Limit cycles of piecewise polynomial perturbations of higher dimensional linear differential systems

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Abstract. The averaging theory has been extensively employed for studying periodic solutions of smooth and nonsmooth differential systems. Here, we extend the averaging theory for studying periodic solutions a class of regularly perturbed non–autonomous n-dimensional discontinuous piecewise smooth differential system. As a fundamental hypothesis, it is assumed that the unperturbed system has a manifold $\mathcal{Z} \subset \mathbb{R}^n$ of periodic solutions satisfying $\dim(\mathcal{Z}) < n$. Then, we apply this result to study limit cycles bifurcating from periodic solutions of linear differential systems, $x' = Mx$, when they are perturbed inside a class of discontinuous piecewise polynomial differential systems with two zones. More precisely, we study the periodic solutions of the following differential system

$$x' = Mx + \varepsilon F_1^n(x) + \varepsilon^2 F_2^n(x),$$

in $\mathbb{R}^{d+2}$ where ε is a small parameter, M is a $(d+2)\times(d+2)$ matrix having one pair of pure imaginary conjugate eigenvalues, m zeros eigenvalues, and $d - m$ non–zero real eigenvalues.

1. Introduction

The analysis of discontinuous piecewise smooth differential systems has recently had a large and fast growth due to its applications in several areas of the knowledge. Such systems model many phenomena in control systems (see [1]), impact on mechanical systems (see [2]), economy (see [17]), biology (see [18]), nonlinear oscillations (see [27]), neuroscience (see [8, 13, 28]), and other fields of science.

Establishing the existence of limit cycles is one of the major problem in the theory of differential systems. The interest in detecting such objects is due to the

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fact that they are non-local invariant sets providing information on the qualitative behavior of the system. The first studies on this subject considered smooth differential systems and, since then, many contributions have been made in this direction (see [15] and the references therein). The study of limit cycles has also been considered for continuous (see, for instance, [4,23,25]) and discontinuous piecewise smooth differential systems (see, for instance, [11,14,19,20,26]). Most of them are concentrated on planar piecewise differential systems.

The *averaging theory* is one of the main tools for studying periodic solutions in regularly perturbed differential systems of the form

$$\dot{x} = F_0(t,\mathbf{x}) + \sum_{i=1}^{k} \varepsilon^i F_i(t,\mathbf{x}) + \varepsilon^{k+1} R(t,\mathbf{x},\varepsilon),\ (t,\mathbf{x},\varepsilon) \in \mathbb{R}\times D\times(-\varepsilon_0,\varepsilon_0), \tag{1.1}$$

where D is an open bounded subset of $\mathbb{R}^n$ and the functions F_i, $i = 0, 1, \dots, k$, and R are T-periodic in the first variable. Here, k is called *order of perturbation* in ε. As a fundamental hypothesis, it is assumed that the *unperturbed system*,

$$\dot{x} = F_0(t,\mathbf{x}), \tag{1.2}$$

has a manifold $\mathcal{Z} \subset \mathbb{R}^n$ of periodic solutions. Roughly speaking, this theory provides a sequence of functions, called *averaged functions*, which have their simple zeros associated with limit cycles of system (1.1).

The averaging theory has been extensively employed for studying periodic solutions of smooth and nonsmooth differential systems. First, considering $F_0 = 0$ (consequently, $\dim(\mathcal{Z}) = n$) one can find in [31, 32] results providing sufficient condition on F_1 ensuring the existence of periodic solutions of system (1.1) under smoothness and boundedness conditions. Topological methods were used in [4] to generalize these results for Lipschitz continuous differential systems. In [23], assuming the weaker hypothesis $\dim(\mathcal{Z}) = n$, the averaging theory was developed at any order for Lipschitz continuous differential systems. Then, in [20, 24], the averaging theory was extended up to order 2 for detecting periodic orbits of discontinuous piecewise smooth differential systems. Some applications of these results can be found in [26,29]. Finally, in [16,22], the averaging theory was developed at any order for a class of discontinuous piecewise smooth systems.

When $\dim(\mathcal{Z}) < n$, the averaging theory has to be combined with other techniques, for instance *Lyapunov-Schmidt reduction method*, to provide sufficient conditions for the existence of periodic solutions. Here, we also obtain a sequence of function, now called *bifurcation functions*, which have their simple zeros associated with limit cycles of system (1.1). In the smooth case, the averaging theory is developed at any order [3, 5, 10]. For the nonsmooth case, the first order averaging theory has been addressed in [30], however it is lacking in a higher order analysis.

In this paper, our first main goal is to develop the averaging theory up to order 2 in ε for a class of discontinuous piecewise smooth differential systems assuming $\dim(\mathcal{Z}) = d < n$. The study of any finite order in ε could be performed in a similar way, however the general expression for higher order bifurcation functions would be more complex because it involves higher derivatives of composite functions. As our second main goal, we apply this result to study the number of limit cycles bifurcating from the periodic orbits of a linear differential system $x' = Mx$, where M is a $(d+2)\times(d+2)$ matrix having one pair of pure imaginary conjugate eigenvalues, m zeros eigenvalues, and $d-m$ real eigenvalues. We focus our attention when this system is perturbed up to order 2 in the small parameter ε inside a class of discontinuous piecewise polynomial functions having two zones.

This paper is organized as follows. In Section 2, we state our main results: Theorem 1, improving the averaging theory for nonsmooth systems; and Theorems 3-5, regarding piecewise polynomial perturbations of higher dimensional linear systems. In Section 3, we provide some preliminary results. The remainder Sections 4-7 are devoted to the proofs of Theorem 1 and Theorems 3-5.

2. Statements of the main results

2.1. Advances on averaging theory

In this subsection we improve the averaging theory of first and second order to study the limit cycles of a class of discontinuous piecewise smooth differential systems.

Let D be an open bounded subset of $\mathbb{R}^{d+1}$ and for a positive real number T we consider the $\mathcal{C}^3$ differentiable functions $F_i^{\pm}:\mathbb{S}^1\times D\to\mathbb{R}^{d+1}$ for $i=0,1,2$, and $R^{\pm}:\mathbb{S}^1\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^{d+1}$ where $\mathbb{S}^1\equiv\mathbb{R}/(\mathbb{Z}T)$. Thus, we define the following T-periodic *discontinuous piecewise smooth differential system*

$$\mathbf{x}' = \begin{cases} F^+(\theta,\mathbf{x},\varepsilon) & \text{if} \quad 0\le\theta\le\phi, \\ F^-(\theta,\mathbf{x},\varepsilon) & \text{if} \quad \phi\le\theta\le T, \end{cases} \tag{2.1}$$

where the prime denotes derivative with respect to the variable $\theta\in\mathbb{S}^1$, and

$$F^{\pm}(\theta,\mathbf{x},\varepsilon) = F_0^{\pm}(\theta,\mathbf{x}) + \varepsilon F_1^{\pm}(\theta,\mathbf{x}) + \varepsilon^2 F_2^{\pm}(\theta,\mathbf{x}) + \varepsilon^3 R^{\pm}(\theta,\mathbf{x},\varepsilon),$$

with $\mathbf{x}\in D$. The set of discontinuity of system (2.1) is given by $\Sigma=\{\theta=0\}\cup\{\theta=\phi\}$.

For $\mathbf{z}\in D$, let $\varphi(\theta,\mathbf{z})$ be the solution of the unperturbed system

$$\mathbf{x}' = F_0(\theta,\mathbf{x}), \tag{2.2}$$

such that $\varphi(0,\mathbf{z})=\mathbf{z}$, where

$$F_0(\theta,\mathbf{x}) = \begin{cases} F_0^+(\theta,\mathbf{x}) & \text{if} \quad 0\le\theta\le\phi, \\ F_0^-(\theta,\mathbf{x}) & \text{if} \quad \phi\le\theta\le T. \end{cases}$$

Clearly,

$$\varphi(\theta,\mathbf{z}) = \begin{cases} \varphi^+(\theta,\mathbf{z}) & \text{if} \quad 0\le\theta\le\phi, \\ \varphi^-(\theta,\mathbf{z}) & \text{if} \quad \phi\le\theta\le T, \end{cases}$$

where $\varphi^{\pm}(\theta,\mathbf{z})$ are the solutions of the systems

$$\mathbf{x}' = F_0^{\pm}(\theta,\mathbf{x}), \tag{2.3}$$

such that $\varphi^{\pm}(0,\mathbf{z}) = \mathbf{z}$.

We assume that there exists a manifold $\mathcal{Z}$ embedded in D such that the solutions starting in $\mathcal{Z}$ are all T-periodic. More precisely, for $p = d+1$, $q \le p$ and V an open bounded subset of $\mathbb{R}^q$, let $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be a $\mathcal{C}^3$ function and define

$$\mathcal{Z} = \{\mathbf{z}_\nu = (\nu,\sigma(\nu)) : \nu \in \overline{V}\}. \tag{2.4}$$

We shall assume that

(H) $\mathcal{Z} \subset D$ and for each $\mathbf{z}_\nu$ the unique solution $\varphi(\theta,\mathbf{z}_\nu)$ such that $\varphi(0,\mathbf{z}_\nu) = \mathbf{z}_\nu$ is T-periodic.

For $\mathbf{z} \in D$ we consider the first order variational equations of systems (2.3) along the solution $\varphi^{\pm}(\theta,\mathbf{z})$, that is

$$Y' = D_{\mathbf{x}} F_0^{\pm}(\theta,\varphi^{\pm}(\theta,\mathbf{z}))\, Y. \tag{2.5}$$

Denote by $Y^{\pm}(\theta,\mathbf{z})$ a fundamental matrix of the differential system (2.5).

Let $\xi : \mathbb{R}^q \times \mathbb{R}^{p-q} \to \mathbb{R}^q$ and $\xi^{\perp} : \mathbb{R}^q \times \mathbb{R}^{p-q} \to \mathbb{R}^{p-q}$ be the orthogonal projections onto the first q coordinates and onto the last $p-q$ coordinates, respectively. For a point $\mathbf{z} \in$ D denote $\mathbf{z} = (u,v) \in \mathbb{R}^q \times \mathbb{R}^{p-q}$. Before defining the bifurcation functions we have to define some auxiliar functions. Let

$$\begin{aligned}
y_0^{\pm}(\theta,\mathbf{z}) &= \varphi^{\pm}(\theta,\mathbf{z}),\\
y_1^{\pm}(\theta,\mathbf{z}) &= Y^{\pm}(\theta,\mathbf{z}) \int_0^\theta Y^{\pm}(s,\mathbf{z})^{-1} F_1^{\pm}(s,\varphi^{\pm}(s,\mathbf{z}))ds,\\
y_2^{\pm}(\theta,\mathbf{z}) &= Y^{\pm}(\theta,\mathbf{z}) \int_0^\theta Y^{\pm}(s,\mathbf{z})^{-1} \Bigg(2F_2^{\pm}(s,\varphi^{\pm}(s,\mathbf{z})) + \\
&\quad 2\frac{\partial F_1^{\pm}}{\partial \mathbf{x}}(s,\varphi^{\pm}(s,\mathbf{z}))y_1^{\pm}(s,\mathbf{z}) + \frac{\partial^2 F_0^{\pm}}{\partial \mathbf{x}^2}(s,\varphi^{\pm}(s,\mathbf{z}))y_1^{\pm}(s,\mathbf{z})^2 \Bigg) ds.
\end{aligned} \tag{2.6}$$

In the formula of $y_2^{\pm}(\theta,\mathbf{z})$, the second derivative $\dfrac{\partial^2 F_0^{\pm}}{\partial \mathbf{x}^2}(s,\varphi^{\pm}(s,\mathbf{z}))$ is a bilinear form defined on $\mathbb{R}^p \times \mathbb{R}^p$ which is applied to a "product" of two vectors, in our case $y_1^{\pm}(s,\mathbf{z})^2$.

