BRANCHING OF LIMIT CYCLES FROM FAMILIES OF PERIODIC SOLUTIONS IN PIECEWISE DIFFERENTIAL SYSTEMS

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Abstract. Consider a differential system on the form

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

where $F_i : \mathbb{S}^1 \times D \to \mathbb{R}^m$ and $R : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^m$ are piecewise C^{k+1} functions and T-periodic in the variable t. Assuming that the unperturbed system $x' = F_0(t,x)$ has a d-dimensional submanifold of periodic solutions with $d < m$ we use the Lyapunov-Schmidt reduction method and the averaging theory to study the existence of limit cycles of the above differential system.

Keywords Lyapunov-Schmidt reduction · periodic solution · averaging method · non-smooth differential system · piecewise smooth differential system

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1. Introduction and Statement of the main result

1.1. **Introduction.** The study of the existence of invariant sets, in especial periodic solutions, is very important for understanding the dynamics of a differential system. A *limit cycle* of a differential system is a periodic solution isolated in the set of all periodic solutions of the differential system. It is well known that there exists a relation between the periodic solutions of a system and the zeros of a Poincaré map and the

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displacement function. In this sense the averaging theory is one of the important tools to detect periodic solutions in m-dimensional systems on the form

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon). \tag{1}$$

A classical introduction to the averaging theory can be found in [21, 22]. Consider the unperturbed system $x' = F_0(t,x)$ and its set of initial conditions whose orbits are periodic denoted here by $\mathcal{Z}$. Assume that the set $\mathcal{Z}$ is a d-dimensional submanifold of $\mathbb{R}^m$ such that either $\dim(\mathcal{Z}) = d = m$) or $\dim(\mathcal{Z}) = d < m$.

When $\dim(\mathcal{Z}) = d = m$ we have many works that study the number of limit cycles of system (1). Assuming that $k \in \{1,2\}$, $F_0 \equiv 0$ and F_1, F_2 are T- periodic functions in the first variable and locally Lipschitz in the second variable Buica and Llibre proved in [3] that the number of limit cycles of (1) is controlled by the number of zeros of some functions called *average functions* that depend on F_1 if $k = 1$ and of F_1 and F_2 if $k = 2$. In [7] the authors studied the case where F_0 is zero or not and F_i are analytic functions for every $k = 1, 2, \dots, n$, and in [16] it was studied the averaging theory at any order when the functions F_i are only continuous and T-periodic on the first variable.

The averaging theory can be extended to discontinuous differential systems. The study of discontinuous differential systems is important in many fields of the applied sciences because many problems of physics, engineering, economics, and biology are modeled using differential equations with discontinuous right-hand side, see for instance [2, 6, 20]. So, there is a natural interest in studying the averaging theory for discontinuous systems. This was the main objective of the works [11, 12, 14, 18].

When $\dim(\mathcal{Z}) = d < m$ only the averaging theory is not enough to study the number of limit cycles of the systems and other techniques need to be employed together, as the *Lyapunov-Schmidt reduction method*. In the case that F_i are smooth functions we have the works [4, 5, 8]. If the functions F_i are not smooth or even continuous we have the works [13, 14], where the authors studied some classes of these systems.

A *piecewise smooth vector field* defined on an open bounded set $U \subset \mathbb{R}^m$ is a function $F : U \to \mathbb{R}^m$ which is smooth except on a set Σ of zero measure, called the *discontinuity set* of the vector field F. We suppose that $U \setminus \Sigma$ is a finite union of disjoint open sets U_i, $i = 1, 2, \dots, n$, where the restriction $F_i = F\big|_{U_i}$ can be extended continuously to $\overline{U_i}$. The orbit of F at a point $p \in U_i$ is defined as usual for a differential system. But if $p \in \Sigma$ then the definition of this orbit through p is more delicate. In [9] Filippov used the theory of differential inclusion (see [1]) to give the definition of what is a local orbit at the points of discontinuity where the set Σ is locally a codimension one embedded submanifold of $\mathbb{R}^m$. If $p \in \Sigma$ and U_p is a small neighborhood of p then we divide $U_p \setminus \Sigma$ in two disjoint open sets U_p^+ and U_p^- and write $F^{\pm}(p) = F\big|_{U_p^{\pm}}(p)$.

In short, let $\mathcal{S} \subset \Sigma$ be an embedded hypersurface in $\mathbb{S}^1 \times D$ and $T_p\mathcal{S}$ denotes the tangent space of $\mathcal{S}$ at the point p. Let $l(p)$ be the segment connecting the vectors $F^+(p)$ and $F^-(p)$ and the *crossing region* of the hypersurface $\mathcal{S}$ is the set $\Sigma^c(\mathcal{S}) = \{p \in \mathcal{S} : l(p) \cap T_p\mathcal{S} = \emptyset\}$. For a point p on the crossing region the local orbit of F at p is given as the concatenation of the local trajectories of $F^{\pm}$ at p. In this case we say that the orbit *crosses* the set of discontinuity and that p is a *crossing point*. When p is not a crossing point we say that p is a *sliding point* and the local trajectory of F at p slides on Σ. For more details on the Filippov conventions see [9, 10].

In what follows we describe how to use the averaging theory and Lyapunov-Schmidt reduction method for computing isolated periodic solutions of the piecewise smooth differential systems. Then, we set the class of non-autonomous discontinuous piecewise smooth differential equations that we are interested as well as our main result (Theorem A).

1.2. **Lyapunov-Schmidt reduction.** Consider the function

$$g(z,\varepsilon)=\sum_{i=0}^{k}\varepsilon^i g_i(z)+\mathcal{O}(\varepsilon^{k+1}), \tag{2}$$

where $g_i : D \to \mathbb{R}^m$ is a C^{k+1} function, $k \geq 1$, for $i = 0, 1, \ldots, k$, in which D an open bounded subset of $\mathbb{R}^m$. For $d < m$, let V be an open bounded subset of $\mathbb{R}^d$ and $\beta : \overline{V} \to \mathbb{R}^{m-d}$ a C^{k+1} function such that

$$\mathcal{Z}=\{z_\alpha=(\alpha,\beta(\alpha)) : \alpha\in\overline{V}\}\subset D. \tag{3}$$

The main hypothesis is

(H_α) the function g_0 vanishes on the d-dimensional submanifold $\mathcal{Z}$ of D.

In [5] the authors used the Lyapunov-Schmidt reduction method to develop the *bifurcation functions of order* i, for $i = 0, 1, \ldots, k$, which for $|\varepsilon| \neq 0$ sufficiently small control the existence of branches of zeros $z(\varepsilon)$ of system (2) that bifurcate from $z(0) \in \mathcal{Z}$. In this subsection we present the results developed in that work and those that we shall need later on.

First we present some notation. Consider the projections onto the first d coordinates and onto the last $m - d$ coordinates denoted by $\pi : \mathbb{R}^d \times \mathbb{R}^{m-d} \to \mathbb{R}^d$ and $\pi^\perp : \mathbb{R}^d \times \mathbb{R}^{m-d} \to \mathbb{R}^{m-d}$, respectively. Also, for a point $z \in \mathcal{Z}$ we write $z = (a, b) \in \mathbb{R}^d \times \mathbb{R}^{m-d}$.

Let L be a positive integer, let $x = (x_1, x_2, \ldots, x_m) \in D$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, \ldots, y_{jm}) \in \mathbb{R}^m$ for $j = 1, \ldots, L$. Given $G : \mathbb{R} \times D \to \mathbb{R}^m$ a sufficiently smooth function, for each $(t, x) \in \mathbb{R} \times D$ we denote by $\partial^L G(t, x)$ a symmetric L–multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^m$, which we denote as $\bigodot_{j=1}^{L} y_j \in \mathbb{R}^{mL}$. The definition of this L–multilinear map is

$$\partial^L G(t,x)\bigodot_{j=1}^{L} y_j=\sum_{i_1,\ldots,i_L=1}^{n}\frac{\partial^L G(t,x)}{\partial x_{i_1}\ldots\partial x_{i_L}}y_{1i_1}\ldots y_{Li_L}.$$

We define ∂^0 as the identity functional.

The bifurcation functions $f_i : \overline{V} \to \mathbb{R}^d$ of order i are defined for $i = 0, 1, \ldots, k$ as

$$f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \ldots c_l! l!^{c_l}} \partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}, \tag{4}$$

where the $\gamma_i : V \to \mathbb{R}^{m-d}$, for $i = 1, 2, \ldots, k$, are defined recursively as

$$\begin{aligned} \gamma_1(\alpha) =&\ -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) =&\ -i! \Delta_\alpha^{-1} \Bigg(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \ldots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{I'} \pi^\perp g_0(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \\ &+ \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \ldots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} \Bigg). \end{aligned} \tag{5}$$

We denote by S_l the set of all l-tuples of non-negative integers $(c_1, c_2, \ldots, c_l)$ such that $c_1 + 2c_2 + \ldots + lc_l = l$, $L = c_1 + c_2 + \ldots + c_l$, and by S_i' the set of all $(i-1)$-tuples of non-negative integers $(c_1, c_2, \ldots, c_{i-1})$ such that $c_1 + 2c_2 + \ldots + (i-1)c_{i-1} = i$, $I' = c_1 + c_2 + \ldots + c_{i-1}$ and $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$.

