LIMIT CYCLES BIFURCATING OF KOLMOGOROV SYSTEMS IN $\mathbb{R}^2$ AND IN $\mathbb{R}^3$

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Abstract. In this work we consider the Kolmogorov system of degree 3 in $\mathbb{R}^2$ and $\mathbb{R}^3$ having an equilibrium point in the positive quadrant and octant, respectively. We provide sufficient conditions in order that the equilibrium point will be a Hopf point for the planar case and a zero-Hopf point for the spatial one. We study the limit cycles bifurcating from these equilibria using averaging theory of second and first order, respectively. We note that the equilibrium point is located in the quadrant or octant where the Kolmogorov systems have biological meaning.

1. Introduction and statement of the main results

The Lotka–Volterra systems, which are a class of polynomial differential systems of degree 2 in the plane, were developed independently by Alfred J. Lotka in 1925 [6] and Vito Volterra in 1926 [13], they initially proposed these models for studying the interactions between two species. Kolmogorov [4] in 1936 extended these systems to arbitrary dimension and degree, which are now called Kolmogorov systems.

The Lotka–Volterra and Kolmogorov systems have been applied to model different natural phenomena such as the time evolution of conflicting species in biology (see for more details May [9]), the evolution of competition between three species (studied by May and Leonard [8]), the evolution of electrons, ions and neutral species in plasma physics [7], chemical reactions [3], hydrodynamics [2], economics [12], etc.

We want to consider the Kolmogorov systems of degree 3 in the plane (resp. in the space) that has an equilibrium (a, b) in the plane (resp. (a, b, c) in the space) in the interior of the first quadrant (resp. octant). We note that the region of ecological interest in Lotka-Volterra and Kolmogorov systems is indeed the first quadrant (resp. octant). It is easy to see that we can consider $(a, b) = (1, 1)$ (resp. $(a, b, c) = (1, 1, 1)$ in the space) doing the rescaling $(x, y) \to (x/a, y/b)$ (resp. $(x, y, z) \to (x/a, y/b, z/c)$).

In short, in this work we consider the following Kolmogorov systems of degree three in the plane, that is,

(1)
$$\begin{aligned}\dot{x} &= -x(a_1(x-1) + a_2(y-1) + a_4(x-1)^2 + a_5(x-1)(y-1) + a_7(y-1)^2),\\ \dot{y} &= -y(b_1(x-1) + b_2(y-1) + b_4(x-1)^2 + b_5(x-1)(y-1) + b_7(y-1)^2),\end{aligned}$$

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and in the space, i.e.

(2)

$$\begin{aligned}\dot{x} &= -x(a_1(x-1)+a_2(y-1)+a_3(z-1)+a_4(x-1)^2+a_5(x-1)(y-1)+\\ &\quad a_6(x-1)(z-1)+a_7(y-1)^2+a_8(y-1)(z-1)+a_9(z-1)^2),\\ \dot{y} &= -y(b_1(x-1)+b_2(y-1)+b_3(z-1)+b_4(x-1)^2+b_5(x-1)(y-1)+\\ &\quad b_6(x-1)(z-1)+b_7(y-1)^2+b_8(y-1)(z-1)+b_9(z-1)^2),\\ \dot{z} &= -z(c_1(x-1)+c_2(y-1)+c_3(z-1)+c_4(x-1)^2+c_5(x-1)(y-1)+\\ &\quad c_6(x-1)(z-1)+c_7(y-1)^2+c_8(y-1)(z-1)+c_9(z-1)^2),\end{aligned}$$

where the dot denotes derivative with respect to the time t.

As far as we know there are no rigorous analytic studies on the existence of periodic solutions for the Kolmogorov systems (1) and (2) coming from a Hopf bifurcation for systems (1), and from a zero-Hopf bifurcation for systems (2). In this paper using the averaging theory of first and second order we shall prove the existence of limit cycles for these systems bifurcating from eihter a Hopf or a zero-Hopf equilibrium point. We recall that a *Hopf equilibrium point* of an autonomous system in $\mathbb{R}^2$ is an isolated equilibrium point with a pair of purely imaginary eigenvalues $\pm i\omega, \omega \in \mathbb{R}^+$. On the other hand, a *zero-Hopf equilibrium point* of an autonomous system in $\mathbb{R}^3$ is an isolated equilibrium point with linear part having one zero eigenvalue and a pair of purely imaginary eigenvalues $\pm i\omega, \omega \in \mathbb{R}^+$. These points are important because under some assumptions a small-amplitude limit cycle emerges from them. We will focus on limit cycles emerging from an equilibrium point (either Hopf or zero-Hopf) in the interior of the positive quadrant or octant of the Kolmogorov systems (1) and (2), respectively.

A *limit cycle* of systems (1) or (2) is a periodic orbit in the set of all periodic orbits of these differential systems. We say that a limit cycle is *unstable* if its associated Poincaré map has at least one positive characterize exponent. On the other hand, we say that a limit cycle is *stable* if its associated Poincaré map has all the characteristic exponents being negative.

