed path

$$\mathbf{u} : [0, 1] \to \mathbb{R}^2 \textit{ in } H^2_{O,+1}([0, 1]; \mathbb{R}^2).$$

Then the function

$$\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') \tag{8.1}$$

is absolutely continuous in $[0,1]$*, and*

$$\frac{d}{dt}\left(\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\right)+F_{\mathbf{u}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')=\mathbf{0}\ \textit{for a.e. } t \textit{ in } (0,1). \tag{8.2}$$

Moreover

$$[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}=\mathbf{0}. \tag{8.3}$$

Brackets in (8.3) indicate the jump of the field inside at the time indicated (difference between $t=1$, and $t=0$), that is

$$[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}=F_{\mathbf{Z}}(t,\mathbf{u}(t),\mathbf{u}'(t),\mathbf{u}''(t))\big|_{t=1^-}-F_{\mathbf{Z}}(t,\mathbf{u}(t),\mathbf{u}'(t),\mathbf{u}''(t))\big|_{t=0^+}$$

Notice that the integrability demanded on those combinations of partial derivatives of F in the statement of Theorem 23 is equivalent to having

$$F_{\mathbf{u}}(t,\mathbf{v},\mathbf{v}',\mathbf{v}'') \text{ and } F_{\mathbf{z}}(t,\mathbf{v},\mathbf{v}',\mathbf{v}'')$$

integrable for every feasible

$$\mathbf{v}\in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

We have however decided to keep the statement as it is for that is exactly the form in which those combinations of partial derivatives will occur in the proof.

Proof of Theorem 23. Take

$$\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2).$$

If $\mathbf{u}$ is a critical closed path of E, then

$$\frac{d}{d\delta}E(\mathbf{u}+\delta\mathbf{U})\bigg|_{\delta=0}=0,$$

that is to say

$$\frac{d}{d\delta}\bigg|_{\delta=0}\int_0^1 F(t,\mathbf{u}(t)+\delta\mathbf{U}(t),\mathbf{u}'(t)+\delta\mathbf{U}'(t),\mathbf{u}''(t)+\delta\mathbf{U}''(t))\,dt=0.$$

This derivative has the form

$$\int_0^1\left[F_{\mathbf{u}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}(t)+F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}'(t)+F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}''(t)\right]dt=0. \tag{8.4}$$

We consider the special subspace $\mathbb{L}$ of variations $\mathbf{U}$ defined by

$$\mathbb{L}=\{\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2):\mathbf{U}(0)=\mathbf{U}(1)=\mathbf{0},\mathbf{U}''\in\{1,t\}^\perp\}, \tag{8.5}$$

where $\{1,t\}^\perp$ is the orthogonal complement, in $L^2([0,1];\mathbb{R}^2)$, of the subspace generated by $\{1,t\}$. Since these orthogonality conditions mean

$$\int_0^1\mathbf{U}''(t)\,dt=\mathbf{U}'(1)-\mathbf{U}'(0)=\mathbf{0},\quad \int_0^1 t\mathbf{U}''(t)\,dt=\mathbf{U}'(1)=\mathbf{0},$$

we can also put

$$\mathbb{L}=\{\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2):\mathbf{U}(0)=\mathbf{U}(1)=\mathbf{U}'(0)=\mathbf{U}'(1)=\mathbf{0}\}.$$

We also set

$$\Psi(t) = \int_0^t F_{\mathbf{u}}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds,$$

$$\Phi(t) = \int_0^t \left[-\Psi(s) + F_{\mathbf{z}}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) \right] ds,$$

two continuous, bounded functions by hypothesis. For $\mathbf{U} \in \mathbb{L}$, an integration by parts in the first term of (8.4) yields

$$\int_0^1 \left[-\Psi(t) \cdot \mathbf{U}'(t) + F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \cdot \mathbf{U}'(t) + F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \cdot \mathbf{U}''(t) \right] dt = 0,$$

because

$$\Psi(t)\mathbf{U}(t)\big|_0^1 = \mathbf{0}$$

for test fields $\mathbf{U} \in \mathbb{L}$. A second integration by parts leads to

$$\int_0^1 \left[-\Phi(t) + F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \right] \cdot \mathbf{U}''(t)\, dt = 0,$$

again because

$$\Phi(t)\mathbf{U}'(t)\big|_0^1 = \mathbf{0}$$

if $\mathbf{U} \in \mathbb{L}$. Due to the arbitrariness of $\mathbf{U} \in \mathbb{L}$, according to (8.5) we conclude that

$$F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') - \Phi(t) = c + Ct \text{ in } (0, 1), \tag{8.6}$$

with c and C constants. In particular, since Φ is absolutely continuous (it belongs to $W^{1,1}((0,1); \mathbb{R}^2)$), we know that $F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'')$ must be absolutely continuous too in $(0, 1)$, and as such, it cannot have jumps in $(0, 1)$, though it could possibly have at the endpoints. By differentiating once in (8.6) with respect to t,

$$\frac{d}{dt} F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') - F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') + \Psi(t) = C \text{ a.e. in } (0, 1),$$

and even further

$$\frac{d}{dt} \left(\frac{d}{dt} F_{\mathbf{Z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') - F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \right) + F_{\mathbf{u}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') = 0 \text{ a.e. in } (0, 1). \tag{8.7}$$

We take this information back to (8.4) for a general

$$\mathbf{U} \in H^2_{O,+1}([0, 1]; \mathbb{R}^2),$$

not necessarily belonging to $\mathbb{L}$. One integration by parts in the second term in (8.4) yields

$$\int_0^1 F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \cdot \mathbf{U}'(t)\, dt = -\int_0^1 \frac{d}{dt} F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'') \cdot \mathbf{U}(t)\, dt + [F_{\mathbf{z}}(t, \mathbf{u}, \mathbf{u}', \mathbf{u}'')]_{t=0} \cdot \mathbf{U}(0).$$

Recall the periodicity conditions for $\mathbf{U}$. Two such integrations by parts in the third term of (8.4) leads to

$$\begin{aligned}\int_0^1 F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}''(t)\,dt =& -\int_0^1 \frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}'(t)\,dt\\ &+[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}'(0)\\ =&\int_0^1 \frac{d^2}{dt^2}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\cdot\mathbf{U}(t)\,dt\\ &-[\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}(0)\\ &+[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}'(0).\end{aligned}$$

In this way (8.4) becomes

$$\int_0^1 \left[\frac{d}{dt}\left(\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\right)+F_{\mathbf{u}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\right]\cdot\mathbf{U}(t)\,dt$$
$$-[\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}(0)+[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}'(0).$$

The integral here vanishes precisely by (8.7), and so we are only left with the contributions on the end-points. Hence, we obtain

$$[\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}(0)-[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}'(0)=0. \tag{8.8}$$

Since vectors $\mathbf{U}'(0)$ and $\mathbf{U}(0)$ can be chosen arbitrarily, and independently of each other, because there is always a path

$$\mathbf{U}\in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

starting in a certain arbitrary vector of $\mathbb{R}^2$ and with any preassigned velocity, we conclude that

$$[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}=\mathbf{0}, \tag{8.9}$$

and

$$[\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}=\mathbf{0}. \tag{8.10}$$

This completes the proof of Theorem 23. Note that this last condition implies that

$$\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')$$

is absolutely continuous in the interval $[0,1]$, including the endpoints. □

The application of Theorem 23 to our situation where

$$F(t,\mathbf{u},\mathbf{z},\mathbf{Z})=\frac{1}{2}(\mathbf{F}^{\perp}(\mathbf{u})\cdot\mathbf{z})^2+\frac{\varepsilon}{2}(|\mathbf{Z}|^2+|\mathbf{z}|^2+|\mathbf{u}|^2)+\mathbf{u}\cdot\mathbf{v}_\varepsilon(t)+\mathbf{z}\cdot\mathbf{v}'_\varepsilon(t)+\mathbf{Z}\cdot\mathbf{v}''_\varepsilon(t)$$

is our key tool. Note that we have dropped out the constant term $(1/2\varepsilon)\|\mathbf{v}_\varepsilon\|^2$ from E_ε as it does not play a role in what follows. The partial derivatives required in the statement of this theorem are

$$F_{\mathbf{Z}} = \varepsilon\mathbf{Z} + \mathbf{v}_\varepsilon''(t),$$
$$F_{\mathbf{z}} = (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{z})\,\mathbf{F}^\perp(\mathbf{u}) + \varepsilon\mathbf{z} + \mathbf{v}_\varepsilon'(t),$$
$$F_{\mathbf{u}} = (\mathbf{F}^\perp(\mathbf{u})\cdot\mathbf{z})\,D\mathbf{F}^\perp(\mathbf{u})\mathbf{z} + \varepsilon\mathbf{u} + \mathbf{v}_\varepsilon(t).$$

Equation (8.2) for critical closed paths in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ for E_ε coming from Theorem 23 involves the combination (8.1) which in our case is

$$\frac{d}{dt}(\varepsilon\mathbf{u}_\varepsilon''(t) + \mathbf{v}_\varepsilon''(t)) - (\mathbf{F}^\perp(\mathbf{u}_\varepsilon(t))\cdot\mathbf{u}_\varepsilon'(t))\,\mathbf{F}^\perp(\mathbf{u}_\varepsilon(t)) - \varepsilon\mathbf{u}_\varepsilon'(t) - \mathbf{v}_\varepsilon'(t), \tag{8.11}$$

which must be an absolutely continuous function in $[0,1]$. Its almost everywhere derivative ought to be, according to system (8.2),

$$-(\mathbf{F}^\perp(\mathbf{u}_\varepsilon(t)\cdot\mathbf{u}_\varepsilon'(t))\,D\mathbf{F}^\perp(\mathbf{u}_\varepsilon(t))\mathbf{u}_\varepsilon'(t) - \varepsilon\mathbf{u}_\varepsilon(t) - \mathbf{v}_\varepsilon(t). \tag{8.12}$$

Here

$$\mathbf{u}_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

is an arbitrary critical closed path of E_ε. In addition, from (8.3) we have

$$[\varepsilon\mathbf{u}_\varepsilon''(t) + \mathbf{v}_\varepsilon''(t)]\big|_{t=0} = \mathbf{0}. \tag{8.13}$$

We need to examine these conditions carefully.