Now, consider

$$g_i(\mathbf{z}) = y_i^{+}(\phi,\mathbf{z}) - y_i^{-}(\phi - T,\mathbf{z}), \text{ for } i = 0,1,2. \tag{2.7}$$

The functions g_1 and g_2 are usually called averaged functions of order 1 and 2, respectively. Finally, assuming that the lower right corner $(p-q)\times(p-q)$ matrix of $Y^{+}(\phi,\nu) - Y^{-}(\phi - T,\nu)$, denoted by Δ_ν, is invertible, we define

$$\gamma(\nu) = -\Delta_\nu^{-1}\xi^{\perp} g_1(\mathbf{z}_\nu). \tag{2.8}$$

Hence, the bifurcation functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ of order 1 and 2 are given, respectively, by

$$
\begin{aligned}
f_1(\nu) &= \xi g_1(\mathbf{z}_\nu), \\
f_2(\nu) &= 2\frac{\partial \xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu) + \frac{\partial^2 \xi g_0}{\partial v^2}(\mathbf{z}_\nu)\gamma(\nu)^2 + 2\xi g_2(\mathbf{z}_\nu).
\end{aligned}
\tag{2.9}
$$

Again, in the formula of f_2, the second derivative $\dfrac{\partial^2 \xi g_0}{\partial v^2}(\mathbf{z}_\nu)$ is a bilinear form defined on $\mathbb{R}^{(p-q)} \times \mathbb{R}^{(p-q)}$. Thus, as before, we say that it is applied to a "product" of two vectors, in our case, $\gamma(\nu)^2$.

Our main result on the periodic solutions of system (2.1) is the following.

Theorem 1. *In addition to hypothesis (H), we assume that for any $\nu \in \overline{V}$ the matrix $Y^+(\phi, \nu) - Y^-(\phi - T, \nu)$ has in the upper right corner the null $q \times (p-q)$ matrix, and in the lower right corner has the $(p-q) \times (p-q)$ matrix Δ_ν with* $\det(\Delta_\nu) \neq 0$. *Then, the following statements hold.*

- (a) *If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $\det(f_1'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system* (2.1) *such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*

- (b) *Assume that $f_1 \equiv 0$. If there exists $\nu^* \in V$ such that $f_2(\nu^*) = 0$ and* $\det(f_2'(\nu^*)) \neq 0$, *then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system* (2.1) *such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*

Theorem 1 is proved in Section 4. The following result is an immediate consequence of Theorem 1.

Corollary 2. *Assume the hypothesis (H) and that $q = p$, in this case $\mathcal{Z} = \overline{V} \subset D$ is a compact bounded p–dimensional manifold. Then, statements (a) and (b) of Theorem 1 hold by taking $f_1 = g_1$ and $f_2 = 2g_2$.*

2.2. Perturbations of higher dimensional linear systems

Consider a $(d+2) \times (d+2)$ matrix M given by

$$
M = \begin{pmatrix} 0 & -1 & 0_{1\times d} \\ 1 & 0 & 0_{1\times d} \\ 0_{d\times 1} & 0_{d\times 1} & \widetilde{M} \end{pmatrix},
$$

where $0_{i\times j}$ denotes a null $i \times j$ matrix. When $0 < m < d$ assume that $\widetilde{M}$ is the diagonal matrix diag$(\mu_1,\ \mu_2,\ \ldots,\ \mu_d)$ with $\mu_1 = \ldots = \mu_m = 0$ and $\mu_{m+1} \neq 0, \ldots, \mu_d \neq 0$. If $m = 0$, then $\widetilde{M}$ is a diagonal matrix with all entries distinct from zero, and if $m = d$ we assume that $\widetilde{M}$ is the null matrix.

Let $L_1 = \{(x,0,z) : x \geq 0,\, \mathrm{z} \in \mathbb{R}^d\}$ and $L_2 = \{(\lambda\cos\phi, \lambda\sin\phi, z) : \lambda \geq 0,\, \mathrm{z} \in \mathbb{R}^d\}$ be two half–hyperplanes of $\mathbb{R}^{d+2}$ sharing the boundary $\{0,0,z) : \mathrm{z} \in \mathbb{R}^d\}$. The set $\Sigma = L_1 \cup L_2$ splits $D \subset \mathbb{R}^{d+2}$ in 2 disjoint open sectors, namely C^+ and C^- (see Figure 1).

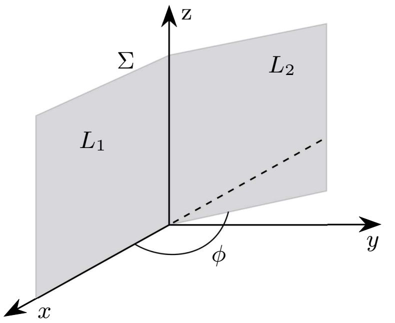

Figure 1: Set of discontinuity Σ.

We will denote by X_λ and Y_λ two polynomials of degree n in the variables $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$, more precisely

$$X_\lambda(x,y,\mathrm{z}) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d}\, x^i y^j z_1^{k_1} \ldots z_d^{k_d}, \text{ and}$$
$$Y_\lambda(x,y,\mathrm{z}) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d}\, x^i y^j z_1^{k_1} \ldots z_d^{k_d},$$

for $\lambda_{ijk_1\ldots k_d} \in \mathbb{R}$ and $i, j, k_1, \ldots, k_d \in \mathbb{N}$. Then, take

(2.10) $$X^\pm = (X_{a^\pm}, X_{b^\pm}, X_{c_1^\pm}, \ldots, X_{c_d^\pm}), \quad Y^\pm = (Y_{\alpha^\pm}, Y_{\beta^\pm}, Y_{\gamma_1^\pm}, \ldots, Y_{\gamma_d^\pm}),$$

and let $\mathcal{X}(x,y,\mathrm{z})$ and $\mathcal{Y}(x,y,\mathrm{z})$ be polynomial vector fields defined by

$$\mathcal{X}(x,y,\mathrm{z}) = X^\pm(x,y,\mathrm{z}) \quad \text{if} \quad (x,y,\mathrm{z}) \in C^\pm,$$
$$\mathcal{Y}(x,y,\mathrm{z}) = Y^\pm(x,y,\mathrm{z}) \quad \text{if} \quad (x,y,\mathrm{z}) \in C^\pm.$$

Now, consider the discontinuous piecewise polynomial differential systems

(2.11) $$(\dot{x}, \dot{y}, \dot{\mathrm{z}}) = M\,(x,y,\mathrm{z}) + \varepsilon\mathcal{X}\,(x,y,\mathrm{z}) + \varepsilon^2\mathcal{Y}\,(x,y,\mathrm{z})\,,$$

where $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$. The dot denotes derivative with respect to the time t, and Σ denotes the set of discontinuity for system (2.11). Also, $M\,(x,y,\mathrm{z})$ is an abuse of notation and denotes the matrix M applied to the vector (x,y,z), which is defined as the product between the matrix M with the column matrix associated with the vector (x,y,z). This abuse of notation will be recurrent throughout the paper.

Denote by $N_i(m,n,\phi)$ the maximum number of limit cycles of system (2.11) that can be detected using averaging theory of order i when $|\varepsilon| \neq 0$ is sufficiently small.

Theorem 3. *Assume* $0 \leq m \leq d$*,* $n \in \mathbb{N}$*, and* $\phi \in (0,2\pi) \setminus \{\pi\}$*. Then,*

(*a*) $N_1(m,n,\phi) = n^{m+1}$ *and*

(*b*) $2n(2n-1)^m \leq N_2(m,n,\phi) \leq (2n)^{m+1}$.

Theorem 3 generalizes the particular case $m = d$ of [26]. Comparing itens (a) and (b) of Theorem 3, we can easily check that $N_2(m,n,\phi) > N_1(m,n,\phi)$ for every $0 \leq m \leq d$, $n \in \mathbb{N}$, and $\phi \in (0,2\pi) \setminus \{\pi\}$.

Notice that, the lower and upper bounds given in statement (b) of Theorem 3 coincide for $m = 0$. In this case, $N_2(0,n,\phi) = 2n$. In general, the lower bound of statement (b) of Theorem 3 is not optimal and can be improved in some cases (see Proposition 5.1).

Theorems 3 is proved in section 5.

If $\phi = \pi$ we note that the maximum number of limit cycles eventually decreases as stated in the following result.

Theorem 4. *Assume* $0 \leq m \leq d$ *and* $\phi = \pi$*. Then,*

(*a*) $N_1(m,n,\pi) = n^{m+1}$ *and*

(*b*) $N \leq N_2(m,n,\pi) \leq (2n)^{m+1}$ *where* $N = (2n-1)^{m+1}$ *if* n *is odd, and* $N = (2n-2)(2n-1)^m$ *if* n *is even.*

Theorem 4 is proved in Section 6.

Comparing itens (a) and (b) of Theorem 4, we can check that $N_2(m,n,\pi) \geq N_1(m,n,\pi)$ for every $0 \leq m \leq d$ and $n \in \mathbb{N}$, with strictly inequality for $n \neq 1$.

When $\phi = 2\pi$, system (2.11) is continuous. In this case $\mathcal{X}(x,y,\mathrm{z}) = X^+(x,y,\mathrm{z})$ and $\mathcal{Y}(x,y,\mathrm{z}) = Y^+(x,y,\mathrm{z})$. So, we get the following result.

Theorem 5. *Assume that* $0 \leq m \leq d$ *and* $\phi = 2\pi$*. Then,*

(*a*) $N_1(m,n,2\pi) = n^m(n-1)/2$ *for all* $m \neq 0$*, and*

$$N_1(0,n,2\pi) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

(*b*) $n \leq N_2(0,n,2\pi) \leq 2n$.

Theorem 5 generalizes the particular cases $m = d = 0$ and $m = d = 1$ of [7] (see Theorems 2 and 3). Moreover, statement (a) of Theorem 5 also generalizes Theorem 1 of [26] when $m = d$. We prove Theorem 5 in Section 7.

3. Preliminary results

In this section we present some preliminaries results that we shall need in Sections 5, 6 and 7. In Section 3.1, we present a change of coordinates so that system (2.11) reads in the standard form (2.1) to apply the averaging method. In Section 3.2, we construct the averaging functions f_1 and f_2 for system (2.11), defined in (2.9). Finally, in Section 3.3 we present some trigonometric relations that will be used in the calculus of the zeros of the functions f_1 and f_2.