Our main objective is to provide explicit sufficient conditions for the existence of limit cycles bifurcating from a Hopf equilibrium of a Kolmogorov system of degree 3 in dimension two, and from a zero-Hopf equilibrium of a Kolmogorov system of degree 3 in dimension three. Limit cycles are important in the dynamics of a differential system see for instance [10], and the Hopf bifurcations are also relevant for understanding such dynamics see for example [11].

Let

$$A = \frac{b_{22}}{2\omega} \quad \text{and} \quad B = \frac{\hat{B}}{8a_2^3\omega^3}, \tag{3}$$

where b_{22} will be introduced in equation (6), and $\hat{B} = (a_1^2 + a_2(2a_3 + b_4) + a_1(a_2 - a_4 - 2b_5))(a_1^3a_5 + a_2^3b_3 + a_1a_2^2(a_3 - b_4) + a_1^2a_2(-a_4 + b_5)) + (2a_1^3a_5 + a_2^2(2a_2a_3 - a_3a_4 + b_4b_5) + a_1a_2(a_4^2 + 2a_3a_5 + a_5b_4 + a_4b_5 - 2b_5^2 + a_2(a_3 - a_4 - b_4 + b_5)) + a_1^2(-2a_5(a_4 + 2b_5) + a_2(-a_4 + a_5 + 3b_5)))\omega^2 + (a_1a_5 + 2a_2b_5 - a_5(a_4 + 2b_5))\omega^4$.

Our first main result concerning the Kolmogorov systems (1) is the following.

Theorem 1. *Using the averaging theory of second order, if* $b_1 = -(a_1^2 + \omega^2)/a_2$ *with* $a_2 \neq 0$, $b_2 = -a_1$ *and* $AB < 0$, *then the Kolmogorov system* (1) *has one small* *limit cycle bifurcating from the Hopf equilibrium point* $(1,1)$. *The limit cycle is stable if* $A>0$ *and unstable if* $A<0$.

Theorem 1 is proved in section 2. The following system in $\mathbb{R}^2$ satisfies the hypotheses of Theorem 1.

Example 2. *The Kolmogorov system*

$$\begin{aligned} \dot{x} &= x(-1+(x-1)^2+(x-1)(y-1)+(y-1)^2+y), \\ \dot{y} &= y(-(x-1)+2(x-1)^2+(x-1)(y-1)-10(y-1)^2), \end{aligned} \tag{4}$$

has one small unstable limit cycle bifurcating from the equilibrium point $(1,1)$.

The details of the example are given in section 2.

Our second main result concerns with Kolmogorov systems (2). To state it we need some extra notation. Let

$\alpha = a_1^2a_3b_1 - a_1^3b_3 + a_1(a_3b_1b_2 - b_3(2a_2b_1+\omega^2)) + b_1(a_2(a_3b_1-b_2b_3)+a_3(b_2^2+\omega^2))$,

$\beta = -a_1^2a_2b_3 - a_2^2b_1b_3 - a_1a_2(-a_3b_1+b_2b_3) + a_3b_2(b_2^2+\omega^2) - a_2(-2a_3b_1b_2 + b_3(b_2^2+\omega^2))$,

$\gamma = -a_3^2b_1 + a_3(a_1-b_2)b_3 + a_2b_3^2$,

$\delta = -\omega(a_3^2b_2^2(b_2^2+\omega^2) + 2a_2a_3b_2(a_3b_1b_2 - b_3(a_1b_2+b_2^2+\omega^2)) + a_2^2(a_3^2b_1^2 - 2a_3b_1(a_1+b_2)b_3 + b_3^2(a_1^2+2a_1b_2+b_2^2+\omega^2)))$,

$S_1 = (CS_3)/E$,

$S_2 = 2C + c_{31}/\omega$,

$S_3 = DS_2 + CF$.

We will also introduce the notation C, D, E, F that due to their complexity will be provided in Appendix B.

Theorem 3. *Using averaging theory of first order, if* $c_1 = \alpha/\gamma$, $c_2 = \beta/\gamma$, $c_3 = -a_1 - b_2$, $FS_2 \neq 0$,$S_1 < 0$ *and* $CD \neq 0$, *then the Kolmogorov system* (2) *has two small limit cycles bifurcating from the zero-Hopf equilibrium point* $(1,1,1)$. *Moreover the following holds.*

(a) *If* $S_2>0$, $S_3/F>0$ *and* $DE<0$, *or* $S_2<0$, $S_3/F>0$, $DE>0$ *and* $(CF+S_3)/D<0$, *or* $S_2>0$, $S_3/F<0$, $DE>0$ *and* $(CF+S_3)/D<0$, *the two limit cycles are unstable. One of them has an unstable manifold of dimension two and the other has a stable manifold and an unstable manifold both of dimension one.*

