ALGEBRAIC AND TOPOLOGICAL CLASSIFICATION OF HOMOGENEOUS QUARTIC VECTOR FIELDS IN THE PLANE

JAUME LLIBRE[1], Y. PAULINA MARTÍNEZ[2] AND CLAUDIO VIDAL[2]

Abstract. We provide canonical forms for the homogeneous polynomials of degree five. Then we characterize all the phase portraits in the Poincaré disk for all quartic homogeneous polynomial differential systems. More precisely, there are exactly 24 different topological phase portraits for the quartic homogeneous polynomial differential systems.

1. Introduction

We consider a family of polynomial vector fields in the plane of the form

$$\dot{x} = P(x,y), \;\; \dot{y} = Q(x,y), \tag{1}$$

where P and Q are homogeneous polynomials of degree four (shortly, they will be called *quartic systems*). This work is divided in two parts. First we are going to give all the possible canonical forms for the homogeneous polynomials of degree five, and secondly, we will characterize all the phase portraits in the Poincaré disk of all homogeneous quartic polynomial differential systems (1). For a definition of the Poincaré disk and the local charts we are going to work, see for instance Chapter 5 of [7].

In general polynomials vector fields are a current topic of research (see for instance [2, 5, 7, 8]). The study of homogenenous polynomial vector fields was initiated by Markus [10] in 1960. He gave a classification for quadratic homogeneous vector fields $\mathcal{X} = (P,Q)$ with P and Q with no common factor. Argemí [1] completed the classification of Markus in 1968, and provided the classification of cubic homogeneous vector fields that have no common factor. Furthermore, for planar homogeneous polynomial vector fields of degree m that have no common factor Argemí gave upper and lower bounds of the numbers of phase portraits.

An algebraic classification of the differential systems (1) when P and Q are homogeneous polynomials of degree 2 was given by Date and Iri in [6]. Their classification of linear binary and cubic forms was obtained using the theory of algebraic invariants (according to Gurevich [9] a binary form can be seen as an homogeneous polynomial in two variables). Gurevich [9] did the classification for third and fourth-order binary forms on the field of complex numbers, and Cima and Llibre in [3] obtained a classification of the fourth-order binary in the real domain using Caley's method. In that paper we find an algebraic classification of homogeneous systems of degree three, and Collins [4] extended this to homogeneous polynomial vector field of degree m.

Our classification of all the possible canonical forms for homogeneous polynomials of degree 5 is based in the results of [3], where the classification of quartic binary forms in the real domain was given. In fact, since a quintic polynomial has a real root we look a homogeneous polynomial of degree five as the product of a linear factor and a homogeneous polynomial of degree 4. The classification of the canonical forms for the homogeneous vector fields (1) of degree 4 will be based in the canonical forms of the homogeneous polynomial of degree 5. Our first result in this direction is the following.

2010 *Mathematics Subject Classification.* Primary 37C10, Secondary 34C05.

Key words and phrases. quartic homogeneous polynomial differential systems, homogeneous polynomial vector fields, phase portraits.