About the zeros of the function (2) the authors proved in [5] the following result.

Theorem 1. *Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the Jacobian matrix $Dg_0(z_\alpha)$. Additionally to hypothesis (H_α) we assume that*

(i) *for each $\alpha \in \overline{V}$, $\det \Delta_\alpha \neq 0$; and*
(ii) *$f_1 = f_2 = \ldots = f_{k-1} = 0$ and f_k is not identically zero.*

If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a branch of zeros $z(\varepsilon)$ with $g(z(\varepsilon), \varepsilon) = 0$ and $|z(\varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.

1.3. **The averaging theory.** In [5], using Theorem 1, the authors studied high order bifurcation of periodic solutions of the following T-periodic C^{k+1} with $k \geq 1$ differential system

$$x' = F(t, x, \varepsilon) = F_0(t, x) + \sum_{i=1}^{k} \varepsilon^i F_i(t, x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t, z) \in \mathbb{S}^1 \times D, \tag{6}$$

where the prime denotes the derivative with respect to the independent variable t, usually called the time. In their work they assumed that the manifold $\mathcal{Z}$, defined in (3), is such that all solutions of the unperturbed system

$$x' = F_0(t, x),$$

starting at points of $\mathcal{Z}$ are T-periodic and $\dim \mathcal{Z} \leq m$.

Consider the variational equation

$$y' = \frac{\partial F_0}{\partial x}(t, x(t, z, 0))y, \tag{7}$$

where $x(t,z,0)$ denotes the solution of system (6) when $\varepsilon = 0$, and we denote a fundamental matrix of system (7) by $Y(t,z)$. The *average function of order* i of system (6) is defined as

$$g_i(z) = Y^{-1}(T,z)\frac{y_i(T,z)}{i!}, \tag{8}$$

where

$$\begin{aligned} y_1(t,z) = &\ Y(t,z)\int_0^t Y(s,z)^{-1}F_1\left(s,x(s,z,0)\right)ds,\\ y_i(t,z) = &\ i!Y(t,z)\int_0^t Y(s,z)^{-1}\Big(F_i\left(s,x(s,z,0)\right)\\ &+\sum_{S_i'}\frac{1}{b_1!\,b_2!2!^{b_2}\dots b_{i-1}!(i-1)!^{b_{i-1}}}\partial^{I'}F_0(s,x(s,z,0))\bigodot_{j=1}^{i-1}y_j(s,z)^{b_j}\\ &+\sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\dots b_l!l!^{b_l}}\partial^{L}F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Big)ds. \end{aligned} \tag{9}$$

Using the functions g_i stated in (8) are defined the functions f_i and γ_i given by (4) and (5), respectively. Under some assumptions and with Theorem 1 it was proved that the simple zeros of the functions f_i provide the existence of isolated periodic solutions of the differential system (6). By a simple zero of a function f we mean a point a such that $f(a) = 0$ and $\det(Df(a)) \neq 0$, where $Df(a)$ denotes the Jacobian matrix of f at the point a.

Remark 1. *The functions $y_i(t,z)$ could be defined recurrently by an integral equation as done in other works (see* [11, 16, 17]*). Indeed, we define*

$$\begin{aligned} y_1(t,z) = &\int_0^t\Big(F_1\left(s,x(s,z,0)\right)+\partial F_0(s,x(s,z,0))y_1(s,z)\Big)ds,\\ y_i(t,z) = &\ i!\int_0^t\Big(F_i\left(s,x(s,z,0)\right)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\dots b_l!l!^{b_l}}\\ &\cdot\partial^L F_{i-l}\left(s,x(s,z,0)\right)\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Big)ds,\ \textit{for}\ i=2,\dots,k, \end{aligned} \tag{10}$$

and it is not difficult to see that solving this integral equations we obtain the formulae (9)*.*

For more details on the results of this subsection 1.2 see [5].

1.4. **Standard form and main result.** Let $n > 1$ be a positive integer. For $i = 0,1,\dots,k$ and $j = 1,2,\dots,n$ let $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}$ and $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ be functions C^{k+1} where D is an open subset of $\mathbb{R}^m$ and $\mathbb{S}^1 \equiv \mathbb{R}/(T\mathbb{Z})$. We define

$$F_i(t,x) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(t) F_i^j(t,x), \; i = 0,1,...,k, \quad \text{and}$$

$$R(t,x,\varepsilon) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(t) R^j(t,x,\varepsilon),$$

where $\chi_A(t)$ is the characteristic function of A defined as

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$$

The notation $t \in \mathbb{S}^1 \equiv \mathbb{R}/(T\mathbb{Z})$ means that all the above functions are T-periodic in the variable t.

Consider the discontinuous and T-periodic differential system

$$x' = F(t,x,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{11}$$

and the submanifold $\mathcal{Z}$ of periodic solutions of the unperturbed system

$$x' = F_0(t,x). \tag{12}$$

The set Σ of discontinuity of system (11) is given by

$$\Sigma = (\{t = 0 \equiv T\} \cup \{t = t_1\} \cup \ldots \cup \{t = t_{n-1}\}) \cap \mathbb{S}^1 \times D,$$

where $0 < t_1 < t_2 < \ldots < t_{n-1} < T$.

For each $j = 1,2,\ldots,n$ and $t \in [t_{j-1}, t_j]$ we have the differential system

$$x' = \sum_{i=0}^{k} \varepsilon^i F_i^j(t,x) + \varepsilon^{k+1} R^j(t,x,\varepsilon). \tag{13}$$

To continue we need to give some definition about system (11). For each $z \in D$ and ε sufficiently small we denote by $x(\cdot,z,\varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^m$ the solution of system (11) such that $x(0,z,\varepsilon) = z$, where $[0, t_{(z,\varepsilon)})$ is the interval of definition for the solution $x(t,z,\varepsilon)$.

Consider the submanifold $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \bar{V}\}$, where V is an open bounded subset of $\mathbb{R}^m$, and $\beta_0 : \overline{V} \to \mathbb{R}^{d-m}$ is a C^k function with $k \geq 1$. Notice that for each $z_\alpha \in \mathcal{Z}$, $(t_i, x(t_i, z_\alpha, 0) \in \Sigma^c)$, for $i \in \{0,1,\ldots,k\}$. Indeed, for each $j = 1,2,\ldots,n$, the set of discontinuity can be locally described by $h_j^{-1}(0)$, where $f : \mathbb{S}^1 \times D \to \mathbb{R}$ is $h_j(t,x) = t - t_j$. It is known that to show that we are in the crossing region it is sufficient to prove that $\langle \nabla h_j(t,x), F^j(t,x)\rangle \langle \nabla h_j(t,x), F^{j+1}(t,x)\rangle > 0$ (see [16]), where $\nabla h_j(t,x)$ denotes the gradient vector of the function $h_j(t,x)$. Here, $\nabla h_j(t,x) = (1,0)$ and $\langle \nabla h_j(t,x), F^j(t,x)\rangle \langle \nabla h_j(t,x), F^{j+1}(t,x)\rangle = 1 > 0$.

In [15], the averaging theory was developed assuming $\dim(\mathcal{Z}) = m$. Here, we are interested in the case $\dim(\mathcal{Z}) < m$. Accordingly, we shall extend the average functions

(8) and the bifurcation functions (4) obtained in [5] to this class of discontinuous differential system, providing then sufficient conditions in order to control which periodic solutions of $\mathcal{Z}$, with $\dim \mathcal{Z} = d < m$, persists to $\varepsilon \neq 0$ sufficiently small.