(b) *If* $S_2<0$, $S_3/F>0$ *and* $DE<0$, *or* $S_2>0$, $S_3/F<0$ *and* $DE<0$, *the two limit cycles are unstable and both have a stable manifold and an unstable manifold of dimension one.*

(c) *If* $S_2<0$, $S_3/F<0$ *and* $DE<0$, *or* $S_2<0$, $S_3/F>0$, $DE>0$ *and* $(CF+S_3)/D>0$, *or* $S_2>0$, $S_3/F<0$, $DE>0$ *and* $(CF+S_3)/D>0$, *one limit cycle is stable with a stable manifold of dimension two and the other limit cycle is unstable with a stable manifold and an unstable manifold of dimension one.*

(d) *If* $S_2>0$, $S_3/F>0$, $DE>0$ *and* $(CF+S_3)/D<0$ *both limit cycles are unstable with unstable manifolds of dimension two.*
(e) *If* $S_2<0$, $S_3/F<0$, $DE>0$ *and* $(CF+S_3)/D>0$ *both limit cycles are stable with stable manifolds of dimension two.*

Theorem 3 is proved in section 3. The following system in $\mathbb{R}^3$ satisfies the hypotheses of statement (a) of Theorem 3.

Example 4. *The Kolmogorov system*

$$(5)\quad \begin{aligned}\dot{x} &= x((y-1)+(z-1)+207/10(x-1)^2+17/10(x-1)(y-1)-53/5(y-1)^2-\\ &\quad 29/10(x-1)(z-1)-189/10(y-1)(z-1)-43/10(z-1)^2),\\ \dot{y} &= y((x-1)-2/5(x-1)^2+42/5(x-1)(y-1)-1/2(y-1)^2+\\ &\quad 82/5(x-1)(z-1)+16/5(y-1)(z-1)+3(z-1)^2),\\ \dot{z} &= z(c_1(x-1)+c_2(y-1)+c_3(z-1)-(3/2)(x-1)^2+13(x-1)(y-1)-\\ &\quad 27/10(y-1)^2+17/10(x-1)(z-1)-69/5(y-1)(z-1)-87/10(z-1)^2),\end{aligned}$$

has two small limit cycles bifurcating from the equilibrium point $(1,1,1)$. *One limit cycle is unstable with an unstable manifold of dimension two and the other is unstable with a stable manifold and an unstable manifold both of dimension one.*

The details of the example are given in section 3.

See the appendix for a summary of the results on the averaging theory that we shall need for proving Theorems 1 and 3.

2. Proof of Theorem 1

Considering system (1) with a finite equilibria $(1,1)$ in the interior of the first quadrant.

First we impose conditions to the parameters so that the Kolmogorov system (1) has a Hopf point at $(1,1)$, that is, the Jacobian matrix at $(1,1)$ has complex conjugate eigenvalues with zero real part $\pm i\omega, \omega\in\mathbb{R}^+$. Doing so we get

$$b_1=-\frac{a_1^2+\omega^2}{a_2},\quad b_2=-a_1,$$

with $\omega>0, a_2=0$, and the linearization around the point $(1,1)$ is

$$M=\begin{pmatrix} a_1 & a_2\\ -(a_1^2+\omega^2)/a_2 & -a_1. \end{pmatrix}$$

We proceed to study the number of limit cycles bifurcating from the Hopf equilibrium point. To do so, we write b_1 and b_2 in the form

$$(6)\qquad b_1=-\frac{a_1^2+\omega^2}{a_2}+\varepsilon b_{11}+\varepsilon^2 b_{12},\quad b_2=-a_1+\varepsilon b_{21}+\varepsilon^2 b_{22}$$

where ε is a parameter to be taken sufficiently small.

We translate the equilibrium point $(1,1)$ to the origin through the change of variables $(x,y)\to(x+1,y+1)=(X,Y)$ and we write the Kolmogorov system into its Jordan normal form J using the auxiliary matrix

$$P = \begin{pmatrix} 1 & 0 \\ -a_1/\omega & -a_2/\omega \end{pmatrix}, \qquad \text{such that } J = PMP^{-1}.$$

and the variables $(u, v) = P(X, Y)$, i.e.

(7)

$$\begin{aligned}
\dot{u} &= (1+u)\left(a_3u^2 - \omega v - \frac{a_4u(a_1u+\omega v)}{a_2} + \frac{a_5}{a_2^2}(a_1u+\omega v)^2\right),\\
\dot{v} &= \frac{-1}{\omega}\Bigg(a_1(1+u)(a_3u^2 - \omega v - \frac{(a_4u(a_1u+\omega v)}{a_2} + \frac{a_5}{a_2^2}(a_1u+\omega v)^2)+\\
&+b_5(a_1u+\omega v)^2 - a_2(\omega(b_4uv + b_{21}\varepsilon v + b_{22}\varepsilon^2 v + \omega u)+\\
&a_1(b_4u^2 + b_{21}\varepsilon u + b_{22}\varepsilon^2 u - \omega v)) + \frac{1}{a_2^2}(a_2 - a_1u - \omega v)(a_2^2u(b_3u + \varepsilon(b_{11} + b_{12}\varepsilon))\Bigg).
\end{aligned}$$