It is also important to stress, for those not used to it, how this result ensures much more regularity for those critical closed paths precisely because they are critical closed paths of a certain functional. Even though paths in our ambient space are just in $H^2([0,1];\mathbb{R}^2)$, critical closed paths of the functional in the statement of this lemma are much more regular. Recall that $H^2_{O,+1}([0,1];\mathbb{R}^2)$ is the completion in $H^2([0,1];\mathbb{R}^2)$ of the subspace of non-self-intersecting paths with rotation index $+1$, i.e. they travel once in a counterclockwise sense. By Theorem 23, the expression in (8.11) is absolutely continuous. Since the last three terms of (8.11) and $\mathbf{v}_\varepsilon'''$ are continuous, we can conclude that $\mathbf{u}_\varepsilon$ is $\mathcal{C}^3$ in $[0,1]$. Moreover, due to the fact that the derivative of (8.11) is equal to (8.12), again by Theorem 23, it follows that $\mathbf{u}_\varepsilon$ is even $\mathcal{C}^4$ in $[0,1]$ because all terms in (8.11), when differentiated with respect to t, are continuous except possibly the first one $\mathbf{u}_\varepsilon'''(t)$, and such a derivative is equal to (8.12) which is continuous. Note how condition (8.13) is redundant with the above information.

Proposition 24. *Critical closed paths $\mathbf{u}_\varepsilon$ of the functional E_ε are $\mathcal{C}^\infty$ in $[0,1]$, and are solutions of the system*

(8.14)

$$\varepsilon(\mathbf{u}_\varepsilon''''-\mathbf{u}_\varepsilon''+\mathbf{u}_\varepsilon)-\frac{d}{dt}[(\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,\mathbf{F}^\perp(\mathbf{u}_\varepsilon)]+(\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,(\mathbf{u}_\varepsilon')^T D\mathbf{F}^\perp(\mathbf{u}_\varepsilon) = -\mathbf{v}_\varepsilon''''+\mathbf{v}_\varepsilon''-\mathbf{v}_\varepsilon$$

in the interval $[0,1]$.

Proof. Regularity conditions for $\mathbf{u}_\varepsilon$ have been stated prior to the statement of the proposition. Equation (8.14) is a consequence, according to equation (8.2), expressing the equality of the derivative of (8.11) with (8.12). A typical bootstrap argument yields the regularity claimed in the statement. □

Equation (8.14) is a key point for counting the critical closed paths of the functional E_ε. We are facing a singularly perturbed, fourth-order ODE system (8.14) with periodic (unknown) boundary conditions. Our plan to count, and eventually find an upper bound for, the number of solutions of (8.14), which we call branches to stress the dependence of ε, proceeds in two steps:

(1) for a fixed such branch (with smooth dependence on ε), understand its asymptotic behavior as $\varepsilon \searrow 0$, to count how many such different asymptotic behavior there might be; and
(2) for a fixed such asymptotic behavior, decide how many branches will converge to it.

The most delicate issue is this last second point. To deal appropriately with it, we need to introduce an important discussion related to variational principles and the role played by end-point conditions.

Second-order variational problems, like the one considered in Theorem 23, are typically studied under fixed, end-point conditions at points $\{0, 1\}$ up to one order less than the highest order explicitly participating in the functional. In our situation, end-point conditions would involve the four values

$$\mathbf{u}(0), \mathbf{u}(1), \quad \mathbf{u}'(0), \mathbf{u}'(1).$$

Our periodicity conditions would demand

$$\mathbf{u}(0) = \mathbf{u}(1), \quad \mathbf{u}'(0) = \mathbf{u}'(1),$$

but these two common values are unknown. There might be several vectors $\mathbf{y}$ and $\mathbf{z}$ for which critical paths for the same functional

$$E(\mathbf{u}) = \int_0^1 F(t, \mathbf{u}(t), \mathbf{u}'(t), \mathbf{u}''(t))\, dt$$

for $\mathbf{u} \in H^2([0, 1]; \mathbb{R}^2)$ under end-point conditions

$$\mathbf{u}(0) = \mathbf{u}(1) = \mathbf{y}, \quad \mathbf{u}'(0) = \mathbf{u}'(1) = \mathbf{z}, \tag{8.15}$$

would also be critical paths for our system (8.14) without imposing such end-point conditions but just periodicity. The calculation of how many branches there may be for a given asymptotic behavior is linked to how many possible values for vectors $\mathbf{y}$ and $\mathbf{z}$ in (8.15) are capable of producing solutions for the problem in Proposition 24 under periodicity.

The following versions of Theorem 23 take into account our discussion above concerning the role played by end-point conditions, and will help us in counting branches of critical paths for our functional E_ε. To this end, we introduce the notation

$$H^2_{O,+1,\mathbf{y}}([0, 1]; \mathbb{R}^2) = \{\mathbf{v} \in H^2_{O,+1}([0, 1]; \mathbb{R}^2) : \mathbf{v}(0) = \mathbf{v}(1) = \mathbf{y}, \mathbf{v}'(0) = \mathbf{v}'(1)\},$$
$$H^2_{O,+1,\mathbf{y},\mathbf{z}}([0, 1]; \mathbb{R}^2) = \{\mathbf{v} \in H^2_{O,+1}([0, 1]; \mathbb{R}^2) : \mathbf{v}(0) = \mathbf{v}(1) = \mathbf{y}, \mathbf{v}'(0) = \mathbf{v}'(1) = \mathbf{z}\},$$

for fixed vectors $\mathbf{y}, \mathbf{z} \in \mathbb{R}^2$.

Theorem 25. *Let the integrand F and the corresponding functional E be as in Theorem 23. Let $\mathbf{y} \in \mathbb{R}^2$ be a given vector. Suppose that $\mathbf{u}$ is a critical path of E over the class* *of feasible paths* $H^2_{O,+1,\mathbf{y}}([0,1];\mathbb{R}^2)$ *just introduced. Then the vector field*

$$\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') \tag{8.16}$$

is absolutely continuous in $(0,1)$,

$$\frac{d}{dt}\left(\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')-F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\right)+F_{\mathbf{u}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')=\mathbf{0}\ \text{a.e. } t \text{ in } (0,1), \tag{8.17}$$

and

$$[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}=\mathbf{0}. \tag{8.18}$$

Notice that the classes of feasible paths $H^2_{O,+1,\mathbf{y}}([0,1];\mathbb{R}^2)$, in this statement, and $H^2_{O,+1,\mathbf{y},\mathbf{z}}([0,1];\mathbb{R}^2)$ are always subsets of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ for every $\mathbf{y}$ and $\mathbf{z}$. In fact, if we add to $H^2_{O,+1}([0,1];\mathbb{R}^2)$ a constraint fixing the starting (and final) vector $\mathbf{y}$, optimality yields a less restrictive set of conditions, which in this situation amounts to just loosing the continuity of the vector field in (8.16) across $t=0$.

Proof of Theorem 25. The proof is exactly the same, word by word, as that of Theorem 23. The only difference revolves around the discussion of (8.8). Under periodic conditions without imposing a particular vector as starting vector (as we are doing here), (8.8) leads to the two vanishing jump conditions (8.9) and (8.10). However, if we insist in that the starting vector for paths is a given, specific vector $\mathbf{y}$, then feasible variations $\mathbf{U}$ in (8.8) must comply with $\mathbf{U}(0)=\mathbf{0}$, and so we are left with

$$[F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')]_{t=0}\cdot\mathbf{U}'(0)=0.$$

The arbitrariness of $\mathbf{U}'(0)$ (which can be chosen freely) leads to the second jump condition (8.10), but we have no longer (8.9). This translates into the continuity of the vector field (8.20) in the open interval $(0,1)$, not including the end-points, precisely because we cannot rely on the jump condition across the end-points. However the differential system (8.21) or (8.2) holds in $(0,1)$ in both situations. □

Again, the application of this last general statement to our particular situation, leads to the following new version of Proposition 24.

Proposition 26. *Critical closed paths* $\mathbf{u}_\varepsilon$ *of the functional* E_ε *over* $H^2_{O,+1,\mathbf{y}}(0,1];\mathbb{R}^2)$ *are* $\mathcal{C}^2$ *in* $[0,1]$, $\mathcal{C}^\infty$ *in* $(0,1)$, *and are solutions of the fourth-order differential system*

$$\varepsilon(\mathbf{u}_\varepsilon''''-\mathbf{u}_\varepsilon''+\mathbf{u}_\varepsilon)-\frac{d}{dt}[(\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,\mathbf{F}^\perp(\mathbf{u}_\varepsilon)]+(\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,(\mathbf{u}_\varepsilon')^T D\mathbf{F}^\perp(\mathbf{u}_\varepsilon)=-\mathbf{v}_\varepsilon''''+\mathbf{v}_\varepsilon''-\mathbf{v}_\varepsilon \tag{8.19}$$

in the interval $(0,1)$.