3.1. Standard form

Let $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Using the change of variables

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{3.1}$$

with $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$, system (2.11) becomes

$$\left(\dot\theta, \dot r, \dot{\mathrm{z}}\right) = (1, 0, \widetilde{M}\mathrm{z}) + \varepsilon A(\theta, r, \mathrm{z}) + \varepsilon^2 B(\theta, r, \mathrm{z}), \tag{3.2}$$

where $A, B : \mathbb{S}^1 \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d+2}$ are piecewise smooth functions given by

$$A = \begin{cases} A^+ & \text{if} \quad 0 \le \theta \le \phi, \\ A^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases} \qquad \text{and} \quad B = \begin{cases} B^+ & \text{if} \quad 0 \le \theta \le \phi, \\ B^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases}$$

where

$$\begin{aligned} A^\pm(\theta, r, \mathrm{z}) = &\quad (A_1^\pm(\theta, r, \mathrm{z}), \ldots, A_{d+2}^\pm(\theta, r, \mathrm{z})), \\ B^\pm(\theta, r, \mathrm{z}) = &\quad (B_1^\pm(\theta, r, \mathrm{z}), \ldots, B_{d+2}^\pm(\theta, r, \mathrm{z})), \end{aligned}$$

with

$$\begin{aligned} A_1^\pm = &\quad \frac{1}{r}\big(X_{b^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\cos\theta - X_{a^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\sin\theta\big), \\ B_1^\pm = &\quad \frac{1}{r}\big(Y_{\beta^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\cos\theta - Y_{\alpha^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\sin\theta\big), \\ A_2^\pm = &\quad X_{a^\pm}(r\cos\theta, r\sin\theta, z)\cos\theta + X_{b^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\sin\theta, \\ B_2^\pm = &\quad Y_{\alpha^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\cos\theta + Y_{\beta^\pm}(r\cos\theta, r\sin\theta, \mathrm{z})\sin\theta, \\ A_{\ell+2}^\pm = &\quad X_{c_\ell^\pm}(r\cos\theta, r\sin\theta, \mathrm{z}), \\ B_{\ell+2}^\pm = &\quad Y_{\gamma_\ell^\pm}(r\cos\theta, r\sin\theta, \mathrm{z}), \end{aligned} \tag{3.3}$$

for $1 \le \ell \le d$. Clearly the discontinuity Σ is now given by

$$\Sigma = \{(0, r, \mathrm{z}) : r \in \mathbb{R}_+, \mathrm{z} \in \mathbb{R}^d\} \cup \{(\phi, r, \mathrm{z}) : r \in \mathbb{R}_+, \mathrm{z} \in \mathbb{R}^d\}.$$

Taking the angle θ as the new time, system (3.2) reads

$$
\begin{aligned}
r' &= \frac{\dot r}{\dot\theta} = \frac{\varepsilon A_2(\theta,r,\mathrm{z}) + \varepsilon^2 B_2(\theta,r,\mathrm{z})}{1+\varepsilon A_1(\theta,r,\mathrm{z}) + \varepsilon^2 B_1(\theta,r,\mathrm{z})},\\
z_\ell' &= \frac{\dot z_\ell}{\dot\theta} = \frac{\mu_\ell z_\ell + \varepsilon A_{\ell+2}(\theta,r,\mathrm{z}) + \varepsilon^2 B_{\ell+2}(\theta,r,\mathrm{z})}{1+\varepsilon A_1(\theta,r,\mathrm{z}) + \varepsilon^2 B_1(\theta,r,\mathrm{z})},
\end{aligned} \tag{3.4}
$$

for $1 \le \ell \le d$. Note that now the prime denotes derivative with respect to the independent variable θ.

Expanding system (3.4) in Taylor series around $\varepsilon = 0$, it can be written as system (2.1) by taking $\mathbf{x} = (r,\mathrm{z}) \in D \subset \mathbb{R}_+ \times \mathbb{R}^d$ and

$$
F_j^\pm(\theta,r,\mathrm{z}) = (F_{j0}^\pm(\theta,r,\mathrm{z}),\dots,F_{jd}^\pm(\theta,r,\mathrm{z})), \quad \text{for} \quad j=0,1,2, \tag{3.5}
$$

where

$$
\begin{aligned}
F_{0\ell}^\pm(\theta,r,\mathrm{z}) =& \quad 0,\\
F_{0\omega}^\pm(\theta,r,\mathrm{z}) =& \quad \mu_\omega z_\omega,\\
F_{1\ell}^\pm(\theta,r,\mathrm{z}) =& \quad A_{\ell+2}^\pm(\theta,r,\mathrm{z}),\\
F_{1\omega}^\pm(\theta,r,\mathrm{z}) =& \quad A_{\omega+2}^\pm(\theta,r,\mathrm{z}) - \mu_\omega z_\omega A_1^\pm(\theta,r,\mathrm{z}),\\
F_{2\ell}^\pm(\theta,r,\mathrm{z}) =& \quad B_{\ell+2}^\pm(\theta,r,\mathrm{z}) - A_1^\pm(\theta,r,\mathrm{z})A_{\ell+2}^\pm(\theta,r,\mathrm{z}),\\
F_{2\omega}^\pm(\theta,r,\mathrm{z}) =& \quad B_{\omega+2}^\pm(\theta,r,\mathrm{z}) + \mu_\omega z_\omega \big(A_1^\pm(\theta,r,\mathrm{z})\big)^2\\
& -A_1^\pm(\theta,r,\mathrm{z})A_{\omega+2}^\pm(\theta,r,\mathrm{z}) - \mu_\omega z_\omega B_1^\pm(\theta,r,\mathrm{z}),
\end{aligned} \tag{3.6}
$$

for $0 \le \ell \le m$ and $m+1 \le \omega \le d$.

When $m = d$ the functions $F_{j\omega}^\pm$, for $j = 0,1,2$, do not be considered.

3.2. Construction of the averaging functions

Now, we shall use the notations introduced in subSection 2.1. Since the unperturbed system (2.2) is continuous, we have $\varphi^+(\theta,\mathbf{z}) = \varphi^-(\theta,\mathbf{z})$. Therefore, when $0 \le m < d$ the solution of system (2.2) is given by

$$
\varphi(\theta,\mathbf{z}) = (r, z_1,\dots,z_m, e^{\mu_{m+1}\theta} z_{m+1},\dots,e^{\mu_d\theta} z_d),
$$

for $\mathbf{z} = (r,\mathrm{z}) = (r,z_1,\dots,z_d)$. Note that if $\mathbf{z}_\nu = (r,z_1,\dots,z_m,0,\dots,0)$ then $\varphi(\theta,\mathbf{z}_\nu) = \mathbf{z}_\nu$ for every $\theta \in \mathbb{S}^1$. Then, taking an open bounded subset $V \subset \mathbb{R}^{m+1}$ and the zero function $\sigma : \overline{V} \to \mathbb{R}^{d-m}$, the manifold $\mathcal{Z}$, defined in (2.4), becomes

$$
\mathcal{Z} = \{\mathbf{z}_\nu = (\nu,0) \in \mathbb{R}^{d+1} : \nu = (r,z_1,\dots,z_m) \in \overline{V}\}.
$$

For $\mathbf{z} \in D$ a fundamental matrix of system (2.5) is

$$Y(\theta, \mathbf{z}) = \begin{pmatrix} \mathrm{Id}_{1+m} & 0 \\ 0 & \Delta \end{pmatrix},$$

where Id_{1+m} is the $(1+m) \times (1+m)$ identity matrix, and Δ is the diagonal matrix $\mathrm{diag}(e^{\mu_{m+1}\theta}, \ldots, e^{\mu_d \theta})$. Since $Y(\theta, \mathbf{z})$ does not depend of $\mathbf{z}$ we denote $Y(\theta, \mathbf{z}) = Y(\theta)$. Then, we have

$$Y(\phi) - Y(\phi - 2\pi) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_\nu \end{pmatrix},$$

where

$$\Delta_\nu = \mathrm{diag}\big(e^{\mu_{m+1}\phi}(1 - e^{-\mu_{m+1}2\pi}), \ldots, e^{\mu_d \phi}(1 - e^{-\mu_d 2\pi})\big). \tag{3.7}$$

According to the notation introduced in Theorem 1 we have $p = d + 1$ and $p - q = d - m$, with $q = m + 1$. Since $\mathcal{Z}$ has dimension $m + 1$, we consider the projections $\xi : \mathbb{R}^{m+1} \times \mathbb{R}^{d-m} \to \mathbb{R}^{m+1}$ and $\xi^\perp : \mathbb{R}^{m+1} \times \mathbb{R}^{d-m} \to \mathbb{R}^{d-m}$, with $u = (r, z_1, \ldots, z_m) \in \mathbb{R}^{m+1}$ and $v = (z_{m+1}, \ldots, z_d) \in \mathbb{R}^{d-m}$.

From (2.6) and (3.6) we have $y_1(\theta, \mathbf{z}) = \big(y_{10}(\theta, \mathbf{z}), \ldots, y_{1d}(\theta, \mathbf{z})\big)$ where

$$\begin{aligned} y_{1\ell}^{\pm}(\theta, \mathbf{z}) = & \int_0^\theta A_{\ell+2}^{\pm}(s, \varphi(s, \mathbf{z}))ds, \\ y_{1\omega}^{\pm}(\theta, \mathbf{z}) = & \int_0^\theta e^{\mu_\omega(\theta - s)}\big(A_{\omega+2}^{\pm}(s, \varphi(s, \mathbf{z})) - \mu_\omega z_\omega A_1^{\pm}(s, \varphi(s, \mathbf{z}))\big)ds, \end{aligned} \tag{3.8}$$

for $0 \le \ell \le m$ and $m + 1 \le \omega \le d$.

Moreover, from (2.7) we have $g_1(\mathbf{z}_\nu) = \big(g_{10}(\mathbf{z}_\nu), \ldots, g_{1d}(\mathbf{z}_\nu)\big)$ with

(3.9)
$$\begin{aligned} g_{1\ell}(\mathbf{z}_\nu) = & \int_0^\phi A_{\ell+2}^{+}(s, \varphi(s, \mathbf{z}_\nu))\, ds + \int_\phi^{2\pi} A_{\ell+2}^{-}(s, \varphi(s, \mathbf{z}_\nu))\, ds, \\ g_{1\omega}(\mathbf{z}_\nu) = & \int_0^\phi e^{\mu_\omega(\phi - s)} A_{\omega+2}^{+}(s, \varphi(s, \mathbf{z}_\nu))\, ds + \int_\phi^{2\pi} e^{\mu_\omega(\phi - 2\pi - s)} A_{\omega+2}^{-}(s, \varphi(s, \mathbf{z}_\nu))\, ds \end{aligned}$$

for $0 \le \ell \le m$ and $m + 1 \le \omega \le d$.

Therefore, the bifurcation function $f_1 : \overline{V} \to \mathbb{R}^{m+1}$, defined in (2.9), is given by

$$f_1(\nu) = \xi g_1(\mathbf{z}_\nu) = \big(f_{10}(\nu), \ldots, f_{1m}(\nu)\big), \tag{3.10}$$

with $f_{1\ell}(\nu) = g_{1\ell}(\mathbf{z}_\nu)$, where $g_{1\ell}$ is given in (3.9) for $0 \le \ell \le m$.

Now, we compute the bifurcation function f_2 defined also in (2.9).

Since g_0 is linear (see (2.6) and (2.7)) we have $\dfrac{\partial^2\xi g_0}{\partial v^2}(\mathbf{z}_\nu)=0$.