Now we write system (7) in such a way that conditions of Theorem 5 are satisfied. First we write system (7) in polar coordinates $(u, v) = (r\cos\theta, r\sin\theta)$ and get the system of equations

(8)

$$\begin{aligned}
\dot{r} &= \cos(\theta)(r\cos(\theta)+1)\left[a_5(a_1r\cos(\theta) + r\omega\sin(\theta))^2 - a_2a_4r^2\cos^2(\theta)(a_1+\right.\\
&\left.\omega\tan(\theta)) + a_2^2a_3r^2\cos^2(\theta) - a_2^2r\omega\sin(\theta)\right]/a_2^2 - \sin(\theta)\left[a_1(r\cos(\theta)+\right.\\
&1)(-a_2a_4r^2\cos(\theta)(a_1\cos(\theta) + \omega\sin(\theta)) + a_5(a_1r\cos(\theta) + r\omega\sin(\theta))^2+\\
&a_2^2r(a_3r\cos^2(\theta) - \omega\sin(\theta))) - (a_2 - r(a_1\cos(\theta) + \omega\sin(\theta)))(a_1a_2r(b_4r\cos^2(\theta)-\\
&\omega\sin(\theta)) - b_5(a_1r\cos(\theta) + r\omega\sin(\theta))^2 - a_2^2b_3r^2\cos^2(\theta) + a_2r\omega\cos(\theta)(b_4r\sin(\theta)+\\
&\omega)) + a_2r\epsilon(a_2 - r(a_1\cos(\theta) + \omega\sin(\theta)))(\cos(\theta)(a_2b_{11} - a_1b_{21}) - b_{21}\omega\sin(\theta))+\\
&a_2r\epsilon^2(a_2 - r(a_1\cos(\theta) + \omega\sin(\theta)))(\cos(\theta)(a_2b_{12} - a_1b_{22}) - b_{22}\omega\sin(\theta))]/(a_2^2\omega^2)\\
\dot{\theta} &= \cos^2(\theta)[R\omega\epsilon\sin(\theta)(a_1^2(a_2 - 3a_5) + a_1a_2(a_2 + 2a_4 - 2b_{21}\epsilon - 2b_{22}\epsilon^2 - 2b_5)+\\
&a_2(a_2(-a_3 + \epsilon(b_{11} + b_{12}\epsilon) + b_4) - \omega^2)) + a_2^2(a_1\epsilon(b_{21} + b_{22}\epsilon) - a_2\epsilon(b_{11} + b_{12}\epsilon)+\\
&\omega^2) + R^2\omega^2\epsilon^2\sin^2(\theta)(3a_1(b_5 - a_5) + a_2(a_4 - b_4))) - a_1R^2\epsilon^2\cos^4(\theta)(a_1^2(a_5 - b_5)+\\
&a_1a_2(b_4 - a_4) + a_2^2(a_3 - b_3)) - R\epsilon\cos^3(\theta)(a_1^3a_5 + R\omega\epsilon\sin(\theta)(3a_1^2(a_5 - b_5)+\\
&2a_1a_2(b_4 - a_4) + a_2^2(a_3 - b_3)) + a_1^2a_2(-a_4 + \epsilon(b_{21} + b_{22}\epsilon) + b_5) + a_1a_2(a_2(a_3-\\
&\epsilon(b_{11} + b_{12}\epsilon) - b_4) + \omega^2) + a_2^3b_3) + \tfrac{1}{4}\omega\epsilon\sin(2\theta)(2R\omega\sin(\theta)(a_1(a_2 - 3a_5) + a_2(a_2+\\
&a_4 - b_{21}\epsilon - b_{22}\epsilon^2 - b_5)) + 2a_2^2b_{21} + 2a_2^2b_{22}\epsilon + R^2\omega^2\epsilon(a_5 - b_5)\cos(2\theta)-\\
&a_5R^2\omega^2\epsilon + b_5R^2\omega^2\epsilon) + \omega^2\sin^2(\theta)(a_2^2 - a_5R\omega\epsilon\sin(\theta)]/(a_2^2\omega)
\end{aligned}$$