For system (12) we consider the fundamental matrix $Y(t,z)$ of the variational system

$$y' = \frac{\partial}{\partial x} F_0(t, x(t,z,0))y, \tag{14}$$

where Y is an $m \times m$ matrix. Notice that, for each $j = 1, 2, \ldots, n$, if $x_j(t,z,\varepsilon)$ denotes the solution of (13) for $t_{j-1} \le t \le t_j$, the function $t \mapsto (\partial x_j/\partial z)(t,z,0)$ is a solution of (14) for $t_{j-1} \le t \le t_j$. Recall that the right product of a solution of the variational equation (14) by constant matrix is still a solution of (14). Therefore, the solution $Y(t,z)$ can be built as follows:

$$Y(t,z) = \begin{cases} Y_1(t,z) & \text{if } 0 = t_0 \le t \le t_1, \\ Y_2(t,z) & \text{if } t_1 \le t \le t_2, \\ \vdots & \\ Y_n(t,z) & \text{if } t_{n-1} \le t \le t_n = T, \end{cases}$$

where

$$\begin{aligned} Y_1(t,z) &= \frac{\partial x_1}{\partial z}(t,z,0), \quad \text{and} \\ Y_j(t,z) &= \frac{\partial x_j}{\partial z}(t,z,0)\left(\frac{\partial x_j}{\partial z}(t_{j-1},z,0)\right)^{-1} Y_{j-1}(t_{j-1},z), \quad \text{for } j = 2,3,\ldots,n. \end{aligned} \tag{15}$$

Our main result says that the simple zeros of the bifurcation functions (4) also controls the branching of isolated periodic solutions of the nonsmooth system (11). The derivatives $\partial^j F_i(s,z)$, which appears in (4), are computed as follows:

$$\frac{\partial^j F_i}{\partial z}(s,z) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(s) \frac{\partial^j F_i^j}{\partial z^j}(s,z).$$

Theorem A. *Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the matrix $Id - Y^{-1}(T,z)$. We assume that the functions defined by (4) and (8) satisfy $f_1 = f_2 = \ldots = f_{k-1} = 0$ and that for each $\alpha \in \overline{V}$, $\det(\Delta_\alpha) \neq 0$. If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$, and that $\det(Df_k(\alpha^*)) \neq 0$, then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of (11) such that $|\varphi(0,\varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

The paper is organized as follows. In section 2 we present the explicit formulae (8) for the average functions of the nonsmooth differential system (11). In section 3 we prove Theorem A and in Section 4 we give two applications of TheoremA.

2. An algorithm for the bifurcation functions

In this section we will provide an algorithm for computing the average functions, defined in (10), for the nonsmooth case. Their expressions are defined recurrently and using *Bell polynomials*, which can be implemented more easily. This is because softwares such as Mathematica and Maple have already implemented these polynomials. In [19], it was proved that the average functions defined for smooth cases can be computed using *Bell polynomials* and in [15] the authors did the same for the nonsmooth case. For each pair of nonnegative integers (p, q), the partial Bell polynomial is defined as

$$B_{p,q}(x_1, x_2, \dots, x_{p-q+1}) = \sum_{\widetilde{S}_{p,q}} \frac{p!}{b_1!b_2!\dots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!}\right)^{b_j},$$

where $\widetilde{S}_{p,q}$ is the set of all $(p-q+1)$-tuple of nonnegative integers $(b_1, b_2, \dots, b_{p-q+1})$ satisfying $b_1 + 2b_2 + \dots + (p-q+1)b_{p-q+1} = p$, and $b_1 + b_2 + \dots + b_{p-q+1} = q$. Moreover, if g and h are sufficiently smooth functions, then using Bell polynomials we have that

$$\frac{d^l}{dx^l} g(h(x)) = \sum_{m=1}^{l} g^{(m)}(h(x)) B_{l,m}(h'(x), h''(x), \dots, h^{(l-m+1)}(x)).$$

2.1. **Average Functions.** In this section we develop a recurrence to compute the average function (8) in the particular case of the discontinuous differential equation (11). So, consider the functions $w_i^j : (t_{j-1}, t_j] \times D \to \mathbb{R}$ defined recurrently for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$, as

(16)

$$\begin{aligned}
w_1^1(t,z) = \;& \int_0^t \Big(F_1^1(s, x(s,z,0)) + \partial F_0^1(s, x(s,z,0)) w_1^1(s,z) \Big) ds, \\
w_i^1(t,z) = \;& i! \int_0^t \Big(F_i^1(s, x(s,z,0)) + \\
& \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^1(s, x(s,z,0)) \bigodot_{m=1}^{l} w_m^1(s,z)^{b_m} \Big) ds, \\
w_i^j(t,z) = \;& w_i^{j-1}(t_{j-1}, z) + i! \int_{t_{j-1}}^t \Big(F_i^j(s, x(s,z,0)) + \\
& \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^j(s, x(s,z,0)) \bigodot_{m=1}^{l} w_m^j(s,z)^{b_m} \Big) ds.
\end{aligned}$$

Since $F_0 \neq 0$ the recurrence defined in (16) is an integral equation and the next lemma solves it using Bell polynomials.

Lemma 2. *For $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$ the recurrence* (16) *can be written as follows*

$$
\begin{aligned}
w_1^1(t,z) &= Y_1(t,z)\int_0^t Y_1^{-1}(s,z)F_1^1(s,x(s,z,0))ds,\\
w_1^j(t,z) &= Y_j(t,z)\Big(Y_j^{-1}(t_{j-1},z)w_1^{j-1}(t_{j-1},z)+\int_{t_{j-1}}^t Y_j^{-1}(s,z)F_1^j(s,x(s,z,0))ds\Big),\\
w_i^1(t,z) &= Y_1(t,z)\int_0^t Y_1^{-1}(s,z)\Big(i!F_i^1\,(s,x(s,z,0))\\
&\quad+\sum_{m=2}^{i}\partial^m F_0^1(s,x(s,z,0)).B_{i,m}(w_1^1,\dots,w_{i-m+1}^1),\\
&\quad+\sum_{l=1}^{i-1}\sum_{m=1}^{l}\frac{i!}{l!}\partial^m F_{i-l}^1(s,x(s,z,0)).B_{l,m}(w_1^1,\dots,w_{l-m+1}^1)\Big)ds,\\
w_i^j(t,z) &= Y_j(t,z)\Big[Y_j^{-1}(t_{j-1},z)w_i^{j-1}(t_{j-1},z)+\int_{t_{j-1}}^t Y_j^{-1}(s,z)\Big(i!F_i^j\,(s,x(s,z,0))\\
&\quad+\sum_{m=2}^{i}\partial^m F_0^j(s,x(s,z,0)).B_{i,m}(w_1^j,\dots,w_{i-m+1}^j),\\
&\quad+\sum_{l=1}^{i-1}\sum_{m=1}^{l}\frac{i!}{l!}\partial^m F_{i-l}^j(s,x(s,z,0)).B_{l,m}(w_1^j,\dots,w_{l-m+1}^j)\Big)ds.\Big]
\end{aligned}
$$

Proof. The idea of the proof is to relate the integral equations (16) to the Cauchy problem and then solve it. For example, if $i = j = 1$ the integral equation is equivalent to the following Cauchy problem

$$\frac{\partial w_1^1}{\partial t}(t,z) = F_1^1\,(t,x(t,z,0)) + \partial F_0^1\,(t,x(t,z,0))\,w_1^1 \quad \text{with} \quad w_1^1(0,z) = 0,$$

and solving this linear differential equation we get the expression of $w_1^1(t,z)$ described in the statement of the lemma. For more details see [15]. □

Now, we provide a formula for the average functions (8) for the class of discontinuous differential systems studied in this paper.

Proposition 3. *For $i = 1, 2, \dots, k$, the average function* (8) *of order i is*

$$g_i(z) = Y_n^{-1}(T,z)\frac{w_i^n(T,z)}{i!}.$$

Proof. For each $i = 1, 2, \dots, k$ we define

$$w_i(t,z) = \sum_{j=1}^{n}\chi_{[t_{j-1},t_j]}(t)w_i^j(t,z).$$

Given $t \in [0,T]$ there exists a positive integer $\bar{k}$ such that $t \in (t_{\bar{k}-1}, t_{\bar{k}}]$ and, therefore, $w_i(t,z) = w_i^{\bar{k}}(t,z)$. By the proof of Proposition 2 of [15] we obtain

$$
\begin{aligned}
w_1(t,z) = & \int_0^t \Big(F_1(s,x(s,z,0)) + \partial F_0(s,x(s,z,0)) w_1(s,z) \Big) ds, \\
w_i(t,z) & \cdot \partial^L F_{i-l}(s,x(s,z,0)) \bigodot_{m=1}^{l} w_m(s,z)^{b_m} \Big) ds.
\end{aligned}
\tag{17}
$$

Since by Remark 1 we can consider the functions (9) given implicitly, we compute the derivatives in the variable t of the functions (17) and (10) for $i = 1$, and we see that the functions $w_1(t,z)$ and $y_1(t,z)$ satisfy the same differential equation. Moreover, for each $i = 2, \ldots, k$, the integral equations (10) and (17), which provide respectively y_i and w_i, are defined by the same recurrence. Then the functions y_i and w_i satisfy the same differential equations for $i = 1, 2, \ldots, k$, and their initial conditions coincide. Indeed, let $i \in \{1, 2, \ldots, k\}$, since $y_i(0,z) = 0$ and, by (17), $w_i(0,z) = 0$, it follows that the initial conditions are the same. Applying the Existence and Uniqueness Theorem on the solutions of the differential system we get $y_i(t,z) = w_i(t,z)$, for all $i \in \{1, 2, \ldots, k\}$. □

2.2. **Bifurcation Functions.** In this section we shall write the bifurcation functions (4) and the functions $\gamma_i(\alpha)$ given by (5) in terms of Bell polynomials.