For the proof, simply notice that not having the vanishing of the jump of (8.11), we cannot rely on the continuity of the third derivative $\mathbf{u}_\varepsilon'''$ across $t=0$ (the only term which is not guaranteed to be continuous across $t=0$ in (8.11)), and so the critical path $\mathbf{u}_\varepsilon$ can only be ensured to belong to $\mathcal{C}^2$ in $[0,1]$.

We can also perform the same analysis in the more restrictive subspace

$$H^2_{O,+1,\mathbf{y},\mathbf{z}}([0,1];\mathbb{R}^2)$$

for fixed vectors $\mathbf{y}$ and $\mathbf{z}$, and find the parallel statements that follow, whose proofs can be very easily adapted from the previous ones. Note how as we place more demands on feasible paths, optimality turns back less regularity through end-points.

Theorem 27. *Let the integrand F and the corresponding functional E be as in Theorem 23. Let $\mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ be given vectors. Suppose that $\mathbf{u}$ is a critical path of E over the class of feasible paths $H^2_{O,+1,\mathbf{y},\mathbf{z}}([0,1];\mathbb{R}^2)$. Then the vector field*

$$\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') - F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') \tag{8.20}$$

is absolutely continuous in $(0,1)$, and

$$\frac{d}{dt}\left(\frac{d}{dt}F_{\mathbf{Z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') - F_{\mathbf{z}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'')\right) + F_{\mathbf{u}}(t,\mathbf{u},\mathbf{u}',\mathbf{u}'') = \mathbf{0} \text{ a.e. } t \text{ in } (0,1). \tag{8.21}$$

Proposition 28. *Critical closed paths $\mathbf{u}_\varepsilon$ of the functional E_ε over $H^2_{O,+1,\mathbf{y},\mathbf{z}}([0,1];\mathbb{R}^2)$ are $\mathcal{C}^1$ in $[0,1]$, $\mathcal{C}^\infty$ in $(0,1)$, and are solutions of the fourth-order differential system*

$$\varepsilon(\mathbf{u}_\varepsilon'''' - \mathbf{u}_\varepsilon'' + \mathbf{u}_\varepsilon) - \frac{d}{dt}[(\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,\mathbf{F}^\perp(\mathbf{u}_\varepsilon)] + (\mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon')\,(\mathbf{u}_\varepsilon')^T D\mathbf{F}^\perp(\mathbf{u}_\varepsilon) = -\mathbf{v}_\varepsilon'''' + \mathbf{v}_\varepsilon'' - \mathbf{v}_e \tag{8.22}$$

in the interval $(0,1)$.

Critical closed paths in Proposition 28 might not be unique. Obviously, however, different solutions cannot coincide in the full interval $(0,1)$. Since we plan to use this result and Proposition 26 to count how many solutions we may have in Proposition 24, in a neighborhood of constant values $\mathbf{y} = \mathbf{p}$ and $\mathbf{z} = \mathbf{q}$, we can rely on periodicity to translate, in an arbitrary way, the independent variable t and to place $t = 0$ where solutions in Proposition 28 are unique for $\mathbf{y}$ and $\mathbf{z}$ in such neighborhoods of appropriate $\mathbf{p}$ and $\mathbf{q}$. We refer to Section 10 below. In addition, and once we have ensured such uniqueness, the dependence of the solution on such initial and final values $\mathbf{y}$ and $\mathbf{z}$ is smooth, even analytic, i.e. if

$$\mathbf{u}(t;\mathbf{y},\mathbf{z},\varepsilon) \tag{8.23}$$

is such unique solution of (8.22) for $(\mathbf{y},\mathbf{z})$ in a suitable vicinity of $(\mathbf{p},\mathbf{q})$, then the dependence of $\mathbf{u}$ in (8.23) on pairs $(\mathbf{y},\mathbf{z})$ is analytic.

9. Asymptotic behavior

For the sake of transparency, and to facilitate a few interesting computations, we recast system (8.14) in its two components

$$\begin{aligned}(ZQ)' + Z(-Q_x x' + P_x y') &= -\varepsilon\alpha_1,\\ (ZP)' + Z(Q_y x' - P_y y') &= \varepsilon\alpha_2.\end{aligned}$$

where

$$\mathbf{F} = (P,Q), \quad \mathbf{u}_\varepsilon = (x,y), \quad \mathbf{v}_\varepsilon = \varepsilon(X,Y)$$
$$Z \equiv \mathbf{F}^\perp(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon' = P(x,y)y' - Q(x,y)x',$$
$$W \equiv \mathbf{F}(\mathbf{u}_\varepsilon)\cdot\mathbf{u}_\varepsilon' = P(x,y)x' + Q(x,y)y',$$
$$\text{Div} \equiv P_x + Q_y, \quad \alpha_1 = \overline{x}'''' - \overline{x}'' + \overline{x}, \quad \alpha_2 = \overline{y}'''' - \overline{y}'' + \overline{y},$$

with

$$\overline{x} = x + X, \quad \overline{y} = y + Y.$$

Note that $Z^2/2$ is precisely the integrand for E_0, and recall that all close paths involved are 1-periodic, $\mathcal{C}^\infty$ and belong to $H^2_{O,+1}([0,1];\mathbb{R}^2)$, so that we can freely differentiate in t as many times as needed. In particular, the two equations of the system of critical closed paths become

$$Z'Q + Z\,\mathrm{Div}\,y' = -\varepsilon\alpha_1, \quad Z'P + Z\,\mathrm{Div}\,x' = \varepsilon\alpha_2. \tag{9.1}$$

Moreover X, Y and their derivatives are uniformly bounded with respect to ε by choice of $\mathbf{v}_\varepsilon$ (recall the properties of $\mathbf{v}_\varepsilon$ in Theorem 13). Note that these functions X and Y are the components of $(1/\varepsilon)\mathbf{v}_\varepsilon$, so that

$$X = (1/\varepsilon)X_\varepsilon, \quad Y = (1/\varepsilon)Y_\varepsilon$$

where $\mathbf{v}_\varepsilon$ was introduced as $\mathbf{v}_\varepsilon = (X_\varepsilon, Y_\varepsilon)$ earlier in the paper.

We manipulate the two equations in (9.1) in two ways:

(1) multiply the first equation by Q, the second by P, and add up the results to find

$$Z'(P^2 + Q^2) = -\varepsilon(\alpha_1 Q - \alpha_2 P) - ZW\,\mathrm{Div}; \tag{9.2}$$

(2) then, multiply the first by P, the second by Q, and subtract the results to have

$$Z^2\,\mathrm{Div} = -\varepsilon(\alpha_1 P + \alpha_2 Q).$$

We remind readers that, according to Proposition 22, we are after periodic solutions of this system for which E_ε is away from zero. We will therefore discard from our consideration those such solutions for which E_ε is arbitrarily small. In particular, we do not need to consider asymptotic behaviors reducing to a point, and so bearing in mind that equilibria of our polynomial, differential system are isolated and they could only be associated with critical closed paths of the kind we are not interested in, we can further multiply (9.2) by Z and divide by $P^2 + Q^2$, to have, taking into account the other equation,

$$(Z^2)' = 2\varepsilon(\alpha_1 x' + \alpha_2 y').$$

Hence system (8.14) can be written in the simplified, equivalent form

$$(Z^2)' = 2\varepsilon(\alpha_1 x' + \alpha_2 y'), \quad Z^2\,\mathrm{Div} = -\varepsilon(\alpha_1 P + \alpha_2 Q). \tag{9.3}$$

The theory of singularly-perturbed differential problems (see for instance [37]) informs us that convergence of solutions of (9.3) to the limit system

$$(Z^2)' = 0, \quad Z^2\,\mathrm{Div} = 0, \tag{9.4}$$

setting $\varepsilon = 0$ in (9.3), takes place pointwise for a.e. $t \in [0,1]$, though there can be small sets where large transition layers may occur; or else, solutions may escape to infinity along system (9.4). The choice of our ambient space $H^2_{O,+1}([0,1];\mathbb{R}^2)$ is most important in this regard.

Lemma 29. *Let $(x_\varepsilon, y_\varepsilon)$ be a family of solutions of (9.3) in the space $H^2_{O,+1}([0,1];\mathbb{R}^2)$,*

$$(Z_\varepsilon^2)' = 2\varepsilon(\alpha_{1,\varepsilon}x'_\varepsilon + \alpha_{2,\varepsilon}y'_\varepsilon), \quad Z_\varepsilon^2\,\mathrm{Div}_\varepsilon = -\varepsilon(\alpha_{1,\varepsilon}P_\varepsilon + \alpha_{2,\varepsilon}Q_\varepsilon). \tag{9.5}$$

For a suitable subsequence (not relabeled),

$$(Z_\varepsilon^2)' \to 0, \quad Z_\varepsilon^2\,\mathrm{Div}_\varepsilon \to 0,$$

pointwise for a.e. $t \in [0,1]$.

Proof. The main idea is to realize that periodic solutions of (9.5) for which the terms involving a fourth-order derivative are not negligible, need to have a frequency going to infinity as ε tends to zero. But this is not possible for paths in our ambient space $H^2_{O,+1}([0,1];\mathbb{R}^2)$ because they cannot have period less than one. We try to make this argument precise in the sequel.