Moreover, as $\xi^\perp g_1(\mathbf{z}_\nu)=\big(g_{1\,m+1}(\mathbf{z}_\nu),\ldots,g_{1\,d}(\mathbf{z}_\nu)\big)$, it follows from (2.8), (3.7) and (3.9) that

$$\gamma(\nu)=\big(\gamma_{m+1}(\nu),\ldots,\gamma_d(\nu)\big),$$

where

(3.11)

$$\gamma_\omega(\nu)=\frac{-1}{1-e^{-\mu_\omega 2\pi}}\left(\int_0^\phi e^{-\mu_\omega s}A^+_{\omega+2}(s,\mathbf{z}_\nu)ds+\int_\phi^{2\pi}e^{-\mu_\omega(2\pi+s)}A^-_{\omega+2}(s,\mathbf{z}_\nu)ds\right),$$

for $m+1\le\omega\le d$. Furthermore, for $v=(z_{m+1},\ldots,z_d)$ we have

$$\frac{\partial\xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu)=\big(\widetilde{G}_{10}(\nu),\ldots,\widetilde{G}_{1m}(\nu)\big),$$

with

$$\widetilde{G}_{1\ell}(\nu)=\sum_{\omega=m+1}^{d}\frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu)\gamma_\omega(\nu), \tag{3.12}$$

where $g_{1\ell}$ is given in (3.9) for $0\le\ell\le m$. Additionally from (2.7) and (2.6) we obtain

$$\xi g_2(\mathbf{z}_\nu)=\xi(y_2^+(\phi,\mathbf{z}_\nu))-\xi(y_2^-(\phi-2\pi,\mathbf{z}_\nu)),$$

where

$$\xi y_2^\pm(\theta,\mathbf{z}_\nu)=2\int_0^\theta\xi\big(F_2^\pm(s,\mathbf{z}_\nu)\big)+\xi\left(\frac{\partial F_1^\pm}{\partial\mathbf{x}}(s,\mathbf{z}_\nu)y_1^\pm(s,\mathbf{z}_\nu)\right)ds,$$

because $F_0^\pm$ is linear.

On the other hand

$$\xi F_2^\pm(s,\mathbf{z}_\nu)=\big(F_{20}^\pm(s,\mathbf{z}_\nu),\ldots,F_{2m}^\pm(s,\mathbf{z}_\nu)\big),\ \text{and}$$

$$\xi\left(\frac{\partial F_1^\pm}{\partial\mathbf{x}}(s,\mathbf{z}_\nu)y_1^\pm(s,\mathbf{z}_\nu)\right)=\big(\widetilde{F}_{10}^\pm(s,\mathbf{z}_\nu),\ldots,\widetilde{F}_{1m}^\pm(s,\mathbf{z}_\nu)\big),$$

being

$$\widetilde{F}_{1\ell}^\pm(s,\mathbf{z}_\nu)=\frac{\partial F_{1\ell}^\pm}{\partial r}(s,\mathbf{z}_\nu)y_{10}^\pm(s,\mathbf{z}_\nu)+\ldots+\frac{\partial F_{1\ell}^\pm}{\partial z_d}(s,\mathbf{z}_\nu)y_{1d}^\pm(s,\mathbf{z}_\nu), \tag{3.13}$$

for $F_{1\ell}^\pm$ and $F_{2\ell}^\pm$ defined in (3.6) for $0\le\ell\le m$. Hence

$$f_2(\nu)=2\,\frac{\partial\xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu)+2\,\xi g_2(\mathbf{z}_\nu)=\big(f_{20}(\nu),\ldots,f_{2m}(\nu)\big), \tag{3.14}$$

where

$$\begin{aligned}f_{2\ell}(\nu)=\ &2\,\widetilde{G}_{1\ell}(\nu)+4\int_0^\phi\big(F_{2\ell}^+(s,\mathbf{z}_\nu)+\widetilde{F}_{1\ell}^+(s,\mathbf{z}_\nu)\big)ds\\&+4\int_\phi^{2\pi}\big(F_{2\ell}^-(s,\mathbf{z}_\nu)+\widetilde{F}_{1\ell}^-(s,\mathbf{z}_\nu)\big)ds,\end{aligned} \tag{3.15}$$

for $0 \le \ell \le m$. See the explicit expression of all functions that appear in (3.15) in the Appendix.

If $m = d$, then the functions $\widetilde{G}_{1\ell}(\nu)$ are not considered because $f_2 = 2g_2$ (see Corollary 2).

3.3. Some trigonometric integrals

In order to study the zeros of the averaging functions f_1 and f_2, we need to know some results about trigonometric integrals. Then, we shall state Lemma 6. The proof of this lemma will be omitted here, but it can easily be proven using some trigonometric relations found in Chapter 2 of [12].

For $p, q \in \mathbb{N}$ and $\phi \in (0, 2\pi]$ consider the functions

$$I_{(p,q,\phi)} = \int_0^\phi \cos^p s \sin^q s \, ds, \quad J_{(p,q,\phi)} = \int_\phi^{2\pi} \cos^p s \sin^q s \, ds. \tag{3.16}$$

Lemma 6. *Let $I_{(p,q,\phi)}$ and $J_{(p,q,\phi)}$ be the functions defined in* (3.16) *for $\phi \in (0, 2\pi]$. Then, the following statements hold.*

(a) *If $\phi \neq \pi$ and $\phi \neq 2\pi$ then $I_{(p,q,\phi)}$, $J_{(p,q,\phi)}$, $\int_0^\phi \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$, and $\int_\phi^{2\pi} \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$ are non–zero;*

(b) *If $\phi = \pi$ then $I_{(p,q,\pi)} = 0$ or $J_{(p,q,\pi)} = 0$ if and only if p is odd.*

Moreover

$$\int_0^\pi \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0 \quad \text{or} \quad \int_\pi^{2\pi} \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0$$

if and only if one of the following statements hold:

(i) *i, j, p and q are odd;*

(ii) *i, p and q are odd, and j is even;*

(iii) *i and p are odd, and q and j are even;*

(iv) *i, p and j are odd, and q is even.*

(c) *If $\phi = 2\pi$ then $I_{(p,q,2\pi)} \neq 0$ if and only if p and q are simultaneously even.*

4. Proof of Theorem 1

The proof of Theorem 1 is based on the next lemma which is a particular case of the *Lyapunov-Schmidt reduction* for a finite dimensional function (see for instance [6]).

Lemma 7. *Assuming $q \le p$ are positive integers, let D and V be open bounded subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Let $g : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^p$ and $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be $\mathcal{C}^3$ functions such that $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^3)$ and $\mathcal{Z} = \{\mathbf{z}_\nu = (\nu, \sigma(\nu)) : \nu \in \overline{V}\} \subset D$. We denote by Γ_ν the upper right corner $q \times (p-q)$ matrix of $D\, g_0(\mathbf{z}_\nu)$, and by Δ_ν the lower right corner $(p-q) \times (p-q)$ matrix of $D\, g_0(\mathbf{z}_\nu)$. Assume that for each $\mathbf{z}_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \ne 0$ and $g_0(\mathbf{z}_\nu) = 0$. We consider the functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ defined in (2.9). Then, the following statements hold.*

(*a*) *If there exists $\nu^* \in V$ with $f_1(\nu^*) = 0$ and $\det(D\, f_1(\nu^*)) \ne 0$, then there exists ν_ε such that $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.*

(*b*) *Assume that $f_1 = 0$. If there exists $\nu^* \in V$ with $f_2(\nu^*) = 0$ and $\det(D\, f_2(\nu^*)) \ne 0$, then there exists ν_ε such that $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.*

The proof of this lemma can be found in [21].

Note that in Lemma 7 the functions g_i for $i = 0, 1, 2$ which appears in the expression of (2.9) and (2.8) are the ones of the function

$$g(z, \varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \mathcal{O}(\varepsilon^3), \tag{4.1}$$

instead of the functions which appear in (2.7).

Proof of Theorem 1. Let $\psi(\theta, \mathbf{z}, \varepsilon)$ be a periodic solution of system (2.1) such that $\psi(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. Similarly let $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ be the solutions of the systems $\mathbf{x}' = F^\pm(\theta, \mathbf{x}, \varepsilon)$ such that $\psi^\pm(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. So

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \le \theta \le \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi \le \theta \le T. \end{cases}$$

Since the vector field (2.1) is T–periodic, it may also read

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \le \theta \le \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi - T \le \theta \le 0. \end{cases}$$

Now, we consider the function $g(\mathbf{z}, \varepsilon) = \psi^+(\phi, \mathbf{z}, \varepsilon) - \psi^-(\phi - T, \mathbf{z}, \varepsilon)$. It is easy to see that the solution $\psi(\theta, \mathbf{z}, \varepsilon)$ is T–periodic in θ if and only if $g(\mathbf{z}, \varepsilon) = 0$. So, from hypothesis (H) we have that $g(\mathbf{z}_{\nu,\varepsilon}) = 0$ for every $\mathbf{z}_{\nu,\varepsilon} \in \mathcal{Z}$.

Using Taylor series to expand the functions $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ in powers of ε we obtain

$$\psi^\pm(\theta, \mathbf{z}, \varepsilon) = y_0^\pm(\theta, \mathbf{z}) + \varepsilon y_1^\pm(\theta, \mathbf{z}) + \varepsilon^2 \frac{y_2^\pm(\theta, \mathbf{z})}{2} + \mathcal{O}(\varepsilon^2), \tag{4.2}$$

where $y_i(\theta, \mathbf{z})$ is given in (2.6). We shall omit the computations for obtaining (4.2), nevertheless they can be found in [23]. Therefore, $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^2)$, where $g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - y_i^-(\phi - T, \mathbf{z})$ for $i = 0, 1, 2$. Moreover

$$Dg_0(\mathbf{z}) = \frac{\partial \varphi^+}{\partial \mathbf{z}}(\phi, \mathbf{z}) - \frac{\partial \varphi^-}{\partial \mathbf{z}}(\phi - T, \mathbf{z}) = Y^+(\phi, \mathbf{z}) - Y^-(\phi - T, \mathbf{z}).$$

So, from hypothesis of Theorem 1 we have that the matrix $Dg_0(\mathbf{z})$ has in the upper right corner the zero $q \times (d-q)$ matrix, and in the lower right corner has the $(p-q) \times (p-q)$ matrix Δ_ν with $\det(\Delta_\nu) \neq 0$.

We conclude the proof of this theorem by applying Lemma 7 to the function $g(\mathbf{z}, \varepsilon)$ defined in (4.1). □

5. Proof of Theorem 3

In order to prove Theorem 3 we shall study the zeros of the averaging functions f_1 and f_2, given in (3.10) and (3.14), respectively, when $\phi \in (0, 2\pi) \setminus \{\pi\}$.

Remark 8. *For sake of simplicity we shall denote by $\lambda_{ijk_1\ldots k_m0}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m}$, and by λ_{ij0} the coefficient of $x^i y^j$ of system (2.11), when $\lambda = a^\pm, b^\pm, c_\ell^\pm$ for all $1 \leq \ell \leq m$.*

From statement (a) of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1m}(\nu))$ where

(5.1)
$$\begin{aligned} f_{10}(\nu) = & \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Big(a^+_{ijk_1\ldots k_m0} I_{(i+1,j,\phi)} \\ & + b^+_{ijk_1\ldots k_m0} I_{(i,j+1,\phi)} + a^-_{ijk_1\ldots k_m0} J_{(i+1,j,\phi)} + b^-_{ijk_1\ldots k_m0} J_{(i,j+1,\phi)} \Big), \\ f_{1\ell}(\nu) = & \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Big(c^+_{\ell,ijk_1\ldots k_m0} I_{(i,j,\phi)} + c^-_{\ell,ijk_1\ldots k_m0} J_{(i,j,\phi)} \Big), \end{aligned}$$

with $\nu = (r, z_1, \ldots, z_m)$ and $1 \leq \ell \leq m$.