Claim 1. *The bifurcation function* (4) *is given by*

$$
f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^{i} \sum_{m=1}^{l} \frac{1}{l!} \partial_b^m \pi g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha), \ldots, \gamma_{l-m+1}(\alpha)),
$$

where

$$
\begin{aligned}
\gamma_1(\alpha) = & \; -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \textit{and} \\
\gamma_i(\alpha) = & \; -\Delta_\alpha^{-1} \Bigg(\sum_{l=0}^{i-1} \frac{i!}{l!} \sum_{m=1}^{l} \partial_b^m \pi^\perp g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha), \ldots, \gamma_{l-m+1}(\alpha)) \\
& + \sum_{m=2}^{i} \partial_b^m \pi^\perp g_0(z_\alpha) B_{i,m}(\gamma_1(\alpha), \ldots, \gamma_{i-m+1}(\alpha)) \Bigg).
\end{aligned}
$$

Proof. The expressions (4) was obtained in [5] using the Faá di Bruno's formula for the L-th derivative of a composite function. This claim follows just by applying the version of the Faá di Bruno's formula in terms of the Bell polynomials (see [19, 5]). □

3. Proof of Theorem A

For $j = 1, 2, \ldots, n$ let $\xi_j(t, t_0, z_0, \varepsilon)$ be the solution of the discontinuous differential system (13) such that $\xi_j(t_0, t_0, z_0, \varepsilon) = z_0$. Then, we define the recurrence

$$\begin{aligned} &x_1(t, z, \varepsilon) = \xi_1(t, 0, z, \varepsilon)\\ &x_j(t, z, \varepsilon) = \xi_j(t, t_{j-1}, x_{j-1}(t_{j-1}, z, \varepsilon), \varepsilon), \quad j = 2, \ldots, n. \end{aligned}$$

Since we are working in the cross region it is easy to see that, for $|\varepsilon| \neq 0$ sufficiently small, each $x_j(t, z, \varepsilon)$ is defined for every $t \in [t_{j-1}, t_j]$. Therefore $x(\cdot, z, \varepsilon) : [0, T] \to \mathbb{R}$ is defined as

$$x(t, z, \varepsilon) = \begin{cases} x_1(t, z, \varepsilon) & \text{if } 0 = t_0 \leq t \leq t_1,\\ x_2(t, z, \varepsilon) & \text{if } t_1 \leq t \leq t_2,\\ \vdots & \\ x_j(t, z, \varepsilon) & \text{if } t_{j-1} \leq t \leq t_j,\\ \vdots & \\ x_n(t, z, \varepsilon) & \text{if } t_{n-1} \leq t \leq t_n = T. \end{cases}$$

Notice that $x(t, z, \varepsilon)$ is the solution of the differential equation (12) such that $x(0, z, \varepsilon) = z$. Moreover, the following equality hold

$$x_j(t_{j-1}, z, \varepsilon) = x_{j-1}(t_{j-1}, z, \varepsilon),$$

for $j = 1, 2, \ldots, n$.

The next lemma expands the solution $x_j(\cdot, z, \varepsilon)$ around $\varepsilon = 0$.

Lemma 4. *For $j \in \{1, 2, \ldots, n\}$ and $t_z^j > t_j$, let $x_j(\cdot, z, \varepsilon) : [t_{j-1}, t_j)$ be the solution of* (13)*. Then*

$$x_j(t, z, \varepsilon) = x_j(t, z, 0) + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} w_i^j(t, z) + \mathcal{O}(\varepsilon^{k+1}).$$

Proof. First, fixed $j \in \{1, 2, \ldots, n\}$, we use the continuity of the solution $x_j(t, z, \varepsilon)$ and the compactness of the set $[t_{j-1}, t_j] \times \overline{D} \times [-\varepsilon_0, \varepsilon_0]$ to get that

$$\int_{t_{j-1}}^{t} R^j(s, x_j(s, z, \varepsilon), \varepsilon) ds = \mathcal{O}(\varepsilon), \quad t \in [t_{j-1}, t_j].$$

Thus, integrating the differential equation (13) from t_{j-1} to t, we get

$$\begin{aligned} x_j(t, z, \varepsilon) &= x_j(t_{j-1}, z, \varepsilon) + \sum_{i=0}^{k} \varepsilon^i \int_{t_{j-1}}^{t} F_i^j(s, x_j(s, z, \varepsilon)) ds + \mathcal{O}(\varepsilon^{k+1}), \quad \text{and}\\ x_j(t, z, 0) &= x_j(t_{j-1}, z, 0) + \int_{t_{j-1}}^{t} F_0^j(s, x_j(s, z, 0)) ds. \end{aligned} \tag{18}$$

By the differentiable dependence of the solutions of a differential system on its parameters the function $\varepsilon \mapsto x_j(t, z, \varepsilon)$ is a C^{k+1} map. Then, the next step is to compute the Taylor expansion of $F_i^j(t,x_j(t,z,\varepsilon))$ around $\varepsilon=0$ and for this we use the Faá di Bruno's Formula about the l-th derivative of a composite function, which guarantees that if g and h are sufficiently smooth functions then

$$\frac{d^l}{d\alpha^l}g(h(\alpha))=\sum_{S_l}\frac{l!}{b_1!\,b_2!2!^{b_2}\ldots b_l!l!^{b_l}}g^{(L)}(h(\alpha))\bigodot_{j=1}^{l}\Big(h^{(j)}(\alpha)\Big)^{b_j},$$

where S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,\ldots,b_l)$ satisfying $b_1+2b_2+\ldots+lb_l=l$, and $L=b_1+b_2+\ldots+b_l$.

For each $i=0,1,...,k-1$, expanding $F_i^j(s,x_j(s,z,\varepsilon))$ around $\varepsilon=0$ we get

(19)
$$\begin{aligned}F_i^j(s,x_j(s,z,\varepsilon))=\ & F_i^j(s,x_j(s,z,0))+\\ & \sum_{l=1}^{k-i}\sum_{S_l}\frac{\varepsilon^l}{b_1!\,b_2!2!^{b_2}\ldots b_l!l!^{b_l}}\partial^L F_i^j(s,x_j(s,z,0))\bigodot_{m=1}^{l}r_m^j(s,z)^{b_m},\end{aligned}$$

where

$$r_m^j(s,z)=\frac{\partial^m}{\partial\varepsilon^m}x_j(s,z,\varepsilon)\Big|_{\varepsilon=0},$$

and for $i=k$

(20)
$$F_k^j(s,x_j(s,z,\varepsilon))=F_k^j(s,x_j(s,z,0))+\mathcal{O}(\varepsilon).$$

Substituting (19) and (20) in (18) we get

$$\begin{aligned}x_j(t,z,\varepsilon)=\ & x_j(t_{j-1},z,\varepsilon)+\int_{t_{j-1}}^{t}\left(\sum_{i=0}^{k}\varepsilon^i F_i^j(s,x_j(s,z,0))ds\right.\\ & +\sum_{i=0}^{k-1}\sum_{l=1}^{k-i}\varepsilon^{l+i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\ldots b_l!l!^{b_l}}\\ & \left.\cdot\partial^L F_i^j(s,x_j(s,z,0))\bigodot_{m=1}^{l}r_m^j(s,z)^{b_m}\right)ds+\mathcal{O}(\varepsilon^{k+1}).\end{aligned}$$

Then, the proof of the lemma ends using the next two claims.

Claim 2. *For $j=1,2,\ldots,n$ we have*

$$x_j(t,z,\varepsilon)=x_j(t,z,0)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}r_i^j(t,z)+\mathcal{O}(\varepsilon^{k+1}).$$

Claim 3. *The equality $r_i^j=w_i^j$ holds for $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$.*

□

Proof of Theorem A. Consider the *displacement function*

(21)
$$h(z,\varepsilon)=x(T,z,\varepsilon)-z=x_n(T,z,\varepsilon)-z$$

It is easy to see that $x(\cdot,\overline{z},\overline{\varepsilon})$ is a T-periodic solution if and only if $h(\overline{z},\overline{\varepsilon})=0$. Moreover, to study the zeros of (21) is equivalent to study the zeros of

$$g(z,\varepsilon)=Y_n^{-1}(T,z)h(z,\varepsilon). \tag{22}$$

From Lemma 4 we have that

$$x_n(T,z,\varepsilon)=x_n(T,z,0)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^n(T,z)+\mathcal{O}(\varepsilon^{k+1}), \tag{23}$$

for all $(t,z)\in\mathbb{S}^1\times D$. Replacing (23) in (22) it follows that

$$\begin{aligned} g(z,\varepsilon)= &\ Y_n^{-1}(T,z)\left(x_n(T,z,0)-z+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^n(T,z)+\mathcal{O}(\varepsilon^{k+1})\right)\\ = &\ Y_n^{-1}(T,z)(x_n(T,z,0)-z)+\sum_{i=1}^{k}g_i(z)+\mathcal{O}(\varepsilon^{k+1})\\ = &\ \sum_{i=0}^{k}g_i(z)+\mathcal{O}(\varepsilon^{k+1}),\end{aligned} \tag{24}$$

where $g_0(z)=Y_n^{-1}(T,z)(x_n(T,z,0)-z)$.