Note that system (8) is periodic in the variable θ with period 2π. We write system (8) by taking as the new independent variable the angular variable θ and we obtain $r' = \frac{\dot{r}}{\dot{\theta}}$ where the prime denotes derivative with respect to θ. We now consider the new variable $R = r/\varepsilon$. We compute R' and we develop the new equation for R' in power series in the variable ε up to second order in the form

$$R' = \varepsilon F_1 + \varepsilon^2 F_2 + O(\varepsilon^3), \tag{9}$$

where

$F_1 = -R[-(a_2^2a_3 - a_1a_2a_4 + a_1^2a_5)R\omega\cos^3\theta + R(a_1^3a_5 + a_1^2a_2(-a_4 + b_5) + a_2(a_2^2b_3 + a_2\omega^2 + a_4\omega^2) + a_1(a_2^2(a_3 - b_4) + a_2\omega^2 - 2a_5\omega^2))\cos^2\theta\sin\theta + \omega\ \sin^2\theta(-a_2^2b_{21} - (a_1a_2 - a_1a_5 - a_2b_5)R\omega\sin\theta) - \cos\theta\sin\theta(a_2^2(-a_2b_{11} + a_1b_{21}) + R\omega(a_1^2(a_2 - 2a_5) + a_2^2b_4 + a_1a_2(a_2 + a_4 - 2b_5) - a_2\omega^2 + a_5\omega^2)\sin\theta)]/(a_2^2\omega^2)$, and

$$F_2 = R[(a_2^2a_3 - a_1a_2a_4 + a_1^2a_5)R\omega\cos^3\theta - R(a_1^3a_5 + a_1^2a_2(-a_4 + b_5) + a_2(a_2^2b_3 + a_2\omega^2 + a_4\omega^2) + a_1(a_2^2(a_3 - b_4) + a_2\omega^2 - 2a_5\omega^2))\cos^2\theta\sin\theta + \omega\sin\theta^2(a_2^2b_{21} + (a_1a_2 - a_1a_5 - a_2b_5)R\omega\sin\theta) + \cos\theta\sin\theta(a_2^2(-a_2b_{11} + a_1b_{21}) + R\omega(a_1^2(a_2 - 2a_5) + a_2^2b_4 + a_1a_2(a_2+a_4-2b_5) - a_2\omega^2 + a_5\omega^2)\sin\theta))(R(a_1^3a_5 + a_2^3b_3 + a_1^2a_2(-a_4+b_5) + a_1a_2(a_2a_3 - a_2b_4 + \omega^2))\cos^3\theta + a_5R\omega^3\sin^3\theta - \omega\cos\theta\sin\theta(a_2^2b_{21} + (a_1(a_2 - 3a_5) + a_2(a_2 + a_4 - b_5))R\omega\sin\theta) + \cos^2\theta(a_2^2(a_2b_{11} - a_1b_{21}) + R\omega(-a_1^2(a_2 - 3a_5) - a_1a_2(a_2 + 2a_4 - 2b_5) + a_2(a_2a_3 - a_2b_4 + \omega^2))\sin\theta)) + a_2^2((a_2^2a_3 - a_1a_2a_4 + a_1^2a_5)R^2\omega\cos^4\theta - R^2(a_1^2a_2(-a_4 + b_4) + a_1^3(a_5 - b_5) + a_2a_4\omega^2 + a_1(a_2^2(a_3 - b_3) - 2a_5\omega^2))\cos^3\theta\sin\theta - R\cos^2\theta\sin\theta(a_1a_2(-a_2b_11 + a_1b_21) - R\omega(a_2^2b_3 + a_1a_2(a_4 - 2b_4) + a_1^2(-2a_5 + 3b_5) + a_5\omega^2)\sin\theta) + \omega\sin^2\theta(a_2^2b_{22} - a_2b_{21}R\omega\sin\theta + b_5R^2\omega^2\sin^2\theta) - \cos\theta\sin\theta(a_2^2(a_2b_{12} - a_1b_{22}) - a_2(a_2b_{11} - 2a_1b_{21})R\omega\sin\theta + (a_1a_5 + a_2b_4 - 3a_1b_5)R^2\omega^2\sin^2\theta))]/(a_2^4\omega^4).$$

Note that the differential system (9) is in normal form (12) for applying the averaging theory with $T = 2\pi$, $x = R$, $t = \theta$ and $\varepsilon^3 R(\theta, x, \varepsilon) = O(\varepsilon^3)$. We also observe that F_1 is C^2 in x and 2π-periodic in θ in an open interval

$$I = \{R : 0 < R < \bar{R}\} \quad \text{for some } \bar{R} > 0.$$

Applying Theorem 5 (see Appendix A) we obtain the averaging function of first order

$$f_1(R) = \frac{b_{21}}{2\omega}R.$$

We can see that $f_1(R)$ has no solution in I. Therefore the averaging method of first order does not provide any small limit cycle bifurcating from the origin. We set $b_{21} = 0$ such that $f_1(R) \equiv 0$ and then can apply the averaging theory of second order. After some calculations using F_2 and that

$$\begin{aligned} y_1 \quad &= -\tfrac{R}{12a_2^2\omega^2}\left(-R\omega(9\sin(\theta) + \sin(3\theta))\left(a_1^2a_5 - a_1a_2a_4 + a_2^2a_3\right) - \right. \\ &2\sin^2(\theta)(2R\omega\sin(\theta)(a_1^2(a_2 - 2a_5) + a_1a_2(a_2 + a_4 - 2b_5) + a_2^2b_4 - a_2\omega^2 \\ &+a_5\omega^2) - 3a_2^3b_{11}) - 4R(\cos^3(\theta) - 1)(a_1^3a_5 + a_1^2a_2(b_5 - a_4) + a_1a_2^2(a_3 - b_4) + \\ &a_1\omega^2(a_2 - 2a_5) + a_2^3b_3 + a_2\omega^2(a_2 + a_4)) + 16R\omega^2\sin^4(\tfrac{\theta}{2})(\cos(\theta) + 2)(a_2b_5 + \\ &\left. a_1(a_5 - a_2))\right), \end{aligned}$$

