QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$ HAVING INVARIANT PLANES WITH TOTAL MULTIPLICITY 9

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Abstract. In this paper we consider the 30 possible configurations for all quadratic polynomial differential systems in $\mathbb{R}^3$ having exactly 9 invariant planes taking into account their multiplicities, which is the maximum number of invariant planes that this kind of systems can have without taking into account the infinite plane. We study the realization and the existence of first integrals for each one of these configurations.

1. Introduction and statement of the main results

Let $\mathbb{R}[x, y, z]$ be the ring of polynomials in the variables x, y and z with coefficients in $\mathbb{R}$. We consider the polynomial differential system in $\mathbb{R}^3$ defined by

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z), \tag{1}$$

where P, Q and R are relatively prime polynomials in $\mathbb{R}[x, y, z]$, and the dot denotes derivative with respect to the independent variable t usually called the *time*.

We can naturally associate to differential system (1) the vector field

$$\mathcal{X} = P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y} + R\,\frac{\partial}{\partial z}.$$

We say that $m = \max\{\deg(P), \deg(Q), \deg(R)\}$ is the *degree* of differential system (1) and of vector field $\mathcal{X}$. If $m = 2$ we say that differential system (1) is *quadratic*.

An *invariant algebraic surface* for the differential system (1) or for the vector field $\mathcal{X}$ is an algebraic surface $f(x, y, z) = 0$, with $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$, such that for some polynomial $K \in \mathbb{C}[x, y, z]$ we have $\mathcal{X}(f) = \langle (P, Q, R), \nabla f \rangle = Kf$. The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$ and if m is the degree of the vector field $\mathcal{X}$, then the degree of K is at most $m - 1$. If the polynomial f is irreducible in $\mathbb{C}[x, y, z]$, then we say that $f = 0$ is an *irreducible invariant algebraic surface*. Note that from this definition if an orbit of the vector field $\mathcal{X}$ has a point on $f = 0$, then the whole orbit is contained in $f = 0$, that is $f = 0$ is invariant by the flow of $\mathcal{X}$. If the degree of f is 1 then we call the invariant algebraic surface $f = 0$ *invariant plane*. We note that system (1) is real and that an invariant algebraic surface can be complex, this is due to the fact that the complex invariant algebraic surfaces play a role in the existence of real first integrals of system (1), because if $f = 0$ is an invariant algebraic surface of system (1) with cofactor K, then the conjugate complex surface $\overline{f} = 0$ is also an invariant

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