From hypothesis (H) the function $g_0(z)$ vanishes on the submanifold $\mathcal{Z}$, therefore hypothesis (H_α) holds for the function (24). In order to take the derivative of $g_0(z)$ with respect to the variable z we have the next claim.

Claim 4. *For every $j\in\{1,2,\dots,n\}$*

$$Y_j(t_j,z)=\frac{\partial x_j}{\partial z}(t_j,z,0).$$

The proof will be done by induction on j. For $j=1$ the claim is exactly the definition. Suppose that the claim is valid for $j=j_0-1$ and we shall prove it for $j=j_0$. Since $x_j(t_{j-1},z,\varepsilon)=x_{j-1}(t_{j-1},z,\varepsilon)$ for all $j=1,2,\dots,n$ we have

$$\begin{aligned} Y_{j_0}(t_{j_0},z) &= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0)\left(\frac{\partial x_{j_0}}{\partial z}(t_{j_0-1},z,0)\right)^{-1}Y_{j_0-1}(t_{j_0-1},z)\\ &= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0)\left(\frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1},z,0)\right)^{-1}\frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1},z,0)\\ &= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0).\end{aligned}$$

Hence if $z\in\mathcal{Z}$ then

$$\begin{aligned} \frac{\partial g_0}{\partial z}(z)= &\ Y^{-1}(T,z)\left(\frac{\partial x}{\partial z}(T,z,0)-Id\right)\\ = &\ Y^{-1}(T,z)(Y(T,z)-Id)\\ = &\ Id-Y^{-1}(T,z),\end{aligned}$$

which has by assumption its lower right corner $(m-d)\times(m-d)$ matrix Δ_α nonsingular. From here, the result follows from Proposition 3 and Theorem 1. □

4. Examples

This section is devoted to present some applications of Theorem A. The first one is as 3D piecewise smooth system for which the plane $y=0$ is the discontinuous manifold and admits a surface $z=f(x,y)$ foliated by periodic solutions. The second one is a 3D piecewise smooth system for which the algebraic variety $xy=0$ is the discontinuous set and the plane $z=0$ has a piecewise constant center. For these systems, we compute some of the bifurcations functions in order to study the persistence of periodic solutions.

4.1. **Nonsmooth perturbation of a 3D system.** Let $f:\mathbb{R}^2\to\mathbb{R}$ and $g:\mathbb{R}^2\to\mathbb{R}$ be differential functions such that $g(x,y)=f(x,y)+x\partial_y f(x,y)-y\partial_x f(x,y)$. Consider the nonsmooth vector field

$$X_\varepsilon(x,y,z)=\begin{cases} X_\varepsilon^+(x,y,z), & y>0 \\ X_\varepsilon^-(x,y,z), & y<0 \end{cases} \tag{25}$$

where

$$X_\varepsilon^+(x,y,z)=\big(-y+\varepsilon(a_0+a_1z)+\varepsilon^2(a_2+a_3z),\ x,\ -z+g(x,y)\big), \quad \text{and}$$

$$X_\varepsilon^-(x,y,z)=\big(-y,\ x+\varepsilon b_1 z+\varepsilon^2(b_2+b_3)z,\ -z+g(x,y)\big),$$

with a_0, a_1, a_2, b_1, b_2, $b_3\in\mathbb{R}$. Denote the discontinuous se by $\Sigma=\{(x,y,z)\in\mathbb{R}^3: y=0\}$.

Notice that the surface $z=g(x,y)$ is an invariant set of the unperturbed vector field X_0. Indeed, considering the function $\hat f(x,y,z)=z-f(x,y)$, we get

$$\langle\nabla\hat f(x,y,z),X_0(x,y,z)\rangle\big|_{z=f(x,y)}=0.$$

Moreover, since $X_0(x,y,f(x,y))=\big(-y,x,x\partial_y f(x,y)-y\partial_x f(x,y)\big)$ we conclude that the invariant set $z=f(x,y)$ is foliate by periodic solutions.

Next result gives suficient conditions in order to guarantee the persistence of some periodic solution. Consider the function

$$f_1(r)=a_1\int_0^\pi f(r\cos\phi,r\sin\phi)\cos\phi d\phi+b_1\int_\pi^{2\pi} f(r\cos\phi,r\sin\phi)\sin\phi d\phi. \tag{26}$$

Theorem 5. *Consider the piecewise vector field* (25)*. Then, for each* $r*>0$*, such that* $f_1(r^*)=0$ *and* $f_1'(r^*)\neq 0$*, there exists a crossing limit cycle* $\varphi(t,\varepsilon)$ *of* X *of period* $T_\varepsilon=2\pi+\mathcal{O}(\varepsilon)$ *such that* $\varphi(t,\varepsilon)=(x^*,y^*,f(x^*,y^*))+\mathcal{O}(\varepsilon)$ *with* $|(x^*,z^*)|=r^*$.

In order to apply Theorem A for proving Theorem 5 we need to write system (25) in the standard form. Considering cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the set of discontinuity becomes $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\}$ with $t_0 = 0, t_1 = \pi$ and $t_2 = 2\pi$. The differential system $(\dot{x}, \dot{y}, \dot{z}) = X_\varepsilon^+(x, y, z)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= \varepsilon(a_0 + a_1 z)\cos\theta + \varepsilon^2(a_2 + a_3 z)\cos\theta, \\ z'(t) &= g(r\cos\theta, r\sin\theta) - z, \\ \theta'(t) &= 1 - \varepsilon\frac{(a_0 + a_1 z)\sin\theta}{r} - \varepsilon^2\frac{(a_2 + a_3 z)\sin\theta}{r}, \end{aligned}$$

and the differential system $(\dot{x}, \dot{y}, \dot{z}) = X_\varepsilon^-(x, y, z)$ becomes

$$\begin{aligned} r'(t) &= \varepsilon b_1 z\sin\theta + \varepsilon^2(b_2 + b_3 z)\sin\theta, \\ z'(t) &= g(r\cos\theta, r\sin\theta) - z, \\ \theta'(t) &= 1 + \varepsilon\frac{b_1 z\cos\theta}{r} + \varepsilon^2\frac{(a_2 + a_3 z)\cos\theta}{r}. \end{aligned} \tag{27}$$

Notice that, for each $j = 1, 2$ and $t_{j-1} \le \theta \le t_j$, we have $\dot{\theta}(t) \ne 0$ for $|\varepsilon| \ne 0$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable. Accordingly, system (27) becomes

$$\begin{aligned} \dot{r}(\theta) &= \frac{r'(t)}{\theta'(t)} = F_{01}(\theta, r, z) + \varepsilon F_{11}(\theta, r, z) + \varepsilon^2 F_2(\theta, r, z) + \mathcal{O}_1(\varepsilon^3), \\ \dot{z}(\theta) &= \frac{z'(t)}{\theta'(t)} = F_{02}(\theta, r, z) + \varepsilon F_{12}(\theta, r, z)) + \varepsilon^2 F_{22}(\theta, r, z) + \mathcal{O}_2(\varepsilon^3). \end{aligned}$$

Considering the notation of Theorem A we have $F_i(\theta, r, z) = (F_{i1}(\theta, r, z), F_{i2}(\theta, r, z))$ for each $i \in \{1, 2\}$. Moreover, for each $i \in \{1, 2\}$ the function $F_i(\theta, r, z)$ is written in the form $F_i(\theta, r, z) = \sum_{j=1}^{2} \chi_{[t_{j-1}, t_j]}(\theta) F_i^j(\theta, r, z)$.