The basic tool is a typical scaling argument. Define

$$u_\varepsilon(s) = x_\varepsilon(\varepsilon^r s), \quad v_\varepsilon(s) = y_\varepsilon(\varepsilon^r s) \tag{9.6}$$

for a positive exponent r to be determined, and $s \in [0,+\infty)$ regarding $(x_\varepsilon, y_\varepsilon)$ defined in all of $[0,+\infty)$ by periodicity. Note that the pair $(u_\varepsilon, v_\varepsilon)$ has period ε^{-r} going to infinity, as $\varepsilon \to 0$. Taking (9.6) to (9.3) leads to the system that $(u_\varepsilon, v_\varepsilon)$ must be a solution of

$$\begin{aligned}
\varepsilon^{-3r}((\overline{P}_\varepsilon v_\varepsilon' - \overline{Q}_\varepsilon u_\varepsilon')^2)' = 2\varepsilon^{1-r} u_\varepsilon'(\varepsilon^{-4r}(u_\varepsilon + U_\varepsilon)'''' - \varepsilon^{-2r}(u_\varepsilon + U_\varepsilon)'' + (u_\varepsilon + U_\varepsilon))\\
+2\varepsilon^{1-r} v_\varepsilon'(\varepsilon^{-4r}(v_\varepsilon + V_\varepsilon)'''' - \varepsilon^{-2r}(v_\varepsilon + V_\varepsilon)'' + (v_\varepsilon + V_\varepsilon)),\\
\varepsilon^{-2r}(\overline{P}_\varepsilon v_\varepsilon' - \overline{Q}_\varepsilon u_\varepsilon')^2(\overline{P}_{x,\varepsilon} + \overline{Q}_{y,\varepsilon}) = -\varepsilon \overline{P}_\varepsilon(\varepsilon^{-4r}(u_\varepsilon + U_\varepsilon)'''' - \varepsilon^{-2r}(u_\varepsilon + U_\varepsilon)'' + (u_\varepsilon + U_\varepsilon))\\
-\varepsilon \overline{Q}_\varepsilon(\varepsilon^{-4r}(v_\varepsilon + V_\varepsilon)'''' - \varepsilon^{-2r}(v_\varepsilon + V_\varepsilon)'' + (v_\varepsilon + V_\varepsilon)),
\end{aligned}$$

where derivatives in these equations are taken with respect to the variable s,

$$\overline{P}_\varepsilon(s) = P(u_\varepsilon(s), v_\varepsilon(s))$$

and similarly with $\overline{Q}$, $\overline{P}_x$, $\overline{Q}_y$. The functions

$$U_\varepsilon(s) = X_\varepsilon(\varepsilon^r s), \quad V_\varepsilon(s) = Y_\varepsilon(\varepsilon^r s)$$

correspond to the components of our rescaled auxiliary path $(X_\varepsilon, Y_\varepsilon)$. If we take the right scaling $r = 1/2$, our system for the pair $(u_\varepsilon, v_\varepsilon)$ becomes

$$\begin{aligned}
((\overline{P}_\varepsilon v_\varepsilon' - \overline{Q}_\varepsilon u_\varepsilon')^2)' =& 2u_\varepsilon'((u_\varepsilon + U_\varepsilon)'''' - \varepsilon(u_\varepsilon + U_\varepsilon)'' + \varepsilon^2(u_\varepsilon + U_\varepsilon))\\
&+ 2v_\varepsilon'((v_\varepsilon + V_\varepsilon)'''' - \varepsilon(v_\varepsilon + V_\varepsilon)'' + \varepsilon^2(v_\varepsilon + V_\varepsilon)),
\end{aligned} \tag{9.7}$$

$$\begin{aligned}
(\overline{P}_\varepsilon v_\varepsilon' - \overline{Q}_\varepsilon u_\varepsilon')^2(\overline{P}_{x,\varepsilon} + \overline{Q}_{y,\varepsilon}) =& - \overline{P}_\varepsilon((u_\varepsilon + U_\varepsilon)'''' - \varepsilon(u_\varepsilon + U_\varepsilon)'' + \varepsilon^2(u_\varepsilon + U_\varepsilon))\\
&- \overline{Q}_\varepsilon((v_\varepsilon + V_\varepsilon)'''' - \varepsilon(v_\varepsilon + V_\varepsilon)'' + \varepsilon^2(v_\varepsilon + V_\varepsilon)).
\end{aligned} \tag{9.8}$$

This is no longer a singularly-perturbed system, since the small parameter ε does not occur in the highest derivatives. The limit system is

$$((\overline{P}v' - \overline{Q}u')^2)' = 2u'u'''' + 2v'v'''', \tag{9.9}$$

$$(\overline{P}v' - \overline{Q}u')^2(\overline{P}_x + \overline{Q}_y) = -\overline{P}u'''' - \overline{Q}v''''. \tag{9.10}$$

Recall that U_ε and V_ε, together with their derivatives, tend to zero, according to our choice of the auxiliary path $(X_\varepsilon, Y_\varepsilon)$.

Periodic solutions $(x_\varepsilon, y_\varepsilon)$ of (9.5), in which neither of the two members of those equations vanish, would correspond to pairs $(u_\varepsilon, v_\varepsilon)$, through (9.6), solutions of (9.7) and (9.8) for which none of the terms not involving a power of ε vanish, and this in turn would correspond to limits (u, v) of (9.9)-(9.10) for which the terms of those two limit equations do not vanish. But these limit solutions (u, v) will have a finite finite period, and so will $(u_\varepsilon, v_\varepsilon)$, given that (9.7)-(9.8) is a regular perturbed system. In this case, the pair $(x_\varepsilon, y_\varepsilon)$ coming from (9.6) will have a period tending to zero with ε. This is impossible for elements $(x_\varepsilon, y_\varepsilon) \in H^2_{O,+1}([0,1];\mathbb{R}^2)$. □

We are therefore entitled to understand all possible asymptotic behaviors of critical closed paths

$$(x_\varepsilon, y_\varepsilon) \in H^2_{O,+1}([0,1];\mathbb{R}^2)$$

through an analysis of the limit system (9.4). The first equation in (9.4) implies that $Z^2 = k^2$, but since we are only interested in the asymptotic behavior for critical closed paths whose value for E_0 stays away from zero, we discard the case $k = 0$. In this case, the second equation in (9.4), implies $\mathrm{Div} = 0$. We would like to understand the nature of the limit system

$$Z^2 = k^2 > 0, \quad \mathrm{Div} = 0. \tag{9.11}$$

We write this system in the form, differentiating the second equation,

$$\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}' = \pm k \neq 0, \quad \nabla \mathrm{Div}(\mathbf{u}) \cdot \mathbf{u}' = 0. \tag{9.12}$$

These are two implicit, first-order systems that become singular when the determinant

$$\nabla \mathrm{Div}(\mathbf{u}) \cdot \mathbf{F}(\mathbf{u})$$

of the matrix of the system

$$\begin{pmatrix} \mathbf{F}^\perp(\mathbf{u}) \\ \nabla \mathrm{Div}(\mathbf{u}) \end{pmatrix}$$

vanishes. These singular points are precisely the contact points of our differential system over the curve $\mathrm{Div} = 0$. The fact that

$$\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}' = \pm k \neq 0$$

shows that $\mathbf{u}_\varepsilon$, for ε sufficiently small, can only turn around, changing $+k$ by $-k$ or viceversa, near those contact points. As a matter of fact, critical closed paths $\mathbf{u}_\varepsilon$ have to turn around whenever one such point is found because at such points, where vectors $\mathbf{F}^\perp$ and $\nabla \mathrm{Div}$ are parallel, the derivative $\mathbf{u}'$ in (9.12) is not defined.

Our above discussion can be summarized in the following statement that classifies all possible asymptotic behaviors for critical closed paths.

Theorem 30. *Assume that all the components of the curve* $\mathrm{Div} = 0$ *are topologically straight lines or ovals. The possible limit behaviors as* $\varepsilon \to 0$ *of branches of critical closed paths of* E_ε*, not coming from zeroes of* E_0*, can be identified in a one-to-one fashion with arcs of the connected components of the curve* $\mathrm{Div} = 0$ *in one of the following possibilities:*

- (a) *If the component is homeomorphic to a straight line, then*
  - (a.1) *the limit behavior is an arc whose endpoints are two contact points and eventually with additional contact points in its interior;*
  - (a.2) *the limit behavior is an arc whose endpoints are one contact point and the infinity, and eventually with additional contact point in its interior;*
  - (a.3) *the limit behavior is the whole component.*
- (b) *If the component is homeomorphic to an oval, then*
  - (b.1) *the limit behavior is an arc whose endpoints are two contact points and eventually with additional contact points in its interior;*

(b.2) *the limit behavior is an arc covering the full oval whose endpoints have to be a single contact point, and eventually with additional contact point in the oval;*

(b.3) *the limit behavior is the whole oval.*

10. Multiplicity

We are concerned in this section about the possibility that various branches of the set of critical closed paths, for ε positive, may coalesce into the same limit behavior as $\varepsilon \searrow 0$, and how they can possibly contribute to the inequality in Proposition 22.