following (13) we get that the averaging function of second order is

$$f_2(R) = R(A + B\,R^2),$$

where A and B are given in (3). Thus, $f_2(R)$ has one positive real root $R^* = +\sqrt{-A/B}$ in I if $0 < -A/B$ and in this case it holds that $df_2/dR(R^*) = -2A \neq 0$. It follows then from Theorem 5 that for $|\varepsilon|$ sufficiently small, system (9) has one 2π-periodic solution $R^*(\theta, \varepsilon)$ such that $R^*(\theta, \varepsilon) \to \sqrt{-A/B}$ when $\varepsilon \to 0$. Moreover, this periodic solution is stable if $A > 0$ and unstable if $A < 0$.

Now we shall go back through the changes of variables and we obtain a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ bifurcating from $(1, 1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x(t, \varepsilon), y(t, \varepsilon)) = (1, 1) + O(\varepsilon)$. This completes the proof of the theorem.

2.1. **Details of Example 2.** Consider system (4) with $a_1 = 1, a_2 = 0, b_1 = -1$ and $b_2 = 0$. Note that it is in the form (1) and the eigenvalues associated to the Jacobian matrix around $(1, 1)$ are equal to $\pm i$. So, $(1, 1)$ is a Hopf equilibrium point.

For this we perturb the parameters b_1 and b_2 as in (6) taking $b_1 = -1 + b_{11}\varepsilon + b_{12}\varepsilon^2$ and $b_2 = b_{22}\varepsilon^2$, for some parameter ε to be taken sufficiently small.

After translating the Hopf equilibrium point to the origin, introducing polar coordinates and setting $r = \varepsilon R$ we get

$$\begin{aligned}
\dot{R} &= \tfrac{1}{2}R\left(\sin\theta(R\varepsilon\sin\theta+1)\left(2\cos\theta\left(b_{11}\varepsilon+b_{12}\varepsilon^2-1\right)+2b_{22}\varepsilon^2\sin\theta+R\varepsilon\sin(2\theta)+\right.\right.\\
&\quad 12R\varepsilon\cos(2\theta)-8R\varepsilon)+2\cos\theta(R\varepsilon\cos\theta+1)(\sin\theta+R\varepsilon\sin\theta\cos\theta+R\varepsilon))\\
\dot{\theta} &= \tfrac{1}{2}\left(\cos\theta(R\varepsilon\sin\theta+1)\left(2\cos\theta\left(b_{11}\varepsilon+b_{12}\varepsilon^2-1\right)+2b_{22}\varepsilon^2\sin\theta+R\varepsilon\sin(2\theta)+\right.\right.\\
&\quad 12R\varepsilon\cos(2\theta)-8R\varepsilon)-2\sin\theta(R\varepsilon\cos\theta+1)(\sin\theta+R\varepsilon\sin\theta\cos\theta+R\varepsilon))\,.
\end{aligned}$$

Finally, we write $R' = \dot{R}/\dot{\theta}$ as in (9) as

$$\begin{aligned}
R' &= -R\varepsilon(\cos\theta(b_{11}\sin\theta+R)+R\sin\theta(7\cos(2\theta)-3))-\tfrac{1}{8}R\varepsilon^2\left(2b_{11}^2\sin(2\theta)+\right.\\
&\quad +b_{11}^2\sin(4\theta)-16b_{11}R\sin\theta-2b_{11}R\sin(3\theta)+14b_{11}R\sin(5\theta)+\\
&\quad +6b_{11}R\cos\theta+2b_{11}R\cos(3\theta)+4b_{12}\sin(2\theta)-4\left(b_{22}-3R^2\right)\cos(2\theta)+4b_{22}+\\
&\quad \left.109R^2\sin(2\theta)-112R^2\sin(4\theta)+49R^2\sin(6\theta)+16R^2\cos(4\theta)-4R^2\right).
\end{aligned}$$

The averaging function of first order is identically zero. The averaging function of second order is

$$f_2(R) = -R(b_{22} - R^2)/2.$$

Then for $b_{22} > 0$ we have the solution $R^* = \sqrt{b_{22}} \neq 0$ and so $df_2/dR = b_{22} > 0$. Therefore, there exists a unique unstable limit cycle bifurcating from the Hopf-point $(1, 1)$.

3. Proof of Theorem 3

The proof of Theorem 3 is analogous to the proof of Theorem 1 in section 2, so we will follow the structure of the previous section.