Proposition 9. *Assume $0 \leq m \leq d$ and $\phi \neq \pi$. Then f_1 has at most n^{m+1} simple zeros and this number can be reached.*

Proof. For each $0 \leq \ell \leq m$ and $\nu = (r, z_1, \ldots, z_m)$, $f_{1\ell}(\nu)$ is a complete polynomial of degree n. Recall that a *complete polynomial of degree k* means a polynomial that appears all its monomials. By Bezout Theorem (see [9]), $f_1(\nu)$ can be at most n^{m+1} simple zeros. Since all the coefficients of $f_1(\nu)$ are independent, we can choose them in order that $f_1(\nu)$ has exactly n^{m+1} zeros with $r > 0$, and $\det f_1'(\nu^*) \neq 0$ for each zero ν^* of $f_1(\nu)$ (that is, ν^* is a simple zero). □

Proposition 10. *Take $0 \leq m \leq d$ and $\phi \neq \pi$. If $f_1 \equiv 0$ then f_2 has at most $(2n)^{m+1}$ simple zeros, and a lower bound for the maximum number of simple zeros is $(2n)(2n-1)^m$.*

Proof. Assume that $f_1 \equiv 0$. From (5.1) it follows that

(5.2)
$$\begin{aligned} & \sum_{i+j=s} a^+_{ijk_1\ldots k_m0} I_{(i+1,j,\phi)} + b^+_{ijk_1\ldots k_m0} I_{(i,j+1,\phi)} \\ & \qquad + a^-_{ijk_1\ldots k_m0} J_{(i+1,j,\phi)} + b^-_{ijk_1\ldots k_m0} J_{(i,j+1,\phi)} = 0, \\ & \sum_{i+j=s} c^+_{\ell,ijk_1\ldots k_m0} I_{(i,j,\phi)} + c^-_{\ell,ijk_1\ldots k_m0} J_{(i,j,\phi)} = 0, \end{aligned}$$

for $1 \leq \ell \leq m$, $0 \leq s \leq n$, $0 \leq k_\ell \leq n$ with $0 \leq k_1 + \ldots + k_m \leq n - s$.

Moreover, $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. In particular, if $m = 0$ then $f_2(\nu) = f_{20}(r)$. Considering the expression for $f_{2\ell}(\nu)$, given in (3.15) for $0 \leq \ell \leq m$, we conclude that $\widetilde{G}_{1\ell}(\nu)$ and $\int_0^\phi \widetilde{F}_{1\ell}^+(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} \widetilde{F}_{1\ell}^-(s, \mathbf{z}_\nu)\, ds$ are complete polynomials of degree $2n - 1$ in the variables $(r, z_1, \ldots, z_m)$, and

$$\int_0^\phi F_{2\ell}^+(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} F_{2\ell}^-(s, \mathbf{z}_\nu)\, ds = \frac{1}{r} \sum_{k=0}^{2n} Q_k(z_1, \ldots, z_m)\, r^k,$$

where $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$, $Q_k(z_1, \ldots, z_m)$ is a complete polynomial of degree $2n - k$ in the variables $(z_1, \ldots, z_m)$ if $m \neq 0$, and $Q_k(z_1, \ldots, z_m)$ is constant if $m = 0$. The above equality is evident if we take into account statement (a) of Lemma 6 and conditions (5.2). Therefore, each $r f_{2\ell}(\nu)$ is a complete polynomial of degree $2n$ in the variables $(r, z_1, \ldots, z_m)$. Since $r > 0$, it is known that $r f_{2\ell}(\nu) = 0$ if and only if $f_{2\ell}(\nu) = 0$ for each $0 \leq \ell \leq m$. Then, by Bezout Theorem, $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \leq m \leq d$.

In order to show that the maximum number is greater than or equal to $(2n)(2n-1)^m$ we provide a particular example. So, take $a_{i00}^\pm \neq 0$, $c_{\ell,00\ldots0k_\ell 0}^\pm \neq 0$, and we take zero all the other coefficients for $1 \leq \ell \leq m$. From (3.15) we obtain $f_{20}(\nu) = f_{20}(r)$ and $f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$\begin{aligned} f_{20}(r) = \quad & \frac{4}{r} \sum_{i=0}^{n} \sum_{p=0}^{n} r^{i+p} \Bigg(a_{i00}^+ a_{p00}^+ I_{(i+p+1,1,\phi)} + a_{i00}^- a_{p00}^- J_{(i+p+1,1,\phi)} \\ & + i\, a_{i00}^+ a_{p00}^+ \int_0^\phi \cos^{i+1} s\, I_{(p+1,0,s)} ds + i a_{i00}^- a_{p00}^- \int_\phi^{2\pi} \cos^{i+1} s I_{(p+1,0,s)} ds \Bigg), \\ f_{2\ell}(r, z_\ell) = \quad & \frac{4}{r} \sum_{i=0}^{n} \sum_{k_l=0}^{n} r^i z_\ell^{k_\ell} \left(a_{i00}^+ c_{\ell,0\ldots0k_\ell 0}^+ I_{(i,1,\phi)} + a_{i00}^- c_{\ell,0\ldots0k_\ell 0}^- J_{(i,1,\phi)} \right) \\ & + 4 \sum_{k_\ell=1}^{n} \sum_{L_\ell=0}^{n} z_\ell^{k_\ell + L_\ell - 1} \Bigg(\tfrac{\phi^2}{2} k_\ell\, c_{\ell,0\ldots0k_\ell 0}^+\, c_{\ell,0\ldots0L_\ell 0}^+ \\ & + \tfrac{(2\pi)^2 - \phi^2}{2} k_\ell\, c_{\ell,0\ldots0k_\ell 0}^-\, c_{\ell,0\ldots0L_\ell 0}^- \Bigg), \end{aligned}$$

where $a_{i00}^+ I_{(i+1,0,\phi)} = -a_{i00}^- J_{(i+1,0,\phi)}$ and $c_{\ell,00\ldots0k_\ell 0}^+ I_{(0,0,\phi)} = -c_{\ell,00\ldots0k_\ell 0}^- J_{(0,0,\phi)}$ for $1 \leq \ell \leq m$ (see (5.2)).

From statement (a) of Lemma 6, $r f_{20}(r)$ is a complete polynomial of degree $2n$ in the variable r, whose coefficients are independent. Furthermore, if $f_{20}(r^*) = 0$ with $r^* > 0$, then $f_{2\ell}(r^*, z_\ell)$ is a polynomial of degree $2n - 1$ in the variable z_ℓ, and all their coefficients are independent for $1 \leq \ell \leq m$. Therefore, By Bezout Theorem, $f_2(\nu)$ has at most $(2n)(2n-1)^m$ simple zeros, and this number can be reached due to the independence of coefficients. □

Proof of Theorem 3. We apply Theorem 1 to the function f_1 of Proposition 9 and we conclude statement (a). Statement (b) is proved applying Theorem 1 to the functions f_1 and f_2 given in Proposition 10. □

5.1. Improving the lower bound

As mentioned in the introduction, the lower bound of statement (b) of Theorem 3 is not optimal and can be improved. From Theorem 1 we need to solve the equation $f_2(\nu) = 0$, assuming $f_1 \equiv 0$. This can be a hard task due to the complexity of f_2. In what follows, we provide a simpler polynomial system for which their simple zeros imply the existence of simple zeros of f_2.

From (3.14) we have $f_2(\nu) = (f_{20}(\nu), \dots, f_{2m}(\nu))$. In (3.15) we can take $\widetilde{G}_{1\ell}(\nu) = 0$ and, since $1/r$ appears as a common factor in the expression of $A_1^{\pm}$ (3.3), we define $\widetilde{A}_1^{\pm} = rA_1^{\pm}$. Finally, for $1 \le \ell \le m$, we assume that $A_{\ell+2}^{\pm} = \delta \widetilde{A}_{\ell+2}^{\pm}$ and $B_{\ell+2}^{\pm} = \delta \widetilde{B}_{\ell+2}^{\pm}$ for $\delta > 0$ sufficiently small. Notice that, the assumption is equivalent to ask that the coefficients of the perturbation (2.10) for $1 \le \ell \le m$ are of order δ.

Now, for $1 \le \ell \le m$, we define

$$
\begin{aligned}
P_\ell(\nu) &= \int_0^{\phi} \widetilde{B}_{\ell+2}^{+}(s, \mathbf{z}_\nu)\, ds + \int_{\phi}^{2\pi} \widetilde{B}_{\ell+2}^{-}(s, \mathbf{z}_\nu)\, ds, \\
Q_\ell(\nu) &= \int_0^{\phi} \widetilde{A}_1^{+}(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^{+}(s, \mathbf{z}_\nu)\, ds + \int_{\phi}^{2\pi} \widetilde{A}_1^{-}(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^{-}(s, \mathbf{z}_\nu)\, ds.
\end{aligned}
\tag{5.3}
$$

Thus, from (3.3), (3.6), (3.8) and (3.13) we have $\int \widetilde{F}_{1\ell}^{\pm}(s, \mathbf{z}_\nu) ds = \mathcal{O}_2(\delta)$ and, therefore,

$$
\frac{r}{4\delta} f_{2\ell}(\nu) = \quad r P_\ell(\nu) - Q_\ell(\nu) + \mathcal{O}(\delta), \quad \text{for } 1 \le \ell \le m.
$$

Hence, taking $\delta > 0$ sufficiently small, we obtain the following proposition.

Proposition 11. *If the polynomial system*

$$
f_{20}(\nu) = 0 \quad \textit{and} \quad r\, P_\ell(\nu) - Q_\ell(\nu) = 0, \quad \textit{for} \ \ 1 \le \ell \le m,
\tag{5.4}
$$

has N isolated solutions, then $N_2(m, n, \phi) \ge N$.

6. Proof of Theorem 4

In this section we study the zeros of the functions f_1 and f_2, given in (3.10) and (3.14), respectively, when $\phi = \pi$. Then, we conclude Theorem 4 applying Theorem 2.1.

From statement (b) of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1\ell}(\nu))$ where

(6.1)
$$
\begin{aligned}
f_{10}(\nu) = & \sum_{i\,odd,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left(a^+_{ijk_1\ldots k_m 0} I_{(i+1,j,\pi)} + a^-_{ijk_1\ldots k_m 0} J_{(i+1,j,\pi)} \right) \\
& + \sum_{i\,even,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left(b^+_{ijk_1\ldots k_m 0} I_{(i,j+1,\pi)} + b^-_{ijk_1\ldots k_m 0} J_{(i,j+1,\pi)} \right), \\
f_{1\ell}(\nu) = & \sum_{i\,even,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left(c^+_{\ell,ijk_1\ldots k_m 0} I_{(i,j,\pi)} + c^-_{\ell,ijk_1\ldots k_m 0} J_{(i,j,\pi)} \right),
\end{aligned}
$$

where $\nu = (r, z_1, \ldots, z_m)$, $1 \le \ell \le m$ and $P = i + j + k_1 + \ldots + k_m$.