Defining $\tilde{f}(\theta, r) = f(r\cos\theta, r\sin\theta)$ and $\tilde{g}(\theta, r) = g(r\cos\theta, r\sin\theta)$ we write explicitly the expressions of F_0, F_1^j and F_2^j for $j \in \{1, 2\}$,

$$\begin{aligned} F_0(\theta, r, z) = &\quad (0,\ \tilde{g}(\theta, r) - z), \\ F_1^1(\theta, r, z) = &\quad \left((a_0 + a_1 z)\cos\theta,\ \frac{(a_0 + a_1 z)\sin\theta}{r}(\tilde{g}(\theta, r) - z)\right), \\ F_1^2(\theta, r, z) = &\quad \left(b_1 z\sin\theta,\ -\frac{b_1 z\cos\theta}{r}(\tilde{g}(\theta, r) - z)\right), \\ F_2^1(\theta, r, z) = &\quad \Big((a_2 + a_3 z)\cos\theta + \frac{(a_0 + a_1 z)^2\sin\theta\cos\theta}{r},\ \frac{\sin\theta}{r^2}\left((a_0 + a_1 z)^2\sin\theta\right. \\ &\quad \left.+ (a_2 + a_3 z)r\right)(\tilde{g}(\theta, r) - z)\Big), \\ F_2^2(\theta, r, z) = &\quad \left((b_2 + b_3 z)\sin\theta - \frac{b_1^2 z^2\sin\theta\cos\theta}{r},\ \frac{\cos\theta}{r^2}\left(b_1^2 z\cos\theta - (b_2 + b_3 z)r\right)(\tilde{g}(\theta, r) - z)\right). \end{aligned}$$

The unperturbed systems is smooth and its solution $(r(\theta,r_0,z_0),z(\theta,r_0,z_0))$ with initial condition (r_0,z_0) is given by

$$r(\theta)=\overline{r}(\theta,r_0,z_0)=r_0,\quad z(\theta)=\overline{z}(\theta,r_0,z_0)=e^{-\theta}\left(z_0+\int_0^\theta e^s\tilde{g}(s,r_0)ds\right). \tag{28}$$

Consequently, a fundamental matrix solution of (14) is given by

$$Y(\theta,r_0,z_0)=\frac{\partial(\overline{r},\overline{z})}{\partial(r_0,z_0)}(\theta,r_0,z_0)=\begin{pmatrix}1 & 0\\ G(\theta,r_0) & e^{-\theta}\end{pmatrix},$$

where $G(\theta,r_0)$ is the derivative of $\overline{z}(\theta,r_0,z_0)$ with respect to the variable r_0. Notice that, from (28), $G(\theta,r_0)$ does not depend on z_0.

Let $\varepsilon_0>0$ be a real positive number and consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(r,\tilde{f}(0,r)):r>\varepsilon_0\}$. Notice that for $(r_0,z_0)=(r_0,\tilde{f}(0,r_0))\in\mathcal{Z}$ we have $z(\theta,r_0,z_0)=\tilde{f}(\theta,r_0)=f(r_0\cos\theta,r_0\sin\theta)$. Indeed, let $w(\theta)=f(r_0\cos\theta,r_0\sin\theta)$. So

$$\begin{aligned}
w'(\theta)&=\partial_x f(r_0\cos\theta,r_0\sin\theta)(-r_0\sin\theta)+\partial_y f(r_0\cos\theta,r_0\sin\theta)(r_0\cos\theta)\\
&=g(r_0\cos\theta,r_0\sin\theta)-f(r_0\cos\theta,r_0\sin\theta)\\
&=g(r_0\cos\theta,r_0\sin\theta)-w(\theta)\\
&=\tilde{g}(\theta,r_0)-w(\theta).
\end{aligned}$$

The second equality holds because $g(x,y)=f(x,y)+x\partial_y f(x,y)-y\partial_x f(x,y)$. Hence, for $(r_0,z_0)\in\mathcal{Z}$ the solution $z(\theta,r_0,z_0)$ is 2π-periodic. Moreover,

$$Id-Y^{-1}(2\pi,r,z)=\begin{pmatrix}0 & 0\\ \star & 1-e^{2\pi}\end{pmatrix}.$$

Consequently, $\Delta_\alpha=1-e^{2\pi}\neq 0$. Accordingly, all the hypotheses of Theorem A are satisfied.

Proof of Theorem 5. Denote by (r,z_r) a point in $\mathcal{Z}$, that is $z_r=\tilde{f}(0,r)$. Notice that the bifurcation function of first order is $f_1(r)=\pi g_1(r,z_r)$, where g_1 is defined in (8). Indeed, from definition $f_1(r)=\pi g_1(r,z_r)+\dfrac{\partial\pi g_0}{\partial b}(r,z_r)\gamma_1(r)$. But

$$g_0(r,z)=Y^{-1}(2\pi,r,z)((r,z(2\pi,r,z)))-(r,z(0,r,z)))=(0,\star),$$

and then $\pi g_0\equiv 0$. Moreover,

$$w_1^1(\theta,r,z)=\left(a_0\sin\theta+a_1\int_0^\theta z(\phi)\cos\phi d\phi,\ G(\theta,r)\left(a_0\sin\theta+a_1\int_0^\theta z(\phi)\cos\phi d\phi\right)-\right.$$
$$\left.e^{-\theta}\int_0^\theta\left(e^{\phi}G(\phi,r)(a_0+a_1z(\phi))\cos\phi+\sin\phi\frac{e^{\phi}(\tilde g(\phi,r)-z(\phi))(a_0+a_1z(\phi))}{r}\right)d\phi\right),$$

$$\begin{aligned}w_1^2(\theta,r,z)=&Y(\theta,r,z)\left[Y^{-1}(\pi,r,z)w_1^1(\pi,r,z)+\int_\pi^\theta Y^{-1}(\phi,r,z)F_1^2(\phi,r(\phi),z(\phi))d\phi\right]\\ =&Y(\theta,r,z)\left(a_1\int_0^\pi z(\phi)\cos\phi d\phi+b_1\int_\pi^\theta z(\phi)\sin\phi d\phi,\right.\\ &\int_0^\pi\frac{e^{\phi}((a_0+a_1z(\phi))(\sin\phi(g(r\cos\phi,r\sin\phi)-z(\phi))-r\cos\phi\,G(\phi,r))}{r}d\phi\\ &\left.+\int_\pi^\theta-\frac{b_1e^{\phi}z(\phi)(\cos\phi(g(r\cos\phi,r\sin\phi)-z(\phi))+r\sin\phi\,G(\phi,r))}{r}d\phi\right).\end{aligned}$$

Since $g_1(r,z)=Y^{-1}(2\pi,r,z)w_1^2(2\pi,r,z)$ and $f_1(r)=\pi g_1(r,z_r)$ it follows that

$$f_1(r)=a_1\int_0^\pi f(r\cos\phi,r\sin\phi)\cos\phi d\phi+b_1\int_\pi^{2\pi}f(r\cos\phi,r\sin\phi)\sin\phi d\phi. \tag{29}$$

So, from Theorem A, each positive simple zero of (26) provides an isolated periodic solution of system (25). This concludes this proof. □

The next result is an application of Theorem 5. We shall use in its statement the concept of Bessel functions, which are defined as the canonical solutions $y(x)$ of Bessel's differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0,\quad \alpha\in\mathbb{C}.$$

This equation has two linearly independent solutions. Using Frobenius' method we obtain one of these solutions, which is called a *Bessel function of the first kind*, and is denoted by $J_\alpha(x)$. More details about this function can be found in [23].

Corollary 6. *Consider the piecewise vector field* (25).

(a) *If $f(x,y)=\cos x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon=2\pi+\mathcal{O}(\varepsilon)$, $\varphi_n(t,\varepsilon)=(x_n^*,y_n^*,\cos(x_n^*))+\mathcal{O}(\varepsilon)$, and $|(x_n^*,z_n^*)|=n\pi/2$.*
(b) *If $f(x,y)=\sin x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon=2\pi+\mathcal{O}(\varepsilon)$, $\varphi_i(t,\varepsilon)=(x_n^*,y_n^*,\sin(x_n^*))+\mathcal{O}(\varepsilon)$, and $|(x_n^*,z_n^*)|=r_n^*$, where each r_n is a zero of the Bessel Function of First Kind, $J_1(r)$.*

Proof. For $f(x,y) = \cos x$, the bifurcation function (29) reads $f_1(r) = -(2b_1 \sin r)/r$, and for $f(x,y) = \cos(x)$, the bifurcation function (29) reads $f_1(r) = a_1 \pi J_1(r)$. Therefore the result follows directly from Theorem 5. □

Notice that Theorem 5 cannot be applied when f_1 is identically zero, which is the case when $f(x,y) = 2x^2 - y^2$ for instance. For these cases we define the function