Consider system (2) with $(1, 1, 1)$ a finite equilibrium point. The conditions to have a zero-Hopf equilibrium point at $(1, 1, 1)$ are

$$c_1 = \alpha/\gamma, \quad c_2 = \beta/\gamma, \quad c_3 = -a_1 - b_2.$$

The Jacobian matrix evaluated at $(1, 1, 1)$ has the eigenvalues $0, \pm i\omega, \omega \in \mathbb{R}^+$ and so it is a zero-Hopf equilibrium point. To study the number of limit cycles bifurcating from this equilibrium point we take

$$c_1 = \alpha/\gamma + \varepsilon c_{11}, \quad c_2 = \beta/\gamma + \varepsilon c_{21}, \quad c_3 = -a_1 - b_2 + \varepsilon c_{31},$$

being ε a small parameter to be taken sufficiently small.

Doing the same steps detailed in the proof of Theorem 1 for the Kolmogorov system (1) in $\mathbb{R}^2$ we arrive to the system in the variables (u, v, w) such that the zero-Hopf equilibrium point is located at the origin and its linear part is in Jordan normal form. Now we want to write the system in such a way that conditions of Theorem 5 are satisfied.

For this we introduce cylindrical coordinates $(u, v, w) \to (r\sin\theta, r\cos\theta, w)$ and get the associated vector field $\mathcal{U} = (\dot r(r,\theta,w), \dot\theta(r,\theta,w), \dot w(r,\theta,w))$. Now we introduce the new variables $(r, w) \to (R\varepsilon, W\varepsilon)$ and we obtain a system in the form

$$\begin{aligned} \dot R &= \dot r/\varepsilon = G_{11}(R,\theta,W)\varepsilon + O(\varepsilon^2),\\ \dot\theta &= \omega + O(\varepsilon),\\ \dot W &= \dot w/\varepsilon = G_{12}(R,\theta,W) + O(\varepsilon^2) \end{aligned}$$

where G_{ij} are 2π-periodic in the variable θ of class C^2 and $O(\varepsilon^2)$ is also of class C^2 and 2π-periodic in the variable θ. We take θ as the new independent variable and write $R' = \dot R/\dot\theta$, $W' = \dot W/\dot\theta$. Doing so, we get

$$R' = \varepsilon F_{11} + O(\varepsilon^2), \quad W' = \varepsilon F_{12} + O(\varepsilon^2). \tag{10}$$

Note that the differential system (10) is in normal form (12) for applying the averaging theory with $T = 2\pi$, $x = (R, W)$, $t = \theta$ and $\varepsilon R(\theta, x, \varepsilon) = O(\varepsilon^2)$. We also remark that F_{11}, F_{12} and R are C^2 in x and 2π-periodic in θ in an open interval $I \times R$. Applying Theorem 5 we obtain the averaging function of first order has the form $f = (f_{11}(R,W), f_{12}(R,W))$ where

$$f_{11}(R,W) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\hat R}{\omega} d\theta = R(C + DW)$$

and

$$f_{12}(R,W) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\hat W}{\omega} d\theta = ER^2 - S_2 W + FW^2,$$

with C, D, E and F explicit in Appendix B.

We can see that $f_{11}(R,W) = 0$ has the solutions $R_1 = 0$ and $W_2 = -C/D$ if $CD \neq 0$. For $R_1 = 0$ the function $f_{12}(R,W) = 0$ has the solution $W_1 = S_2/F$ if $S_2 F \neq 0$ and for $W = W_2$ the function $f_{12}(R,W) = 0$ has the positive solution $R_2 = \sqrt{-S_1}/D$ if $S_1 < 0$.

Now we compute the Jacobian matrix of f and we get

$$\begin{pmatrix} C + DW & DR \\ 2ER & -S_2 + 2FW \end{pmatrix}.$$

By evaluating the determinant of the Jacobian matrix at the solution (R_1, W_1) we get that it is equal to $S_2 S_3/F$ and at the solution (R_2, W_2) we get that it is equal to $-2CS_3/D$. It follows from Theorem 5 that if $S_3 \neq 0$ then for $|\varepsilon|$ sufficiently small, system (10) has two 2π-periodic solutions $(R_1(\theta,\varepsilon), W_1(\theta,\varepsilon))$ and $(R_2(\theta,\varepsilon), W_2(\theta,\varepsilon))$ such that $(R_j(\theta,\varepsilon), W_j(\theta,\varepsilon)) \to (R_j, W_j)$ for $j = 1, 2$ when $\varepsilon \to 0$.

Moreover, the Jacobian matrix evaluated at the periodic solution (R_1, W_1) has eigenvalues equal to S_2 and S_3/F. Since the eigenvalues of the Jacobian matrix evaluated at the periodic solutions (R_1, W_1) provide the stability of the fixed point corresponding to the Poincaré map defined in a neighborhood of the periodic solution, if $S_2 > 0$ and $S_3/F > 0$ then the periodic solution is unstable with an unstable manifold of dimension two. If $S_2 < 0$ and $S_3/F > 0$, or $S_2 > 0$ and $S_3/F < 0$ then the periodic solution is unstable with a stable and an unstable manifold of dimension one. Finally, if $S_2 < 0$ and $S_3/F < 0$ then the periodic solution is stable with a stable manifold of dimension two.