Proposition 12. *Take $0 \le m \le d$ and $\phi = \pi$. Then, f_1 has at most n^{m+1} simple zeros and this number can be reached.*

Proof. This proof is analogously to the proof of Proposition 9, noticing that for each $0 \le \ell \le m$, $f_{1\ell}(\nu)$ is a complete polynomial of degree n in the variables $(r, z_1, \ldots, z_m)$ and all their coefficients are independent. $\square$

Proposition 13. *Assume $0 \le m \le d$ and $\phi = \pi$. If $f_1 \equiv 0$ then f_2 has at most $(2n)^{m+1}$ simple zeros, and the lower bound for the number of simple zeros is $(2n-1)^{m+1}$ if n is odd, and $(2n-2)(2n-1)^m$ if n is even.*

Proof. Assume that $f_1 \equiv 0$. From (6.1) it follows that

(6.2)
$$
\begin{aligned}
& \sum_{i\,odd,\,i+j=s} a^+_{ijk_1\ldots k_m 0}\, I_{(i+1,j,\pi)} + a^-_{ijk_1\ldots k_m 0}\, J_{(i+1,j,\pi)} \\
& + \sum_{i\,even,\,i+j=s} b^+_{ijk_1\ldots k_m 0}\, I_{(i,j+1,\pi)} + b^-_{ijk_1\ldots k_m 0}\, J_{(i,j+1,\pi)} = 0, \\
& \sum_{i\,even,\,i+j=s} c^+_{\ell,ijk_1\ldots k_m 0}\, I_{(i,j,\pi)} + c^-_{\ell,ijk_1\ldots k_m 0}\, J_{(i,j,\pi)} = 0,
\end{aligned}
$$

for $1 \le \ell \le m,\ 0 \le s \le n,\ 0 \le k_\ell \le n$ with $0 \le k_1 + \ldots + k_m \le n - s$.

Moreover, $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. If $m = 0$ then $f_2(\nu) = f_{20}(r)$. Analogously to the proof of Proposition 10 we conclude that $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \le m \le d$.

Now, we provide a particular example to exhibit the lower bound for the maximum number of simple zeros. So, take $a^{\pm}_{i00} \neq 0$, $c^{\pm}_{\ell,00\ldots 0k_\ell 0} \neq 0$, and take zero all the other coefficients for $1 \le \ell \le m$. From (3.15) we obtain $f_{20}(\nu) = f_{20}(r)$ and

$f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$
\begin{aligned}
f_{20}(r) = \; & \frac{4}{r}\Bigg(\sum_{i\, even,\, i=0}^{n} \; \sum_{p\, odd,\, p=0}^{n} r^{i+p}\big(a^+_{i00}a^+_{p00} I_{(i+p+1,1,\pi)} + a^-_{i00}a^-_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^+_{i00}a^+_{p00}\int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + i\, a^-_{i00}a^-_{p00}\int_{\pi}^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds\big) \\
& + \sum_{i\, odd,\, i=0}^{n} \; \sum_{p\, even,\, p=0}^{n} r^{i+p}\big(a^+_{i00}a^+_{p00} I_{(i+p+1,1,\pi)} + a^-_{i00}a^-_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^+_{i00}a^+_{p00}\int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + \; i\, a^-_{i00}a^-_{p00}\int_{\pi}^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds\big) \\
& + \sum_{i\, odd,\, i=0}^{n} \; \sum_{p\, odd,\, p=0}^{n} r^{i+p}\big(a^+_{i00}a^+_{p00} I_{(i+p+1,1,\pi)} + \; a^-_{i00}a^-_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^+_{i00}a^+_{p00}\int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + i a^-_{i00}a^-_{p00}\int_{\pi}^{2\pi} \cos^{i+1} s I_{(p+1,0,s)} ds\big)\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
f_{2\ell}(r, z_\ell) = \; & \frac{4}{r}\sum_{i=0}^{n}\sum_{k_l=0}^{n} r^i z_\ell^{k_\ell}\big(a^+_{i00}c^+_{\ell,0\ldots 0k_\ell 0} I_{(i,1,\phi)} + a^-_{i00}c^-_{\ell,0\ldots 0k_\ell 0} J_{(i,1,\phi)}\big) \\
& + \sum_{k_\ell=1}^{n}\sum_{L_\ell=0}^{n} z_\ell^{k_\ell+L_\ell-1} k_\ell \Bigg(\tfrac{\phi^2}{2} c^+_{\ell,0\ldots 0k_\ell 0} c^+_{\ell,0\ldots 0L_\ell 0} \\
& + \tfrac{(2\pi)^2-\phi^2}{2} c^-_{\ell,0\ldots 0k_\ell 0} c^-_{\ell,0\ldots 0L_\ell 0}\Bigg),
\end{aligned}
$$

for $1 \le \ell \le m$, where $a^+_{i00} I_{(i+1,0,\pi)} = -a^-_{i00} J_{(i+1,0,\pi)}$ if i is odd and $c^+_{\ell,00\ldots 0k_\ell 0}$ $I_{(0,0,\pi)} = -c^-_{\ell,00\ldots 0k_\ell 0} J_{(0,0,\pi)}$ (see (6.2)). Therefore, from statement (b) of Lemma 6, $r f_{20}(r)$ is a complete polynomial in the variable r of degree $2n-1$ if n is odd, and $2n-2$ if n is even, and its coefficients are independent. Furthermore, if $f_{20}(r^*) = 0$ with $r^* > 0$, then $f_{2\ell}(r^*, z_\ell)$ is a polynomial of degree $2n-1$ in the variable z_ℓ for each $1 \le \ell \le m$. Then, the number of simple zeros with $r > 0$ of $f_2(\nu)$ can be $(2n-1)^{m+1}$ if n is odd, and $(2n-2)(2n-1)^m$ if n is even. By the independence of all coefficients these numbers can be reached. □

Proof of Theorem 4. From Theorem 1 and Proposition 12, statement (a) holds, and applying Theorem 1 to the functions f_1 and f_2 given in Proposition 13 we conclude statement (b). □

7. Proof of Theorem 5

When $\phi = 2\pi$ system (2.11) is continuous. Then, considering the cylindrical coordinates given in (3.1), and taking θ as the new time, system (2.11) can be written as system (2.1) that is,

$$\mathbf{x}' = F^{+}(\theta, \mathbf{x}, \varepsilon), \quad \text{for } 0 \leq \theta \leq 2\pi,$$

where

$$F^{+}(\theta, \mathbf{x}, \varepsilon) = F_0^{+}(\theta, \mathbf{x}) + \varepsilon F_1^{+}(\theta, \mathbf{x}) + \varepsilon^2 F_2^{+}(\theta, \mathbf{x}) + \varepsilon^3 R^{+}(\theta, \mathbf{x}, \varepsilon),$$

for $\mathbf{x} = (r, \mathrm{z})$ and $\mathrm{z} = (z_1, \ldots, z_d)$, with F_j^{+} given in (3.5) and (3.6) for $j = 0, 1, 2$.

From statement (c) of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1m}(\nu)$ with

$$\begin{aligned} f_{10}(\nu) &= \sum_{i\,odd,\, j\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m}\, a^{+}_{ijk_1\ldots k_m 0} I_{(i+1,j,2\pi)} \\ &\quad + \sum_{i\,even,\, j\,odd,\, P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m}\, b^{+}_{ijk_1\ldots k_m 0} I_{(i,j+1,2\pi)}, \\ f_{1\ell}(\nu) &= \sum_{i,\, j\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m}\, c^{+}_{\ell,ijk_1\ldots k_m 0} I_{(i,j,2\pi)}, \end{aligned} \tag{7.1}$$

where $\nu = (r, z_1, \ldots, z_m)$, $1 \leq \ell \leq m$ and $P = i + j + k_1 + \ldots + k_m$.

Proposition 14. *Assume $0 \leq m \leq d$ and $\phi = 2\pi$. If $m \neq 0$ then f_1 has at most $n^m(n-1)/2$ simple zeros and this number can be reached. If $m = 0$ then f_1 has at most $(n-1)/2$ simple zeros if n is odd, and $(n-2)/2$ if n is even, and these numbers can be reached.*

Proof. We have $f_{10}(\nu) = r\widetilde{f}_{10}(\nu)$ with

$$\widetilde{f}_{10}(\nu) = h_1 + r^2 h_3 + r^4 h_5 + r^6 h_7 + \ldots + \begin{cases} r^{n-1} h_n & \text{if n is odd,} \\ r^{n-2} h_{n-1} & \text{if n is even,} \end{cases}$$

where

$$\begin{aligned} h_k = & \sum_{k_1+\ldots+k_m=0}^{n-k} z_1^{k_1} \ldots z_m^{k_m} \Bigg(\sum_{i\,odd,\, j\,even,\, i+j=k} a^{+}_{ijk_1\ldots k_m 0}\, I_{(i+1,j,2\pi)} \\ & + \sum_{i\,even,\, j\,odd,\, i+j=k} b^{+}_{ijk_1\ldots k_m 0}\, I_{(i,j+1,2\pi)} \Bigg). \end{aligned}$$

If $m \neq 0$ then $\widetilde{f}_{10}(\nu)$ and $f_{1\ell}(\nu)$ are polynomials in the variables $(r, z_1, \ldots, z_m)$ of degree $n-1$ and n, respectively, for $1 \leq \ell \leq m$. From Bezout Theorem the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)$. Since the exponents of r in the function $\widetilde{f}_{10}(\nu)$ are always even numbers, the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)/2$. In what follows we provide a particular example to prove that this number is reached.

First if n is even we take $a^+_{10k_10} \neq 0$, $b^+_{01k_10} \neq 0$, $c^+_{1,ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and take zero all the other coefficients in other that $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(z_1)$, $f_{11}(\nu) = f_{11}(r)$, and $f_{1\ell}(\nu) = f_{11}(z_\ell)$, where

$$\begin{aligned}
\widetilde{f}_{10}(z_1) &= \sum_{k_1=0}^{n-1} z_1^{k_1} \left(a^+_{10k_10} I_{(2,0,2\pi)} + b^+_{01k_10} I_{(0,2,2\pi)}\right), \\
f_{11}(r) &= \sum_{i,j\, even,\, i+j=0}^{n} r^{i+j}\, c^+_{1,ij0}\, I_{(i,j,2\pi)}, \\
f_{1\ell}(z_\ell) &= \sum_{k_\ell=0}^{n} z_\ell^{k_\ell}\; c^+_{\ell,00k_\ell 0}\, I_{(0,0,2\pi)},
\end{aligned}$$

for $2 \leq \ell \leq m$. Thus, $\widetilde{f}_{10}(z_1)$ is a complete polynomial of degree $n-1$ in the variable z_1, $f_{11}(r)$ is an even polynomial of degree n in the variable r, and $f_{1\ell}(z_\ell)$ is a complete polynomial of degree n in the variable z_ℓ for all $2 \leq \ell \leq m$. Since the exponents of r in $f_{11}(r)$ is even, then $f_1(\nu)$ can have $n^m(n-1)/2$ simple zeros with $r > 0$.

On the other hand, if n is odd we take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and we take zero all the other coefficients and then we obtain $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(r)$ and $f_{1\ell}(\nu) = f_{1\ell}(z_\ell)$, where

$$\begin{aligned}
\widetilde{f}_{10}(r) = &\quad h_1 + rh_2 + r^2 h_3 + \ldots + r^{n-1} h_n, \\
f_{1\ell}(\nu) = &\quad \sum_{k_\ell=0}^{n} {z_\ell}^{k_\ell}\, c_{\ell,00k_\ell 0}\, I_{(0,0,2\pi)},
\end{aligned}$$

for $1 \leq \ell \leq m$. Then, $\widetilde{f}_{10}(r)$ is a polynomial of degree $n-1$ in the variable r, whose exponents are always even. In a similar way $f_{1\ell}(z_\ell)$ is a polynomial of degree n in the variable z_ℓ for $1 \leq \ell \leq m$. Therefore, $f_1(\nu)$ can have $n^m(n-1)/2$ simple zeros with $r > 0$.

