QUADRATIC SYSTEMS WITH AN INVARIANT ALGEBRAIC CURVE OF DEGREE 3 AND A DARBOUX INVARIANT

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ABSTRACT. The planar quadratic systems having a Darboux invariant defined by invariant straight lines of total multiplicity two or by an invariant conic have been studied in [13] and [14], respectively. Here we shall present the normal forms of the planar quadratic systems having an invariant cubic. Moreover we classify the phase portraits in the Poincare disc of all planar quadratic polynomial differential systems with invariant cubic curve and having a Darboux invariant defined by it.

1. Introduction and statements of the results

Even after hundreds of studies on the topology of real planar quadratic vector fields the complete characterization of their phase portraits is a quite complex task. This family of systems depends on twelve parameters but, after affine transformations and time rescaling, we arrive at families with five parameters, which is still a big number of parameters. Many subclasses have been considered.

Denote by $\mathbb{R}[x,y]$ the ring of the real polynomials in the variables x and y. Consider the differential system in $\mathbb{R}^2$ given by

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P, Q \in \mathbb{R}[x,y]$. Here the dot denotes derivative with respect to the *time* t and the degree of system (1) is $m = \max\{\deg P, \deg Q\}$.

When $m = 2$ we say that system (1) is a *quadratic polynomial differential system* or simply a *quadratic system*. More than one thousand papers have been published about quadratic systems, see for instance [15] for a bibliographical survey. The quadratic systems appear in the modeling of many natural phenomena described in different branches of science, in biological and physical applications. Besides the applications the quadratic systems became a matter of interest for the mathematicians. Considering algebraic invariant curves, some authors have published on the subject, for example, [3] and [12]. In the first one the authors studied cubic systems with invariant straight lines of total multiplicity eight that have three distinct infinite singularities. The second paper is dedicated to study the normal forms and global phase portraits of quadratic and cubic integrable systems when they have two nonconcentric circles as invariant algebraic curves.

In this paper we assume that the polynomials P and Q are coprime, otherwise system (1) can be reduced to a linear or constant system doing a rescaling of the time variable.

The first objective of this paper is to characterize all quadratic systems having invariant cubics. Then using the normal forms obtained, we investigate which systems have a Darboux invariant of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ if the cubic is the product of three straight

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