(30)

$$\begin{aligned} f_2(r) = &\int_0^\pi \Bigg(a_1 \cos s \Big(G(s,r) \int_0^s \cos\phi (a_0 + a_1 \tilde{f}(\phi,r)) d\phi \\ &- e^{-s} \int_0^s e^\phi (a_0 + a_1 \tilde{f}(\phi,r))(r \cos\phi\, G(\phi,r) + (\tilde{f}(\phi,r) - \tilde{g}(\phi,r)) d\phi \\ &+ a_2 + a_3 \tilde{f}(\phi,r) + \frac{\sin s}{r}(a_0 + a_1 \tilde{f}(s,r))^2 \Big) \Bigg) ds \\ &+ \frac{e^{-2\pi}(1+e^\pi)}{2(1-e^{2\pi})}(a_1 e^\pi - b_1) \Big[\int_0^\pi e^\phi G(\phi,r) \cos\phi (a_0 + a_1 \tilde{f}(\phi,r)) d\phi \\ &+ \int_0^\pi \frac{e^\phi \sin\phi}{r}(a_0 + a_1 \tilde{f}(\phi,r))(\tilde{g}(\phi,r) - \tilde{f}(\phi,r)) d\phi \\ &+ b_1 \int_\pi^{2\pi} e^\phi G(\phi,r) \sin\phi \tilde{f}(\phi,r) d\phi + \frac{b_1}{r} \int_\pi^{2\pi} e^\phi \cos\phi (\tilde{g}(\phi,r) - \tilde{f}(\phi,r)) d\phi \Big] \\ &+ \int_\pi^{2\pi} \Bigg(\frac{2}{r}(-b_1^2 \cos s (\tilde{f}(s,r))^2 + \sin s (b_2 + b_3 \tilde{f}(s,r))) \\ &+ 2b_1 \sin s \Big(G(s,r) \int_0^\pi \cos\phi (a_0 + a_1 \tilde{f}(\phi,r)) + b_1 G(s,r) \int_\pi^s \sin\phi \tilde{f}(\phi,r) d\phi \\ &+ e^{-s} \Big(\int_0^\pi -e^\phi \cos\phi\, G(\phi,r)(a_0 + a_1 \tilde{f}(\phi,r)) + \frac{e^\phi \sin\phi}{r}(\tilde{g}(\phi,r) - \tilde{f}(\phi,r)) d\phi \\ &+ b_1 \int_\pi^s e^\phi \left(\frac{\cos\phi}{r}(\tilde{f}(\phi,r) - \tilde{g}(\phi,r)) - G(\phi,r) \sin\phi \right) d\phi \Big) \Big) \Bigg) ds. \end{aligned}$$

Theorem 7. *Consider the piecewise vector field* (25)*. Assume that* $f_1 \equiv 0$*. Then, for each* $r* > 0$*, such that* $f_2(r^*) = 0$ *and* $f_2'(r^*) \neq 0$*, there exists a crossing limit cycle* $\varphi(t,\varepsilon)$ *of* X *of period* T_ε *such that* $T_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$*,* $\varphi(t,\varepsilon) = (x^*, y^*, f(x^*,y^*)) + \mathcal{O}(\varepsilon)$*, and* $|(x^*, z^*)| = r^*$*.*

Proof. As we saw before $\pi g_0 \equiv 0$. So, from (4), we compute the bifurcation function of order 2 as

$$f_2(r) = \frac{\partial \pi g_1}{\partial b}(r, z_r)\gamma_1(r) + \pi g_2(r, z_r), \tag{31}$$

where $\gamma_1(r) = -\dfrac{1}{1-e^{2\pi}}\pi^{\perp}g_1(r,z_r)$ and

$$\pi^{\perp}g_1(r,z_r) = \int_0^{\pi} \frac{e^{\phi}((a_0+a_1\tilde{f}(\phi,r))(\sin\phi(g(r\cos\phi,r\sin\phi)-\tilde{f}(\phi,r))-r\cos\phi G(\phi,r,z))}{r}\,d\phi$$
$$-b_1\int_{\pi}^{2\pi} \frac{e^{\phi}\tilde{f}(\phi,r)(\cos\phi(g(r\cos\phi,r\sin\phi)-\tilde{f}(\phi,r))+r\sin\phi G(\phi,r,z))}{r}\,d\phi.$$

From Proposition 3, we have $g_2(r,z_r) = Y^{-1}(2\pi,r,z)w_2^2(2\pi,r,z)/2$, where $w_i^j(2\pi,r,z)$ is given in Lemma 2. All these functions may be computed to get (31) as (30). Again, from Theorem A, each positive simple zero of (30) provides an isolated periodic solution of system (25). This concludes this proof. □

The next result is an application of Theorem 7.

Corollary 8. *Consider the piecewise vector field* (25) *and let* $f(x,y) = 2x^2-y^2$. *Assuming* $a_1^2+b_1^2\neq 0$ *define*

(32)
$$A_0 = \frac{-80b_2(1-e^{\pi})}{(1+e^{\pi})4(15a_1b_1-b_1^2-14a_1^2)-5\pi(1-e^{\pi})(b_1^1+10a_1^2)},$$
$$A_1 = \frac{40a_0((1+e^{\pi})(b_1-a1)-a1\pi(1-e^{\pi})}{(1+e^{\pi})4(15a_1b_1-b_1^2-14a_1^2)-5\pi(1-e^{\pi})(b_1^1+10a_1^2)},$$

and $D=-4A_1^3-27A_0^2$.

(i) *If* $D>0$ *then the piecewise smooth vector field admits at least one limit cycle. Moreover, if* $A_1<0$ *and* $A_0>0$, *then the piecewise smooth vector field admits at least two limit cycles;*
(ii) *If* $D\leq 0$ *and* $A_0<0$, *then the piecewise smooth vector field admits at least one limit cycle.*

Moreover, in both cases we have a limit cycle $\varphi(t,\varepsilon)$ *of* X *of period* T_ε *such that* $T_\varepsilon = 2\pi+\mathcal{O}(\varepsilon)$, $\varphi(t,\varepsilon) = (x_n^*,y_n^*,2(x_n^*)^2-(y_n^*)^2)+\mathcal{O}(\varepsilon)$, *and* $|(x_n^*,z_n^*)| = r_n^*$.

Proof. For $f(x,y)=2x^2-y^2$ the bifurcation function (30) becomes

(33)
$$f_2(r) = -2b_2+\frac{a_0\left(\left(e^{\pi}(1-\pi)+1+\pi\right)a_1-\left(1+e^{\pi}\right)b_1\right)}{e^{\pi}-1}r$$
$$+\frac{\left(-\left(e^{\pi}(56-50\pi)+56+50\pi\right)a_1^2+60\left(1+e^{\pi}\right)a_1b_1-\left(e^{\pi}(4-5\pi)+4+5\pi\right)b_1^2\right)}{40\left(e^{\pi}-1\right)}r^3.$$

Dividing f_2 by $a_1^2+b_1^2\neq 0$, we see that the equation $f_2(r)=0$ is equivalent to $\tilde{f}_2(r)\doteq A_0+A_1r+r^3=0$, where A_0 and A_1 are given in (32).

Notice that $\tilde{f}_2(r)$ is a polynomial function of degree 3, so it has at least one real root and can be written as $\tilde{f}_2(r) = r^3-(r_1+r_2+r_3)r^2+(r_1r_2+r_1r_3+r_2r_3)r-r_1r_2r_3$, where $r_i, i=1,2,3$ are the zeros of the polynomial. Moreover, the sign of its discriminant $D=-4A_1^3-27A_0^2$ carries information about its number of real roots.

If $D > 0$ the polynomial $\tilde{f}_2(r)$ has three simple real roots r_1, r_2 and r_3. Since the polynomial has no quadratic term, it follows that $r_1 + r_2 + r_3 = 0$ and then at least one of these roots must be positive. Moreover, if $A_1 < 0$ and $A_0 > 0$ then there are two changes of sign between the terms of the polynomial and then by *Descartes Sign Theorem* we get the two positive roots.

If $D \leq 0$ then there is a pair of complex roots or a double real root. In both cases the condition $A_0 < 0$ implies that at least one root is positive.

Now, from Theorem A, each positive simple zero of (33) provides an isolated periodic solution of system (25). This concludes this proof. □

4.2. **Nonsmooth perturbation of a nonsmooth center.** In this example we consider a discontinuous differential system in $\mathbb{R}^3$ defined in 4 zones ($n = 4$). Consider the nonsmooth vector field

$$X(u,v,w) = \begin{cases} X_1(u,v,w) & \text{if } u > 0 \text{ and } v > 0, \\ X_2(u,v,w) & \text{if } u < 0 \text{ and } v > 0, \\ X_3(u,v,w) & \text{if } u < 0 \text{ and } v < 0, \\ X_4(u,v,w) & \text{if } u > 0 \text{ and } v < 0, \end{cases} \tag{34}$$

where

$$\begin{aligned} X_1(u,v,w) &= (-1 + \varepsilon(a_1 x + b_1),\, 1,\, -w + \varepsilon(c_1 x + d_1)), \\ X_2(u,v,w) &= (-1 + \varepsilon(a_2 x + b_2),\, -1,\, -w + \varepsilon(c_2 x + d_2)), \\ X_3(u,v,w) &= (1 + \varepsilon(a_3 x + b_3),\, -1,\, -w + \varepsilon(c_3 x + d_3)), \\ X_4(u,v,w) &= (1 + \varepsilon(a_4 x + b_4),\, 1,\, -w + \varepsilon(c_4 x + d_4)), \end{aligned}$$

with $a_j, b_j, c_j, d_j \in \mathbb{R}$ for all j.