If $m = 0$ then $\nu = r$ and $f_1(\nu) = r\widetilde{f}_{10}(r)$. So the number of simple zeros can be $n-1$ if n is odd, and $n-2$ if n is even. Since the exponent of r in $\widetilde{f}_{10}$ is even, the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ is $(n-1)/2$ if n is odd, and $(n-2)/2$ if n is even.

Now, we exhibit a particular example where the maximum number of simple zeros of $f_1(\nu)$ can be reached. Take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$ and we take zero all the other coefficients so that $\widetilde{f}_{10}(r)$ is an even polynomial in the variable r of degree $n-1$ if n is odd, and $n-2$ is n if even. So, the number of simple zeros of $f_1(\nu)$ with $r > 0$ can be $(n-1)/2$ if n is odd, and $(n-2)/2$ if n is even.

In both particular cases, $m \neq 0$ and $m = 0$, the coefficients of $f_1(\nu)$ are independent. Therefore, the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ can be reached. $\square$

Now, we emphasize that the averaging function f_2 of the continuous system (2.11), for $\phi = 2\pi$, is given by $f_2(\nu) = \big(f_{20}(\nu), \dots, f_{2m}(\nu)\big)$ being

$$f_{2\ell}(\nu) = 2\widetilde{G}_{1\ell}(\nu) + 4\int_0^{2\pi} \big(F_{2\ell}^{+}(s,\mathbf{z}_\nu) + \widetilde{F}_{1\ell}^{+}(s,\mathbf{z}_\nu)\big)\, ds, \tag{7.2}$$

for $0 \le \ell \le m$, $F_{2\ell}^{+}$, $\widetilde{G}_{1\ell}$ and $\widetilde{F}_{1\ell}^{+}$ given in (3.6), (3.12) and (3.13), respectively.

Proposition 15. *Assume $m = 0$ and $\phi = 2\pi$. If $f_1 \equiv 0$ then f_2 has at most $2n$ simple zeros. Moreover, the lower bound for the number of simple zeros is n.*

Proof. If $m = 0$ then $\nu = r$ and $f_1(\nu) = f_{10}(r)$. Assume that $f_1 \equiv 0$. From (7.1) we obtain

$$\sum_{i\,odd,\,j\,even,\,P=s}^{n} a_{ij0}^{+} I_{(i+1,j,2\pi)} + \sum_{i\,even,\,j\,odd,\,P=s}^{n} b_{ij0}^{+} I_{(i,j+1,2\pi)} = 0, \tag{7.3}$$

where $P = i + j$ and $0 \le s \le n$.

Furthermore, by (7.2) we have $f_2(\nu) = f_{20}(r)$. Therefore, from statement (c) of Lemma 6 and (7.3), we conclude that $\widetilde{G}_{10}(\nu)$ and $\displaystyle\int_0^{2\pi} \widetilde{F}_{10}(s,\mathbf{z}_\nu)\, ds$ are complete polynomials of degree $2n-1$ in the variable r, and

$$\int_0^{2\pi} F_{20}^{+}(s,\mathbf{z}_\nu)\, ds = \sum_{s=0}^{N_1} R_s\, r^{2s+1} + \frac{1}{r}\sum_{k=0}^{n} Q_k\, r^{2k},$$

where $\mathbf{z}_\nu = (r, 0, \dots, 0) \in \mathbb{R}^{d+1}$, R_s and Q_k are constants, $N_1 = \dfrac{n-2}{2}$ if n is even, and $N_1 = \dfrac{n-1}{2}$ if n is odd. Therefore, $r\displaystyle\int_0^{2\pi} F_{20}^{+}(s,\mathbf{z}_\nu)\, ds$ is an even polynomial in the variable r. Since $r > 0$ it follows that $rf_2(\nu) = 0$ if and only if $f_2(\nu) = 0$. By Bezout Theorem the maximum number of simple zeros of $f_2(\nu)$ is $2n$.

In order to exhibit the lower bound for the number of simple zeros of $f_2(\nu)$, we provide a particular example. Then, take $a_{ij0}^{\pm} \neq 0$, $b_{ij0}^{\pm} \neq 0$, $\alpha_{ij0}^{\pm} \neq 0$, and $\beta_{ij0}^{\pm} \neq 0$ and we take zero all the other coefficients in such a way that $f_{20}(r) = 4\displaystyle\int_0^{2\pi} F_{20}^{+}(r,\mathbf{z}_\nu) d\theta$. Therefore, $rf_{20}(r)$ is a polynomial in r of degree $2n$. Since $rf_{20}(r)$ is an even polynomial in r, then the number of simple zeros of $f_2(\nu)$ with $r > 0$ can be n, and this number can be reached due to the independence of all coefficients. $\square$

Proof of Theorem 5. Applying Theorem 2.1 to the function f_1 given in Proposition 14, statement (a) holds. We apply Theorem 2.1 to the function f_2 given in Proposition 15 and we conclude statement (b). $\square$

Appendix

In this appendix we shall exhibit some general expression of functions that appears in subSection 3.2.

We denote by $\lambda_{ijk_1\ldots k_m1_\omega}$ the coefficient of $x^iy^jz_1^{k_1}\ldots z_m^{k_m}z_\omega$, and by $\lambda_{ijk_1\ldots k_m0}$ the coefficient of $x^iy^jz_1^{k_1}\ldots z_m^{k_m}$ in system (2.11) when $\lambda = a^\pm, b^\pm, \alpha^\pm, \beta^\pm, c_\ell^\pm, \gamma_\ell^\pm$ for all $0 \le \ell \le m$ and $m+1 \le \omega \le d$. Recall that $\nu = (r, z_1, \ldots, z_m)$ and $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$.

For the next expressions, take $P = i + j + k_1 + \ldots + k_m$ and $Q = p + q + L_1 + \ldots + L_m$.

From (3.3) and (3.9) we obtain

$$\begin{aligned}\frac{\partial g_{10}}{\partial z_\omega}(\mathbf{z}_\nu) = \; & \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\Big(a^+_{ijk_1\ldots k_m1_\omega}\int_0^\phi e^{\mu_\omega s}\cos^{i+1}s\,\sin^j s\,ds \\ & + b^+_{ijk_1\ldots k_m1_\omega}\int_0^\phi e^{\mu_\omega s}\cos^i s\,\sin^{j+1}s\,ds \\ & + a^-_{ijk_1\ldots k_m1_\omega}\int_\phi^{2\pi} e^{\mu_\omega s}\cos^{i+1}s\,\sin^j s\,ds \\ & + b^-_{ijk_1\ldots k_m1_\omega}\int_\phi^{2\pi} e^{\mu_\omega s}\cos^i s\,\sin^{j+1}s\,ds\Big), \\ \frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu) = \; & \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\Big(c^+_{l,ijk_1\ldots k_m1_\omega}\int_0^\phi e^{\mu_\omega s}\cos^i s\sin^j s\,ds \\ & + c^-_{\ell,ijk_1\ldots k_m1_\omega}\int_\phi^{2\pi} e^{\mu_\omega s}\cos^i s\sin^j s\,ds\Big),\end{aligned}$$

for $1 \le \ell \le m$ and $m+1 \le \omega \le d$.

From (3.11) we get

$$\begin{aligned}\gamma_\omega(\nu) = \; & \frac{-1}{1-e^{-\mu_\omega 2\pi}}\sum_{P=0}^{n} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\Big(c^+_{\omega,ijk_1\ldots k_m0}\int_0^\phi e^{-\mu_\omega s}\cos^i s\,\sin^j s\,ds \\ & + c^-_{\omega,ijk_1\ldots k_m0}\int_\phi^{2\pi} e^{-\mu_\omega(2\pi+s)}\cos^i s\,\sin^j s\,ds\Big),\end{aligned}$$

for $m+1 \le \omega \le d$.

From the above equalities and (3.12) we obtain for $1 \le \ell \le m$ that

$$\widetilde{G}_{10}(\nu) = \sum_{\omega=m+1}^{d} \left[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(a^{+}_{ijk_1\dots k_m 1_\omega} \int_0^{\phi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \right.\right.$$
$$+ \, b^{+}_{ijk_1\dots k_m 1_\omega} \int_0^{\phi} e^{\mu_\omega s} \cos^{i} s \, \sin^{j+1} s \, ds + \, a^{-}_{ijk_1\dots k_m 1_\omega} \int_{\phi}^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds$$
$$\left.\left. + \, b^{-}_{ijk_1\dots k_m 1_\omega} \int_{\phi}^{2\pi} e^{\mu_\omega s} \cos^{i} s \, \sin^{j+1} s \, ds \right) \right]$$
$$\left[\frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(c^{+}_{\omega,ijL_1\dots L_m 0} \int_0^{\phi} e^{-\mu_\omega s} \cos^{i} s \, \sin^j s \, ds \right.\right.$$
$$\left.\left. + \, c^{-}_{\omega,ijL_1\dots L_m 0} \int_{\phi}^{2\pi} e^{-\mu_\omega (2\pi+s)} \cos^{i} s \, \sin^j s \, ds \right) \right],$$

$$\widetilde{G}_{1\ell}(\nu) = \sum_{\omega=m+1}^{d} \left[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(c^{+}_{\ell,ijk_1\dots k_m 1_\omega} \int_0^{\phi} e^{\mu_\omega s} \cos^{i} s \sin^j s \, ds \right.\right.$$
$$\left.\left. + \, c^{-}_{\ell,ijk_1\dots k_m 1_\omega} \int_{\phi}^{2\pi} e^{\mu_\omega s} \cos^{i} s \sin^j s \, ds \right) \right]$$
$$\left[\frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{L_1} \dots z_m^{L_m} \left(c^{+}_{\omega,ijL_1\dots L_m 0} \int_0^{\phi} e^{-\mu_\omega s} \cos^{i} s \, \sin^j s \, ds \right.\right.$$
$$\left.\left. + \, c^{-}_{\omega,ijL_1\dots L_m 0} \int_{\phi}^{2\pi} e^{-\mu_\omega (2\pi+s)} \cos^{i} s \, \sin^j s \, ds \right) \right].$$

Now, from (3.6) we compute

$$\begin{aligned}
\frac{\partial F^{\pm}_{10}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu)) =& \frac{1}{r} \sum_{P=0}^{n} (i+j) \, r^{i+j} z_1^{k_1} \dots z_m^{k_m} \\
& \left(a^{\pm}_{ijk_1\dots k_m 0} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 0} \cos^{i} s \sin^{j+1} s \right), \\
\frac{\partial F^{\pm}_{10}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu)) =& \sum_{P=0}^{n} k_\rho \, r^{i+j} z_1^{k_1} \dots z_\rho^{k_\rho - 1} \dots z_m^{k_m} \\
& \left(a^{\pm}_{ijk_1\dots k_m 0} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 0} \cos^{i} s \sin^{j+1} s \right), \\
\frac{\partial F^{\pm}_{10}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu)) =& \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \\
& \left(a^{\pm}_{ijk_1\dots k_m 1_\omega} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 1_\omega} \cos^{i} s \sin^{j+1} s \right),
\end{aligned}$$

$$\frac{\partial F^{\pm}_{1\ell}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu)) = \frac{1}{r}\sum_{P=0}^{n}(i+j)\, r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\ell,ijk_1\dots k_m0}\cos^i s\sin^j s,$$
$$\frac{\partial F^{\pm}_{1\ell}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu)) = \sum_{P=0}^{n} k_\rho\, r^{i+j} z_1^{k_1}\dots z_p^{k_\rho-1}\dots z_m^{k_m}\, c^{\pm}_{\ell,ijk_1\dots k_m0}\cos^i s\sin^j s,$$
$$\frac{\partial F^{\pm}_{1\ell}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu)) = \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\ell,ijk_1\dots k_m1_\omega}\cos^i s\sin^j s,$$

for $1\le \ell\le m$, $1\le\rho\le m$ and $m+1\le\omega\le d$.