To this goal, as at the end of Section 8, select a point $\mathbf{p}$, in an arbitrary way, in the part of the curve $\mathrm{Div} = 0$ furnishing the particular asymptotic limit we are focusing on, and $\mathbf{q}$ the corresponding tangent vector, at $\mathbf{p}$, to the same curve $\mathrm{Div} = 0$, coming from the convergence of those branches of critical closed paths that we would like to count. Let $\mathbf{u}(t; \mathbf{y}, \mathbf{z}, \varepsilon)$ be the unique solution of system (8.22) for positive ε and $\mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ in a neighborhood of the pair $(\mathbf{p}, \mathbf{q})$ guaranteed according to the discussion at the end of Section 8. Recall that, in addition to verifying (8.14), we also have

$$\mathbf{u}(0; \mathbf{y}, \mathbf{z}, \varepsilon) = \mathbf{u}(1; \mathbf{y}, \mathbf{z}, \varepsilon) = \mathbf{y}, \quad \mathbf{u}'(0; \mathbf{y}, \mathbf{z}, \varepsilon) = \mathbf{u}'(1; \mathbf{y}, \mathbf{z}, \varepsilon) = \mathbf{z}.$$

Our goal is to show an upper bound, independent of ε, on the number of possible values of pairs $(\mathbf{y}, \mathbf{z})$ for which the path $\mathbf{u}(t; \mathbf{y}, \mathbf{z}, \varepsilon)$ is in reality a solution of (8.14) in Proposition 24 which is the one yielding the critical paths we are interested in. To do so, we proceed in two principal successive steps:

- For each $\mathbf{y}$ fixed sufficiently close to $\mathbf{p}$, find an upper bound on how many values for $\mathbf{z}$ around $\mathbf{q}$, one could find so that $\mathbf{u}(t; \mathbf{y}, \mathbf{z}, \varepsilon)$ is in fact a solution of (8.19) in Proposition 26. Let $\mathbf{z}_j(\mathbf{y})$, $j = 1, 2, \dots, m_1$, denote such solutions for some positive integer m_1 independent of $\mathbf{y}$ and ε, which are analytic for ε sufficiently small, and $\mathbf{y}$ around $\mathbf{p}$ (see Lemma 33).
- For each $j \in \{1, 2, \dots, m_1\}$, determine an upper bound on the number of possible values of $\mathbf{y}$ so that $\mathbf{u}(t; \mathbf{y}, \mathbf{z}_j(\mathbf{y}), \varepsilon)$ can possibly be a solution of (8.14) in Proposition 24. Let m_2 be such an upper bound independent of ε (see Lemma 32).

As a consequence, we would conclude that a maximum of $m_1 m_2$ branches

$$\mathbf{u}(t; \mathbf{y}_{ij}, \mathbf{z}_j(\mathbf{y}_{ij}), \varepsilon), \quad 1 \le j \le m_1, 1 \le i \le m_2,$$

can converge to the asymptotic limit considered. We are going to prove in this section that in fact we can take $m_1 = m_2 = n$, the degree of the system.

Theorem 31. *There cannot be more than n^2 branches of critical closed paths in Proposition 24 converging to any given of the asymptotic behaviors of Theorem 30.*

We will start by showing that for each possible fixed j so that

$$\mathbf{u}(t; \mathbf{y}, \varepsilon) \equiv \mathbf{u}(t; \mathbf{y}, \mathbf{z}_j(\mathbf{y}), \varepsilon) \tag{10.1}$$

is a solution of (8.19) in Proposition 26, we can have, at most, n possible values of $\mathbf{y}$, sufficiently close to $\mathbf{p}$ in terms of ε, for which (10.1) is indeed one of the critical paths in Proposition 24. Afterwards, we will concentrate on showing that for each such fixed

$\mathbf{y}$ arbitrarily close to $\mathbf{p}$, there cannot be more than n possible values of j, so that for $\mathbf{z}_j(\mathbf{y})$ sufficiently close to $\mathbf{q}$, the path in (10.1) is one of the critical paths in Proposition 26.

Since system (8.19) with ε positive is analytic in its initial conditions and parameters, though it is only $\mathcal{C}^2$ on t, its solutions $\mathbf{u}(t;\mathbf{y},\varepsilon)$ in (10.1), selected as indicated in the previous paragraph, depend analytically on end-point conditions $\mathbf{y}$ and parameter ε. Therefore for s sufficiently small and for ε fixed, positive and sufficiently small, we can write

$$\mathbf{u}(t;\mathbf{y}+s\overline{\mathbf{y}},\varepsilon)=\mathbf{u}(t;\mathbf{y},\varepsilon)+sD_{\mathbf{y}}\mathbf{u}(t;\mathbf{y},\varepsilon)\overline{\mathbf{y}}+\mathbf{R}(t,s,\varepsilon,\mathbf{y},\overline{\mathbf{y}}), \tag{10.2}$$

with

$$\lim_{s\to 0}\frac{\mathbf{R}(t,s,\varepsilon,\mathbf{y},\overline{\mathbf{y}})}{s}=0$$

for $t\in[0,1]$ and given ε, $\mathbf{y}$, $\overline{\mathbf{y}}$. Note that $\overline{\mathbf{y}}$ can be taken to be unitary. In what follows, and for the sake of simplicity, we use the following notation

$$\mathbf{V}_\varepsilon(t)=D_{\mathbf{y}}\mathbf{u}(t;\mathbf{y}(\varepsilon),\varepsilon)\overline{\mathbf{y}}(\varepsilon),\quad \mathbf{U}_\varepsilon(t)=\mathbf{u}(t;\mathbf{y}(\varepsilon),\varepsilon)$$

once $\mathbf{y}$ and $\overline{\mathbf{y}}$ have been chosen (below) appropriately. Since we can also use the same argument with first and second derivatives with respect to t, we can also write

$$\lim_{s\to 0}\frac{1}{s}\|\mathbf{u}(t;\mathbf{y}+s\overline{\mathbf{y}},\varepsilon)-\mathbf{U}_\varepsilon(t)-s\mathbf{V}_\varepsilon(t)\|_{H^2([0,1];\mathbb{R}^2)}=0$$

for given ε, $\mathbf{y}$, $\overline{\mathbf{y}}$.

Suppose we are dealing with with a finite set of critical closed paths

$$\mathbf{u}_{\varepsilon,i},\quad i=1,2,\dots,m_\varepsilon,$$

having one of the possible asymptotic limits of Theorem 30, and coming from (10.1) for a fixed j. Even though we could have that m_ε could increase with ε, we will focus our attention on a fixed number m, independent of ε but otherwise arbitrary, of such

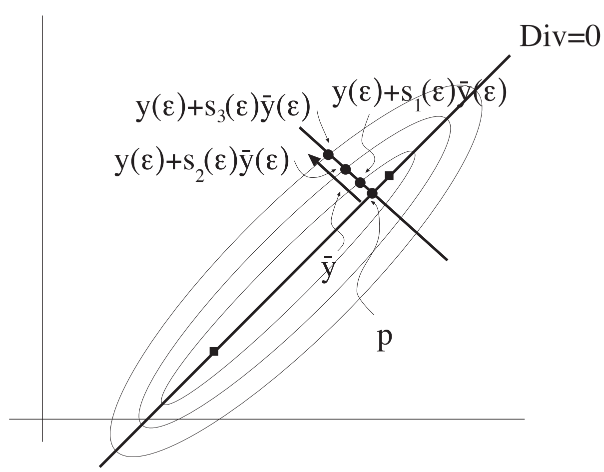

FIGURE 1. Critical closed paths and a transversal section. The square dots are two contact points.

branches of critical closed paths of E_ε. By changing a little bit the starting vector and taking advantage of the periodicity, if necessary, we can put

$$\mathbf{u}_{\varepsilon,i}(t) = \mathbf{u}(t; \mathbf{y}(\varepsilon) + s_i(\varepsilon)\overline{\mathbf{y}}(\varepsilon), \varepsilon) \tag{10.3}$$

for values

$$s_i(\varepsilon), \quad i = 1, 2, \dots, m,$$

and a certain unique (independent of i) unit vector

$$\overline{\mathbf{y}}(\varepsilon), \quad \|\overline{\mathbf{y}}(\varepsilon)\| = 1.$$

(See Figure 1). Under our assumptions, we have that

$$\mathbf{y}(\varepsilon) + s_i(\varepsilon)\overline{\mathbf{y}}(\varepsilon) \text{ and } \mathbf{y}(\varepsilon)$$

converge to $\mathbf{p}$ as $\varepsilon \searrow 0$, and consequently $s_i(\varepsilon) \to 0$ as $\varepsilon \to 0$. In this way, taking into account (10.3), we can write

$$E'_\varepsilon(\mathbf{u}(t; \mathbf{y}(\varepsilon) + s_i(\varepsilon)\overline{\mathbf{y}}(\varepsilon), \varepsilon)) = \mathbf{0}. \tag{10.4}$$

By (10.2) we have

$$\mathbf{u}(t; \mathbf{y}(\varepsilon) + s_i(\varepsilon)\overline{\mathbf{y}}(\varepsilon), \varepsilon) = \mathbf{U}_\varepsilon(t) + s_i(\varepsilon)\mathbf{V}_\varepsilon(t) + \mathbf{R}(t, s_i(\varepsilon), \varepsilon, \mathbf{y}(\varepsilon), \overline{\mathbf{y}}(\varepsilon)), \tag{10.5}$$

with

$$\lim_{\varepsilon \to 0} \frac{\mathbf{R}_{\varepsilon,i}(t)}{s_i(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\mathbf{R}(t, s_i(\varepsilon), \varepsilon, \mathbf{y}(\varepsilon), \overline{\mathbf{y}}(\varepsilon))}{s_i(\varepsilon)} = \mathbf{0}.$$

Bear in mind that $\mathbf{y}(\varepsilon) \to \mathbf{p}$, and that $\overline{\mathbf{y}}(\varepsilon) \to \overline{\mathbf{y}}$ (at least for an appropriate sequence of values for ε) where $\overline{\mathbf{y}}(\varepsilon)$ and $\overline{\mathbf{y}}$ are unitary vectors.