Writing in cylindrical coordinates $u = r\cos\theta$, $v = r\sin\theta$, $w = w$, the set of discontinuity is $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\} \cup \{\theta = t_2\} \cup \{\theta = t_3\}$ with $t_0 = 0, t_1 = \pi/2, t_2 = \pi, t_3 = 3\pi/2$ and $t_4 = 2\pi$. For each $j = 1,2,3,4$ the differential system $(\dot{u}, \dot{v}, \dot{w}) = X_j(u,v,w)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= g_j(\theta) + \sum_{i=1}^{k} \varepsilon^i (a_{ij} r \cos^2\theta + b_{ij}\cos\theta), \\ w'(t) &= -w + \sum_{i=1}^{k} \varepsilon^i (c_{ij} r\cos\theta + d_{ij}\cos\theta), \\ \theta'(t) &= \frac{1}{r}\left(\widehat{g}_j(\theta) - \sum_{i=1}^{k} \varepsilon^i (a_{ij} r\cos\theta\sin\theta + b_{ij}\sin\theta)\right), \end{aligned}$$

where

$$
\begin{aligned}
g_1(\theta) &= \sin\theta - \cos\theta, & \widehat{g}_1(\theta) &= \sin\theta + \cos\theta,\\
g_2(\theta) &= -(\sin\theta + \cos\theta), & \widehat{g}_2(\theta) &= \sin\theta - \cos\theta,\\
g_3(\theta) &= -\sin\theta + \cos\theta, & \widehat{g}_3(\theta) &= -(\sin\theta + \cos\theta),\\
g_4(\theta) &= \sin\theta + \cos\theta, & \widehat{g}_4(\theta) &= -\sin\theta + \cos\theta.
\end{aligned}
$$

Notice that, for each $j = 1,2,3,4$ and $t_{j-1} \le \theta \le t_j$, $\dot{\theta}(t) \neq 0$ for $|\varepsilon|$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable by doing $r'(\theta) = \dot{r}(t)/\dot{\theta}(t)$ and $w'(\theta) = \dot{w}(t)/\dot{\theta}(t)$. Taking θ as the new independent time variable we have

$$
\begin{aligned}
r'(\theta) &= F_{01}^j(\theta,z) + \varepsilon F_{11}^j(\theta,z) + \mathcal{O}_1(\varepsilon^2),\\
w'(\theta) &= F_{02}^j(\theta,z) + \varepsilon F_{12}^j(\theta,z) + \mathcal{O}_2(\varepsilon^2).
\end{aligned} \tag{35}
$$

Here, $z = (r,w)$ and the prime denotes the derivative with respect to θ. The expressions of F_{01}^j and F_{02}^j for $j = 1,2,3,4$ are given by

$$
F_{01}^1 = \frac{r(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta},\; F_{02}^1 = \frac{-rw}{\sin\theta + \cos\theta},\; F_{01}^2 = \frac{r(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta},\; F_{02}^2 = \frac{rw}{\cos\theta - \sin\theta},
$$

$$
F_{01}^3 = \frac{r(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta},\; F_{02}^3 = \frac{rw}{\sin\theta + \cos\theta},\; F_{01}^4 = \frac{r(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta},\; F_{02}^4 = \frac{-rw}{\cos\theta - \sin\theta}.
$$

The expressions of F_{11}^j and F_{12}^j for $j = 1,2,3,4$ are also easily computed. Nevertheless, we shall omit these expressions because of their size.

For each $j \in \{1,2,3,4\}$, the differential system (35) is 2π-periodic in the variable θ and is written in the standard form with

$$
F_i^j(\theta,z) = \left(F_{i1}^j(\theta,z), F_{i2}^j(\theta,z)\right),
$$

for $i = 0,1$. Now, for each $j \in \{1,2,3,4\}$ we compute the solution $x_j(\theta,z,0)$ of the unperturbed system

$$
\dot{r}(\theta) = F_{01}^j(\theta,z), \quad \dot{w}(\theta) = F_{02}^j(\theta,z).
$$

and this solution is

$$
\begin{aligned}
x_1(\theta,z,0) &= \left(\tfrac{r}{\sin\theta+\cos\theta}, we^{-\frac{r\sin\theta}{\sin\theta+\cos\theta}}\right),\\
x_2(\theta,z,0) &= \left(\tfrac{-r}{\cos\theta-\sin\theta}, we^{-\frac{r\sin\theta}{\cos\theta-\sin\theta}-2r}\right),\\
x_3(\theta,z,0) &= \left(\tfrac{-r}{\sin\theta+\cos\theta}, we^{-\frac{r\sin\theta}{\sin\theta+\cos\theta}-2r}\right),\\
x_4(\theta,z,0) &= \left(\tfrac{r}{\cos\theta-\sin\theta}, we^{-\frac{r\sin\theta}{\cos\theta-\sin\theta}-4r}\right).
\end{aligned}
$$

We note that in each quadrant the denominators of these four solutions never vanish.

Let $0 < r_0 < r_1$ be positive real numbers and consider the set $\mathcal{Z} \subset \mathbb{R}^2$ such that $\mathcal{Z} = \{(\alpha, 0) : r_0 < \alpha < r_1\}$. The solution $x(\theta, z, 0)$ of the unperturbed system $x'(\theta) = F_0(\theta, z)$ satisfies $x(\theta, z, 0) = x_j(\theta, z, 0)$, for $\theta \in [t_{j-1}, t_j]$, and $x(2\pi, z, 0) - x(0, z, 0) = (0, z(1 - e^{-4r}))$. Consequently, for each $z_\alpha \in \mathcal{Z}$, the solution $x(\theta, z, 0)$ is 2π-periodic and system (34) satisfies hypothesis (H). Moreover, the fundamental matrix $Y(\theta, z)$ is given by

$$Y(\theta,z) = \begin{cases} Y_1(\theta,z) & \text{if } 0 = t_0 \le \theta \le \pi/2, \\ Y_2(\theta,z) & \text{if } \pi/2 \le \theta \le \pi, \\ Y_3(\theta,z) & \text{if } \pi \le \theta \le 3\pi/2, \\ Y_4(\theta,z) & \text{if } 3\pi/2 \le \theta \le 2\pi, \end{cases}$$

where $Y_j(t, z)$ are defined by (15). So

$$Y_1(\theta,z) = \begin{pmatrix} \dfrac{1}{g_4(\theta)} & 0 \\ -\dfrac{e^{-\frac{r\sin\theta}{g_4(\theta)}} w \sin\theta}{g_4(\theta)} & e^{-\frac{r\sin\theta}{g_4(\theta)}} \end{pmatrix},$$

$$Y_4(\theta,z) = \begin{pmatrix} \dfrac{1}{g_3(\theta)} & 0 \\ -\dfrac{e^{-\frac{r\sin\theta}{g_3(\theta)}-4r} w(\sin\theta + 4g_3(\theta))}{g_3(\theta)} & e^{-\frac{r\sin\theta}{g_3(\theta)}-4r} \end{pmatrix}.$$

Hence,

$$Y_1(0,z)^{-1} - Y_4(2\pi,z)^{-1} = \begin{pmatrix} 0 & 0 \\ -4w & 1 - e^{4r} \end{pmatrix},$$

and then $\det(\Delta_\alpha) = 1 - e^{4r} \neq 0$ if $z_\alpha = (\alpha, 0) \in \mathcal{Z}$. Thus, we can compute the bifurcation functions (4) for system (34). For doing this we first obtain the functions (16) corresponding to this system,

$$\begin{aligned} g_0(\theta,z) = \quad & (0, w(1-e^{4r})), \\ w_1^4(2\pi,z) = \quad & \Big(\frac{1}{2} r(r(a_1+a_2+a_3+a_4) + 2(b_1-b_2-b_3+b_4)), \\ & \frac{1}{3} e^{-4r}(-r^2 w(6a_1+3a_2+2a_3) - 3r(w(4b_1-2b_2-b_3) \\ & + e^{2r}(-e^{2r}c_4 + c_2 + c_3) + c_1) + 3(e^r-1)(e^r(c_2+d_2) \\ & + e^{2r}(c_3-d_3) + e^{3r}(d_4-c_4) + c_1 + d_1))\Big), \end{aligned}$$

and

$$g_1(z) = Y_4(2\pi,z)^{-1} w_1^4(2\pi,z). \tag{36}$$

So, the bifurcation function (4) corresponding to the function (36) becomes

$$f_1(\alpha) = \frac{1}{2}\alpha(\alpha(a_1 + a_2 + a_3 + a_4) + 2(b_1 - b_2 - b_3 + b_4)),$$

which has a simple zero α^*. So, from Theorem A, we get the existence of an isolated periodic solution of system (35) for ε sufficiently small.

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