Note that when $m=d$ we do not consider the functions $\dfrac{\partial F^{\pm}_{10}}{\partial z_\omega}$ and $\dfrac{\partial F^{\pm}_{1\ell}}{\partial z_\omega}$.

Now, from (3.3) and (3.8) we get

$$\begin{aligned}
y^{\pm}_{10}(s,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\left(a^{\pm}_{ijk_1\dots k_m0} I_{(i+1,j,s)} + b^{\pm}_{ijk_1\dots k_m0} I_{(i,j+1,s)}\right),\\
y^{\pm}_{1\rho}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\rho,ijk_1\dots k_m0} I_{(i,j,s)},\\
y^{\pm}_{1\omega}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\omega,ijk_1\dots k_m0}\int_0^s e^{\mu_\omega(s-\tau)}\cos^i\tau\,\sin^j s\, d\tau,
\end{aligned}$$

for $1\le\rho\le m$ and $m+1\le\omega\le d$. Therefore

$$\int \frac{\partial F^{\pm}_{10}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu))y^{\pm}_{10}(s,\mathbf{z}_\nu)\,ds = \frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n}(i+j)\, r^{i+j+p+q}\, z_1^{k_1+L_1}\dots z_m^{k_m+L_m}$$
$$\Big(a^{\pm}_{ijk_1\dots k_m0}a^{\pm}_{pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\, I_{(p+1,q,s)}ds$$
$$+\, b^{\pm}_{ijk_1\dots k_m0}a^{\pm}_{pqL_1\dots L_m0}\int\cos^{i}s\sin^{j+1} s\, I_{(p+1,q,s)}ds$$
$$+\, a^{\pm}_{ijk_1\dots k_m0}b^{\pm}_{pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\, I_{(p,q+1,s)}ds$$
$$+\, b^{\pm}_{ijk_1\dots k_m0}b^{\pm}_{pqL_1\dots L_m0}\int\cos^{i}s\,\sin^{j+1} s\, I_{(p,q+1,s)}ds\Big),$$

$$\int \frac{\partial F^{\pm}_{10}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu))y^{\pm}_{1\rho}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n}\sum_{Q=0}^{n} k_\rho r^{i+j+p+q} z_1^{k_1+L_1}\dots z_\rho^{k_\rho+L_\rho-1}\dots z_m^{k_m+L_m}$$
$$\Big(a^{\pm}_{ijk_1\dots k_m0}\, c^{\pm}_{\rho,pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\, I_{(p,q,s)}\,ds$$
$$+\, b^{\pm}_{ijk_1\dots k_m0}\, c^{\pm}_{\rho,pqL_1\dots L_m0}\int\cos^{i}s\sin^{j+1} s\, I_{(p,q,s)}\,ds\Big),$$

$$\int \frac{\partial F_{10}^{\pm}}{\partial z_{\omega}}(s,\varphi(s,\mathbf{z}_{\nu}))y_{1\omega}^{\pm}(s,\mathbf{z}_{\nu})ds = \sum_{P=0}^{n-1}\sum_{Q=0}^{n} r^{i+j+p+q} z_1^{k_1+L_1}\ z_2^{k_2+L_2} z_3^{k_3+L_3}\dots z_m^{k_m+L_m}$$

$$\Bigg(a_{ijk_1\dots k_m 1_\omega}^{\pm} c_{\omega,pqL_1\dots L_m 0}^{\pm} \int \cos^{i+1} s \sin^{j} s \left(\int_0^s e^{\mu_\omega (s-\tau)} \cos^p \tau \sin^q \tau d\tau\right) ds$$

$$+ b_{ijk_1\dots k_m 1_\omega}^{\pm} c_{\omega,pqL_1\dots L_m 0}^{\pm} \int \cos^{i} s \sin^{j+1} s \left(\int_0^s e^{\mu_\omega (s-\tau)} \cos^p \tau \sin^q \tau d\tau\right) ds\Bigg),$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial r}(s,\varphi(s,\mathbf{z}_{\nu}))y_{10}^{\pm}(s,\mathbf{z}_{\nu})ds = \frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n}(i+j)\, r^{i+j+p+q} z_1^{k_1+L_1} z_2^{k_2+L_2}\dots z_m^{k_m+L_m}$$

$$\Bigg(c_{\ell,ijk_1\dots k_m 0}^{\pm} a_{pqL_1\dots L_m 0}^{\pm} \int \cos^{i} s \sin^{j} s\, I_{(p+1,q,s)}\, ds$$

$$+ c_{\ell,ijk_1\dots k_m 0}^{\pm} b_{pqL_1\dots L_m 0}^{\pm} \int \cos^{i} s \sin^{j} s\, I_{(p,q+1,s)}\, ds\Bigg),$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial z_{\rho}}(s,\varphi(s,\mathbf{z}_{\nu}))y_{1\rho}^{\pm}(s,\mathbf{z}_{\nu})ds = \sum_{P=0}^{n}\sum_{Q=0}^{n} k_\rho\, r^{i+j+p+q} z_1^{k_1+L_1}\dots z_\rho^{k_\rho+L_\rho-1}\dots z_m^{k_m+L_m}$$

$$c_{\ell,ijk_1\dots k_m 0}^{\pm}\, c_{\rho,pqL_1\dots L_m 0}^{\pm} \int \cos^{i} s \sin^{j} s\, I_{(p,q,s)} ds,$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial z_{\omega}}(s,\varphi(s,\mathbf{z}_{\nu}))y_{1\omega}^{\pm}(s,\mathbf{z}_{\nu})ds = \sum_{P=0}^{n-1}\sum_{Q=0}^{n} r^{i+j+p+q} z_1^{k_1+L_1}\ z_2^{k_2+L_2} z_3^{k_3+L_3}\dots z_m^{k_m+L_m}$$

$$c_{\ell,ijk_1\dots k_m 1_\omega}^{\pm}\, c_{\omega,pqL_1\dots L_m 0}^{\pm} \int \cos^{i} s \sin^{j} s \left(\int_0^s e^{\mu_\omega (s-\tau)} \cos^p \tau \sin^q \tau d\tau\right) ds,$$

for $1 \le \ell \le m$, $1 \le \rho \le m$ and $m+1 \le \omega \le d$.

Moreover, from (3.6) we get

$$\int_0^{\phi} F_{20}^{+}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \alpha^{+}_{ijk_1\ldots k_m 0}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i+1,j,\phi)}$$

$$+\sum_{P=0}^{n} \beta^{+}_{ijk_1\ldots k_m 0}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1\ldots k_m 0} b^{+}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+2,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1\ldots k_m 0} a^{+}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b^{+}_{ijk_1\ldots k_m 0} b^{+}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1\ldots k_m 0} b^{+}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+2,\phi)},$$

$$\int_{\phi}^{2\pi} F_{20}^{-}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \alpha^{-}_{ijk_1\ldots k_m 0}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i+1,j,\phi)}$$

$$+\sum_{P=0}^{n} \beta^{-}_{ijk_1\ldots k_m 0}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i,j+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{-}_{ijk_1\ldots k_m 0} b^{-}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+2,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{-}_{ijk_1\ldots k_m 0} a^{-}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b^{-}_{ijk_1\ldots k_m 0} b^{-}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{-}_{ijk_1\ldots k_m 0} b^{-}_{pqL_1\ldots L_m 0}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p,j+q+2,\phi)},$$

$$\int_0^{\phi} F_{2\ell}^{+}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \gamma^{+}_{\ell,ijk_1\ldots k_m} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b^{+}_{ijk_1\ldots k_m 0} c^{+}_{\ell,pqL_1\ldots L_m 0} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1\ldots k_m 0} c^{+}_{\ell,pqL_1\ldots L_m 0} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+1,\phi)},$$

$$\int_{\phi}^{2\pi} F_{2\ell}^{-}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \gamma_{\ell,ijk_1\ldots k_m 0}^{-} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i,j,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b_{ijk_1\ldots k_m 0}^{-} c_{\ell,pqL_1\ldots L_m 0}^{-} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0} a_{ijk_1\ldots k_m 0}^{-} c_{\ell,pqL_1\ldots L_m 0}^{-} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p,j+q+1,\phi)},$$

for $1 \le \ell \le m$.

On the other hand when the perturbation is continuous that is, $\phi = 2\pi$, we have

$$\widetilde{G}_{10}(\nu) = \sum_{\omega=m+1}^{d}\left[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\right.$$

$$\left.\left(a_{ijk_1\ldots k_m 1_\omega}^{+}\int_0^{2\pi} e^{\mu_\omega s}\cos^{i+1} s\,\sin^j s\,ds + b_{ijk_1\ldots k_m 1_\omega}^{+}\int_0^{2\pi} e^{\mu_\omega s}\cos^i s\sin^{j+1} s\,ds\right)\right]$$

$$\left[\frac{-1}{1-e^{-\mu_\omega 2\pi}}\sum_{Q=0}^{n} r^{i+j} z_1^{L_1}\ldots z_m^{L_m}\; c_{\omega,ijL_1\ldots L_m 0}^{+}\int_0^{2\pi} e^{-\mu_\omega s}\cos^i s\,\sin^j s\,ds\right],$$

and

$$\int_0^{2\pi} F_{20}^{+}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0,\,i\,odd,\,j\,even}^{n} \alpha_{ijk_1\ldots k_m 0}^{+} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i+1,j,2\pi)}$$

$$+\sum_{P=0,\,i\,even,\,j\,odd}^{n} \beta_{ijk_1\ldots k_m 0}^{+} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j+1,2\pi)}$$

$$-\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\sum_{p,q\,even,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+} r^{i+j+p+q} z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\sum_{p,q\,odd,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+}\, r^{i+j+p+q} z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\sum_{p\,even,q\,odd,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+}\, r^{i+j+p+q} z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\ \sum_{p\,odd,q\,even,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+2,j+q,2\pi)}$$
$$+\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\ \sum_{p,q\,even,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}a^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$+\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\ \sum_{p,q\,odd,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}a^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$+\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\ \sum_{p\,odd,q\,even,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}a^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$+\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\ \sum_{p\,even,q\,odd,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}a^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$-\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\ \sum_{p,q\,even,Q=0}^{n} b^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$-\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\ \sum_{p,q\,odd,Q=0}^{n} b^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$-\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\ \sum_{p\,odd,q\,even,Q=0}^{n} b^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$-\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\ \sum_{p\,even,q\,odd,Q=0}^{n} b^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$
$$+\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\ \sum_{p,q\,even,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$
$$+\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\ \sum_{p,q\,odd,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$

$$+\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\sum_{p\,even,q\,odd,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$
$$+\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\sum_{p\,odd,q\,even,Q=0}^{n} a^{+}_{ijk_1\ldots k_m0}b^{+}_{pqL_1\ldots L_m0}\,r^{i+j+p+q}z_1^{k_1+L_1}\ldots$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}.$$

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