All these preparations are directed to the proof of the following fundamental fact.

Lemma 32. *For each possible asymptotic limit γ given in Theorem 30, there cannot be more than n critical paths $\mathbf{u}_{\varepsilon,i} = \mathbf{u}(t; \mathbf{y}_{ij}, \mathbf{z}_j(\mathbf{y}_{ij}), \varepsilon)$ of the form* (10.1) *for each fixed possible solution $\mathbf{z}_j(\mathbf{y})$ in Proposition 24, converging to γ, i.e. with $\mathbf{y}_{ij}$ converging to $\mathbf{p}$, when $\varepsilon \to 0$.*

Proof. Recall that even though the number m_ε could increase as $\varepsilon \to 0$, we focus on a fixed number m of such critical paths for a fixed ε sufficiently small.

We proceed in two steps. Firstly, we will show that $m \leq 2n + 1$. After that, we argue that indeed, $m \leq n$. The arbitrariness of m will prove our statement.

We would like to consider two families of real functions

$$P_\varepsilon(r) = \langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), \overline{\mathbf{W}}_\varepsilon \rangle, \quad \mathbf{W}_\varepsilon = \frac{\mathbf{V}_\varepsilon}{\|\mathbf{V}_\varepsilon\|}, \quad r = \|\mathbf{V}_\varepsilon\| s, \tag{10.6}$$

$$g_\varepsilon(r) = \langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon + \mathbf{R}_\varepsilon(r)), \overline{\mathbf{W}}_\varepsilon \rangle, \quad \mathbf{R}_\varepsilon(r) = \mathbf{R}(t, r, \varepsilon, \mathbf{y}(\varepsilon), \overline{\mathbf{y}}(\varepsilon)), \tag{10.7}$$

where the path $\overline{\mathbf{W}}_\varepsilon$ in our ambient space $H^2_{O,+1}([0,1]; \mathbb{R}^2)$ will be chosen appropriately later.

Note that

$$g_\varepsilon(r_i(\varepsilon)) = 0$$

if

$$\mathbf{U}_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon + \mathbf{R}_\varepsilon(r_i(\varepsilon))$$

is a critical closed path of E_ε. But $g_\varepsilon(r)$ can be zero for other values of r not corresponding to any of such critical closed paths.

The polynomial $P_\varepsilon(r)$ from (10.6) can be written as

$$\begin{aligned}P_\varepsilon(r) =&\langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), \overline{\mathbf{W}}_\varepsilon\rangle\\ &= \left(\langle E'_0(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), \overline{\mathbf{W}}_\varepsilon\rangle + \langle \varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), \overline{\mathbf{W}}_\varepsilon\rangle + \langle \mathbf{v}_\varepsilon, \overline{\mathbf{W}}_\varepsilon\rangle\right),\end{aligned}$$

because $E'_\varepsilon(\mathbf{u}) = E'_0(\mathbf{u}) + \varepsilon\mathbf{u} + \mathbf{v}_\varepsilon$.

We note that $E'_0(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon)$ cannot be identically zero, otherwise there would be a continuum of closed paths which are solutions of our polynomial differential system (1.1), but this is not possible due to the free election of the closed path $\overline{\mathbf{y}}$ of (10.5). Then the polynomial $P_\varepsilon(r)$ is non-zero.

From (10.7), we can write

$$\begin{aligned}g_\varepsilon(r) =&\langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon) + E''_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon + a_\varepsilon\mathbf{R}_\varepsilon(r))\mathbf{R}_\varepsilon(r), \overline{\mathbf{W}}_\varepsilon\rangle\\ =&P_\varepsilon(r) + \langle E''_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon + a_\varepsilon\mathbf{R}_\varepsilon(r))\mathbf{R}_\varepsilon(r), \overline{\mathbf{W}}_\varepsilon\rangle.\end{aligned}$$

Define the vectors

$$\mathbf{e}_i = E''_\varepsilon(\mathbf{U}_\varepsilon + r_i\mathbf{W}_\varepsilon + a_\varepsilon\mathbf{R}_\varepsilon(r_i))\mathbf{R}_\varepsilon(r_i), \quad i = 1, 2, \ldots, m,$$

and take $\overline{\mathbf{W}}_\varepsilon$ orthogonal (in $H^2_{O,+1}([0,1];\mathbb{R}^2)$) to the subspace generated by $\{\mathbf{e}_i : i = 1, 2, \ldots, m\}$. Conclude that

$$0 = g_\varepsilon(r_i) = P_\varepsilon(r_i), \quad i = 1, 2, \ldots, m.$$

Since the degree of $P_\varepsilon(r)$ is $2n+1$, we conclude that $m \le 2n+1$.

Improving to $m \le n$ is not difficult. Suppose that we could find m distinct branches of critical paths having one of the asymptotic limits described in Theorem 30 as organized above. Another way of focusing on this set of branches of critical paths is by saying that the corresponding constant

$$\pm k_\varepsilon = \mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}' \tag{10.8}$$

for all such critical paths $\mathbf{u} = \mathbf{u}_{\varepsilon,i}$ stays away from zero (see (9.12)) and that each sign in (10.8) corresponds asymptotically to half the unit interval $[0,1]$. Without loss of generality, we may assume that

$$\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}' - k_\varepsilon \to 0 \text{ in } (0,1/2), \quad -\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}' - k_\varepsilon \to 0 \text{ in } (1/2,1), \tag{10.9}$$

when $\varepsilon \to 0$, where k_ε are constants uniformly away from zero.

If we denote by $\mathbf{u} = \mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon$, the first term in $P_\varepsilon(r)$ is

$$\langle E'_0(\mathbf{u}), \overline{\mathbf{W}}_\varepsilon\rangle = \int_0^1 (\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}')[(D\mathbf{F}^\perp(\mathbf{u})\overline{\mathbf{W}}_\varepsilon) \cdot \mathbf{u}' + \mathbf{F}^\perp(\mathbf{u}) \cdot \overline{\mathbf{W}}'_\varepsilon]\, dt,$$

which can also be written in the form

$$\begin{aligned}&\int_0^{1/2} (\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}')[(D\mathbf{F}^\perp(\mathbf{u})\overline{\mathbf{W}}_\varepsilon) \cdot \mathbf{u}' + \mathbf{F}^\perp(\mathbf{u}) \cdot \overline{\mathbf{W}}'_\varepsilon]\, dt +\\ &\int_{1/2}^1 -(\mathbf{F}^\perp(\mathbf{u}) \cdot \mathbf{u}')[-(D\mathbf{F}^\perp(\mathbf{u})\overline{\mathbf{W}}_\varepsilon) \cdot \mathbf{u}' - \mathbf{F}^\perp(\mathbf{u}) \cdot \overline{\mathbf{W}}'_\varepsilon]\, dt,\end{aligned}$$

or even with a change of variables in the second integral, it can be written as

$$\int_0^{1/2}\Big\{(\mathbf{F}^\perp(\mathbf{u}(t))\cdot\mathbf{u}'(t))[(D\mathbf{F}^\perp(\mathbf{u}(t))\mathbf{W}_\varepsilon(t))\cdot\mathbf{u}'(t)+\mathbf{F}^\perp(\mathbf{u}(t))\cdot\overline{\mathbf{W}}'_\varepsilon(t)]-$$
$$-(\mathbf{F}^\perp(\mathbf{u}(t+1/2))\cdot\mathbf{u}'(t+1/2))[-(D\mathbf{F}^\perp(\mathbf{u}(t+1/2)\overline{\mathbf{W}}_\varepsilon(t+1/2))\cdot\mathbf{u}'(t+1/2)]$$
$$-(\mathbf{F}^\perp(\mathbf{u}(t+1/2))\cdot\mathbf{u}'(t+1/2))[-\mathbf{F}^\perp(\mathbf{u}(t+1/2))\cdot\overline{\mathbf{W}}'_\varepsilon(t+1/2)]\Big\}\,dt.$$

The factors within brackets in the integrand of this last integral are polynomials in r of at most degree n, and the other terms in $P_\varepsilon(r)$, namely

$$\left(\langle\varepsilon(\mathbf{U}_\varepsilon+r\mathbf{W}_\varepsilon),\overline{\mathbf{W}}_\varepsilon\rangle+\langle\mathbf{v}_\varepsilon,\overline{\mathbf{W}}_\varepsilon\rangle\right),$$

are polynomials in r of degree 1, if we assume the existence of more than n roots for $P_\varepsilon(r)$ for ε sufficiently small, then since the first factors

$$(\mathbf{F}^\perp(\mathbf{u}(t))\cdot\mathbf{u}'(t)),\quad -(\mathbf{F}^\perp(\mathbf{u}(t+1/2))\cdot\mathbf{u}'(t+1/2))$$

which tend to $k_\varepsilon\neq 0$ as $\varepsilon\to 0$, we reach a contradiction. We can conclude that $P_\varepsilon(r)$ cannot have more than n roots under (10.8). The lemma is proved. □

We now turn to the issue of finding an upper bound on how many values of $\mathbf{z}$, arbitrarily close to $\mathbf{q}$ in terms of ε, for each fixed $\mathbf{y}$ sufficiently close to $\mathbf{p}$, we could find so that

$$\mathbf{u}(t,\mathbf{z},\varepsilon)\equiv\mathbf{u}(t;\mathbf{y},\mathbf{z},\varepsilon) \tag{10.10}$$

is a solution of (8.19) in Proposition 26. We can go through similar arguments as in the proof of Lemma 32 where (10.2) and (10.3) would be replaced, respectively, by

$$\mathbf{u}(t;\mathbf{z}+s\overline{\mathbf{z}},\varepsilon)=\mathbf{u}(t;\mathbf{z},\varepsilon)+sD_{\mathbf{z}}\mathbf{u}(t;\mathbf{z},\varepsilon)\overline{\mathbf{z}}+\mathbf{R}(t,s,\varepsilon,\mathbf{z},\overline{\mathbf{z}}),$$

and

$$\mathbf{u}_{\varepsilon,j}(t)=\mathbf{u}(t;\mathbf{z}(\varepsilon)+s_j(\varepsilon)\overline{\mathbf{z}}(\varepsilon),\varepsilon), \tag{10.11}$$

under (10.10). In a similar way, as in the previous lemma, we have

$$\lim_{s\to 0}\frac{\mathbf{R}(t,s,\varepsilon,\mathbf{z},\overline{\mathbf{z}})}{s}=0$$

for $t\in[0,1]$ and given ε, $\mathbf{z}$, $\overline{\mathbf{z}}$. We introduce the notation

$$\mathbf{V}_\varepsilon(t)=D_{\mathbf{z}}\mathbf{u}(t;\mathbf{z}(\varepsilon),\varepsilon)\overline{\mathbf{z}}(\varepsilon),\quad \mathbf{U}_\varepsilon(t)=\mathbf{u}(t;\mathbf{z}(\varepsilon),\varepsilon)$$

once $\mathbf{z}$ and $\overline{\mathbf{z}}$ have been chosen appropriately.

We would have the following parallel lemma to Lemma 32 exactly with the same proof.

Lemma 33. *For each possible asymptotic limit γ given in Theorem 30, there cannot be more than n critical paths $\mathbf{u}_{\varepsilon,j}$ of the form* (10.11) *in Proposition 26, for each fixed $\mathbf{y}$, converging to γ, i.e. with $\mathbf{z}_j(\mathbf{y})$ converging to $\mathbf{q}$ if $\mathbf{y}$ converges to $\mathbf{p}$ when $\varepsilon\to 0$.*

Proof. As in the proof of Lemma 32, we focus on a fixed number m of the critical paths converging to γ for ε sufficiently small. Again we consider two families of functions

$$Q_\varepsilon(r)=\langle E'_\varepsilon(\mathbf{U}_\varepsilon+r\mathbf{W}_\varepsilon),\overline{\mathbf{W}}_\varepsilon\rangle,\quad \mathbf{W}_\varepsilon=\frac{\mathbf{V}_\varepsilon}{\|\mathbf{V}_\varepsilon\|},\quad r=\|\mathbf{V}_\varepsilon\|s,$$
$$h_\varepsilon(r)=\langle E'_\varepsilon(\mathbf{U}_\varepsilon+r\mathbf{W}_\varepsilon+\mathbf{R}_\varepsilon(r)),\overline{\mathbf{W}}_\varepsilon\rangle,\quad \mathbf{R}_\varepsilon(r)=\mathbf{R}(t,r,\varepsilon,\mathbf{z}(\varepsilon),\overline{\mathbf{z}}(\varepsilon)),$$

where the path $\overline{\mathbf{W}}_\varepsilon$ in our ambient space $H^2_{O,+1}([0,1];\mathbb{R}^2)$ is chosen as in Lemma 32 in order that the signs of the derivatives $h'_\varepsilon(r_i(\varepsilon))$ alternate in the zeros $r_i(\varepsilon)$ of the function $h_\varepsilon(r)$ with ε sufficiently small. Again, as in the proof of Lemma 32, $Q_\varepsilon(r)$ is a polynomial of degree $2n+1$ in the variable r, and we get that $m \leq 2n+1$. After following the arguments of the Lemma 32, we obtain that $m \leq n$. □

According to the initial discussion in this section and Lemmas 32 and 33, the proof of Theorem 31 follows.

11. Counting the number of critical closed paths of E_ε

For ε positive and sufficiently small, and $0 < a_\varepsilon < b_\varepsilon$, we focus on the number of critical closed paths of E_ε in sets of the form

$$\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}.$$

The values a_ε and b_ε are selected complying with the following conditions:

(i) a_ε and b_ε are non-critical values of E_ε;
(ii) each limit cycle of our differential system identifies one, and only one, of the connected components of $\{E_\varepsilon \leq a_\varepsilon\}$;
(iii) if CP_ε is the set of critical paths of E_ε for branches having one of the limit behaviors described in Theorem 30, then

$$\max_{\mathbf{u}\in CP_\varepsilon} E_\varepsilon(\mathbf{u}) \leq b_\varepsilon;$$

(iv) the level set $\{E_\varepsilon \leq b_\varepsilon\}$ has a single connected component.

Condition (ii) is ensured by Theorem 22; conditions (iii) and (iv) are essentially a requirement on the largeness of b_ε. We will express our bound in terms of the following parameters, in addition to the degree n of the system:

- M: the number of connected components of the curve $\text{Div} = 0$;
- N: the number of contact points of the differential system.

Theorem 34. *Under the assumptions and notation of Theorem 2, and for every choice of numbers a_ε and b_ε as indicated above*

$$-\Sigma(\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}) \leq n^2(M+N),$$

and so our differential system cannot have more than $n^2(M+N)$ limit cycles.

Proof. From Theorem 30, we must compute the number of limit behaviors which are contained in the components of the curve $\text{Div} = 0$.

Let $L \geq 0$ and integer. Assume that the connected component i, for

$$i \in \{0, 1, \ldots, L\},$$

of $\text{Div} = 0$ homeomorphic to a straight line contains $x_i \geq 0$ contact points (note that $x_0 = 0$ always). Then it can have at most $x_i + 1$ limit behaviors. Therefore the number of limit critical closed paths contained in the components of Div $= 0$ homeomorphic to a straight line is at most

$$L + \sum_{i=0}^{L} x_i.$$

Let $O \geq 0$ be an integer. Suppose $y_j \geq 0$ is the number of contact points in the j-th component, for

$$j \in \{0, 1, \ldots, O\}$$

of Div $= 0$ homeomorphic to an oval. Note that $y_0 = 0$ always. Then we can have at most

$$\sum_{j=0}^{O} y_j$$

different limit behaviors, all of which are bounded.

In summary, the number of limit critical closed paths contained in the components of Div $= 0$ is at most

$$L + \sum_{i=0}^{L} x_i + \sum_{j=0}^{O} y_j \leq M + N.$$

However, by Theorem 31, each such possible limit behavior must be multiplied by the corresponding multiplicity factor n^2. Hence, we will have at most $n^2(M+N)$ critical closed paths of E_ε for ε sufficiently small. □

From Theorems 31 and 34, Theorem 2 follows.

12. Non-generic situation

In this section we focus on the treatment of polynomial differential systems (1.1) for which either the curve Div $= 0$ has singular points, i.e. the system

$$P_{xx} + Q_{yx} = P_{xy} + Q_{yy} = 0, \quad P_x + Q_y = 0 \tag{12.1}$$

has some solutions; or our initial differential system (1.1) has non-countable, infinitely many contact points, i.e. system (1.2) has a continuum of solutions. Note that equations (12.1) and (1.2) involve the partial derivatives of Div. Our argument revolves around the idea that the vector fields $\mathbf{F}$ of such systems can be uniformly approximated by a sequence $\mathbf{F}_\delta$ of non-singular polynomial vector fields without increasing the degree, and in such a way that the divergence curve of $\mathbf{F}_\delta$ has no singularities, and finitely many contact points with system (1.1).

We can definitely apply our previous results to the family of functionals

$$\begin{aligned} E_{\varepsilon,\delta}(\mathbf{u}) = \int_0^1 \Bigg[& \frac{1}{2}(\mathbf{F}_\delta^\perp(\mathbf{u}(t)) \cdot \mathbf{u}'(t))^2 + \frac{\varepsilon}{2}(|\mathbf{u}''(t)|^2 + |\mathbf{u}'(t)|^2 + |\mathbf{u}(t)|^2) \\ & + (\mathbf{u}(t) \cdot \mathbf{v}_{\varepsilon,\delta}(t) + \mathbf{u}'(t) \cdot \mathbf{v}'_{\varepsilon,\delta}(t) + \mathbf{u}''(t) \cdot \mathbf{v}''_{\varepsilon,\delta}(t)) \\ & \frac{1}{2\varepsilon}(|\mathbf{v}''_{\varepsilon,\delta}(t)|^2 + |\mathbf{v}'_{\varepsilon,\delta}(t)|^2 + |\mathbf{v}_{\varepsilon,\delta}(t)|^2) \Bigg] \, dt, \end{aligned}$$

where the dependence of the smooth paths $\mathbf{v}_{\varepsilon,\delta}$ on δ is as regular as necessary.

Note that our initial results on the bound

$$H(n) \leq \#\{E_\varepsilon \leq a_\varepsilon\}$$

for some sequence of values a_ε away from zero, are valid regardless of whether $\mathbf{F}$ is generic. On the other hand, because $E_{\varepsilon,\delta} \to E_\varepsilon$ as $\delta \to 0$ uniformly over bounded subset of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ (ε is kept fixed here), we can select numbers $a_{\varepsilon,\delta}$ in such a way that $a_{\varepsilon,\delta} \to a_\varepsilon$, as $\delta \to 0$, and

$$\#\{E_{\varepsilon,\delta} \leq a_{\varepsilon,\delta}\} \leq C(n)$$

independently of ε and δ, because this upper bound requires genericity.

Again, because of the convergence,

$$E_{\varepsilon,\delta} \to E_\varepsilon \text{ as } \delta \to 0$$

uniformly over bounded subset of $H^2_{O,+1}([0,1];\mathbb{R}^2)$, we would have the convergence

$$\{E_{\varepsilon,\delta} \leq a_{\varepsilon,\delta}\} \to \{E_\varepsilon \leq a_\varepsilon\}$$

in the sense of epi-convergence or, equivalently, in the sense of Painlevé-Kuratowski (see, for instance, [4]). But since new connected components cannot be created through this convergence of sets, we conclude that

$$H(n) \leq \#\{E_\varepsilon \leq a_\varepsilon\} \leq \liminf_{\delta \to 0} \#\{E_{\varepsilon,\delta} \leq a_{\varepsilon,\delta}\} \leq C(n),$$

and we have the same bound in terms of the degree of the system for a non-generic differential vector field $\mathbf{F}$.

Notice that we are not claiming anything about the relationship between limit cycles of $\mathbf{F}$ and limit cycles of $\mathbf{F}_\delta$, or of critical closed paths for E_ε and of $E_{\varepsilon,\delta}$.

13. Appendix 1. The 16-th Hilbert problem restricted to algebraic limit cycles

Associated with the polynomial differential system (1.1), there is the *polynomial vector field*

$$\mathbf{F} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{13.1}$$

The algebraic curve $f(x,y) = 0$ of $\mathbb{R}^2$ is called an *invariant algebraic curve* of the polynomial vector field $\mathbf{F}$, or of the polynomial differential system (1.1), if for some polynomial $K = K(x,y)$, we have

$$\mathbf{F}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$.

Since on the points of an algebraic curve $f = 0$, the gradient $\nabla f = (f_x, f_y)$ of the curve is orthogonal to the vector field $\mathbf{F}$ (see (13.1)), this vector field is tangent to the curve $f = 0$. Hence, the curve $f = 0$ is formed by orbits of the vector field $\mathbf{F}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f = 0$ satisfying (13.1) for some polynomial K: it is *invariant* under the flow defined by the vector field $\mathbf{F}$.

The next well-known result tell us that we can restrict our attention to the irreducible invariant algebraic curves, for a proof see for instance [31]. Here, as it is usual, $\mathbb{R}[x,y]$ denotes the ring of all polynomials in the variables x and y, and coefficients in $\mathbb{R}$.

Proposition 35. *Let $f \in \mathbb{R}[x,y]$, and let $f = f_1^{m_1} \cdots f_r^{m_r}$ be its factorization in irreducible factors over $\mathbb{R}[x,y]$. Then for a polynomial vector field $\mathbf{F}$, $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor K_{f_i}. Moreover $K_f = m_1 K_{f_1} + \ldots + m_r K_{f_r}$.*

Consider the space Σ' of all real polynomial vector fields (13.1) of degree n having real irreducible invariant algebraic curves.

An *algebraic limit cycle* is an oval of an algebraic curve which is a limit cycle of a polynomial differential system (1.1).

A simpler version of the second part of the 16th Hilbert's problem with respect to the number of limit cycles is: *Is there an uniform upper bound for the maximal number of algebraic limit cycles of any polynomial vector field of Σ'?* We cannot provide an answer to this question for general real algebraic curves, but we give the answer for the following class of algebraic curves.

We say that a set $f_j = 0$, for $j = 1, \ldots, k$, of irreducible algebraic curves is *generic* if it satisfies the following five conditions:

(i) There are no points at which $f_j = 0$ and all its first derivatives vanish (i.e. $f_j = 0$ is a non–singular algebraic curve).
(ii) The highest order homogeneous terms of f_j have no repeated factors.
(iii) If two curves intersect at a point in the affine plane, they are transversal at this point.
(iv) There are no more than two curves $f_j = 0$ meeting at any point in the affine plane.
(v) There are no two curves having a common factor in the highest order homogeneous terms.

The next result was proved by Llibre, Ramírez and Sadovskaia [32] in 2010.

Theorem 36. *For a polynomial vector field $\mathbf{F}$ of degree $n \geq 2$ having all its irreducible invariant algebraic curves generic, the maximum number of algebraic limit cycles is at most $1 + (n-1)(n-2)/2$ if n is even, and $(n-1)(n-2)/2$ if n is odd. Moreover these upper bounds are sharp.*

For cubic polynomial vector fields having all their irreducible invariant algebraic curves generic, Theorem 36 says that one is the maximum number of algebraic limit cycles, but there are examples of cubic polynomial vector fields having two algebraic limit cycles. Such vector fields have non–generic invariant algebraic curves. For example, the polynomial differential system of degree 3

$$\dot{x} = 2y(10 + xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3,$$

has two algebraic limit cycles contained into the invariant algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$, see Proposition 19 of [35].

Until now, all polynomial vector fields having non–generic invariant algebraic curves and more algebraic limit cycles than the upper bounds given in Theorem 36 for the generic case, have odd degree, and at most one more limit cycle than the upper bound of Theorem 36. So, in [32] we conjectured the following.

Conjecture. *The maximum number of algebraic limit cycles that a polynomial differential system of degree n can have is* $1 + (n-1)(n-2)/2$.

Note that the conjectures is true when n is even and we restrict the algebraic limit cycles to generic invariant algebraic curves.

After [32], three other paper have appeared related to the algebraic limit cycles of polynomial differential systems. One due to the same authors [33], and other two due to Xiang Zhang [51].

14. Appendix 2. More on Hilbert's 16th problem

14.1. **On the configuration of the limit cycles of the polynomial differential systems.** A *topological configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Given a topological configuration of limit cycles $C = \{C_1, \ldots, C_n\}$, the curve C_i is *primary* if there is no curve C_j of C contained into the bounded region limited by C_i.

Two topological configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are (*topologically*) *equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^{n} C_i\right) = \left(\cup_{i=1}^{m} C'_i\right)$. Of course, for equivalent configurations of limit cycles C and C', we have that $n = m$.

We say that a polynomial differential system (1.1) *realizes* the configuration C of limit cycles if the set of all limit cycles of (1.1) is equivalent to C.

Llibre and Rodríguez [34] proved the following result in 2004.

Theorem 37. *Let $C = \{C_1, \ldots, C_n\}$ be a topological configuration of limit cycles, and let r be its number of primary curves. Then the following statements hold.*

(a) *The configuration C is realizable by some polynomial differential system.*
(b) *The configuration C is realizable as algebraic limit cycles by a polynomial differential system of degree $\leq 2(n+r)-1$. Moreover, such a polynomial differential system has a first integral of Darboux type.*

Statement (a) of Theorem 37 follows immediately from statement (b).

Statement (a) of Theorem 37 was solved, for the first time, by Schecter and Singer [43], and Sverdlove [47], but they do not provide an explicit polynomial vector field satisfying the given configuration of limit cycles, as it was provided in the proof of statement (b) of Theorem 37.

If $f = f(x, y)$ is a polynomial we denote its partial derivatives with respect to the variables x and y as f_x and f_y, respectively. Christopher [13] proved the following result in 2001.

Theorem 38. *Let $f = 0$ be a non–singular algebraic curve of degree n, and D, a first degree polynomial chosen so that the straight line $D = 0$ lies outside all bounded* *components of $f = 0$. Choose the constants α and β so that $\alpha D_x + \beta D_y \neq 0$, then the polynomial differential system of degree n,*

$$\dot{x} = \alpha f - D f_y, \qquad \dot{y} = \beta f + D f_x,$$

has all the bounded components of $f = 0$ as hyperbolic limit cycles. Furthermore, the differential system has no other limit cycles.

Theorem 38 improves a similar result due to Winkel [49], but the polynomial differential system obtained by Winkel has degree $2n - 1$.

Given a topological configuration of k limit cycles we can consider an equivalent topological configuration formed by circles. Consider then the algebraic curve $f = 0$ formed by the product of all the circles. Applying Theorem 38 to the curve $f = 0$, we obtain a polynomial differential system of degree $n = 2k$, which realizes the given topological configuration of k limit cycles with algebraic limit cycles. A difference between the polynomial differential systems of Theorems 37 and 38, is that the first always has a first integral, and the second, in general, has no first integrals.

In short, both Theorems 37 and 38 show that any topological configuration of limit cycles is realizable with algebraic limit cycles for some polynomial differential system, and provide the degree of such polynomial differential systems. But there are many questions which remains open, as for instance: *what are the possible topological configurations of limit cycles realizable for the polynomial differential systems of a given degree*? This question is definitely more difficult than the question to provide a uniform upper bound for the maximum number of limit cycles that the polynomial differential systems of a given degree can have.

14.2. **Limit cycles and the inverse integrating factor.** Another useful result on the limit cycles of a C^1 differential system in the plane is the following one due to Giacomini, Llibre and Viano [21], see an easier proof in [34]. This result has been used in the proofs of Theorems 36 and 37. First we need a definition.

Let U be an open subset of $\mathbb{R}^2$. A function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of a C^1 vector field $\mathbf{F}$ defined on U if V verifies the linear partial differential system

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V$$

in U.

Theorem 39. *Let X be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V : U \to \mathbb{R}$ be an inverse integrating factor of X. If γ is a limit cycle of X, then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

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