On the other hand, the Jacobian matrix evaluated at (R_2, W_2) has eigenvalues equal to $-\dfrac{CF+S_3\pm\sqrt{S_4}}{2D}$ where $S_4=(CF+S_3)^2+8CDS_3$. Since $S_1 = CS_3/E < 0$, then taking into account that $CDS_3 = DES_1$ we get that if $DE < 0$ then the periodic solution is unstable with a stable and an unstable manifold of dimension one. If $DE > 0$ and $(CF + S_3)/D > 0$ then the periodic solution is stable with a stable manifold of dimension two. Finally, if $DE > 0$ and $(CF + S_3)/D < 0$ then the periodic solution is unstable with a unstable manifold of dimension two.

Combining the above information of the eigenvalues of the Jacobian matrix for both (R_1, W_1) and (R_2, W_2) we get statements (a)–(e) in the theorem.

Now we shall go back through the changes of variables and we obtain two periodic solutions $(x_j(t,\varepsilon), y(t,\varepsilon), z_j(t,\varepsilon))$ $(j = 1,2)$ bifurcating from $(1,1,1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t,\varepsilon), y(t,\varepsilon), z_j(t,\varepsilon)) = (1,1,1) + O(\varepsilon)$ for $j = 1,2$. This completes the proof of the theorem.

3.1. **Details on Example 4.** . Take system (5). It is in the form (2), with $c_1 = -2, c_2 = 0$ and $c_3 = 0$. Moreover, the singular point $(1,1,1)$ is a zero-Hopf equilibrium point (the eigenvalues of the Jacobian matrix at this point are $0, \pm i$). We write

$$c_1 = -2 + c_{11}\varepsilon, \quad c_2 = c_{21}\varepsilon, \quad c_3 = c_{31}\varepsilon = -\varepsilon/5,$$

with ε a small parameter to be taken sufficiently small. Thus, following the procedure we did it in the proof of Theorem 3, translating the zero-Hopf equilibrium point at the origin, writing the Jacobian matrix at this point in Jordan normal form, introducing cylinder coordinates and taking the variables R, W applying the averaging theory of first order we get the functions

$$f_{11}(R,W) = R(1+2W), \quad f_{12}(R,W) = (-R^2 + 2/5W(-9+10W))/2. \tag{11}$$

From (11) we have two solutions namely $s_1 = (0, 9/10)$ and $s_2 = (\sqrt{14/5}, -1/2)$, whose determinant of the Jacobian matrix is equal to 126/25 and 28/5, respectively. Moreover, the eigenvalues of s_1 are 14/5 and 9/5 so it is unstable with an unstable manifold of dimension two. On the other hand, the eigenvalues of s_2 are $-(38 + 2\sqrt{921})/20 > 0$ and $-(38 - 2\sqrt{921})/20 < 0$ so s_2 is unstable with a stable and an unstable manifolds of dimension one. Then, through the averaging method of first order we have detected two small limit cycles bifurcating from the zero-Hopf equilibrium point $(1,1,1)$ of system (5) when $\varepsilon = 0$.

Appendix A. Averaging Theory

We summarize the averaging theory of second order which provides sufficient conditions for the existence of periodic orbits for a periodic differential system depending on small parameters. See [1] for additional details and of the prof of the result stated in this appendix.

Theorem 5. *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{12}$$

where $F_1, F_2\colon \mathbb{R}\times D \to \mathbb{R}^n$, $R\colon \mathbb{R}\times D\times(-\varepsilon_f,\varepsilon_f)\to\mathbb{R}^n$ are continuous and T-periodic functions in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following assumptions hold:

(a) *$F_1(t,\cdot)\in C^1(D)$ for all $t\in\mathbb{R}$, F_1, F_2, R and D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ε. We define $f_1, f_2\colon D\to\mathbb{R}^n$ as*

$$\begin{aligned} f_1(z) &= \int_0^T F_1(s,z)\,ds, \\ f_2(z) &= \int_0^T \left[D_zF_1(s,z)y_1(s,z)+F_2(s,z)\right]ds \end{aligned} \tag{13}$$

where

$$y_1(s,z)=\int_0^s F_1(t,z)\,dt.$$

(b) *For $V\subset D$ an open and bounded set and for each $\varepsilon\in(-\varepsilon_f,\varepsilon_f)\setminus\{0\}$, there exists $a\in V$ such that if $f_1(z)\not\equiv 0$ then $f_1(a)=0$ and $d_B(f,a)\neq 0$, where $d_B(f,a)$ is the Brouwer degree of the function $f_1\colon V\to\mathbb{R}^n$ at the fixed point a, and if $f_1(z)\equiv 0$ and $f_2(z)\not\equiv 0$, then $f_2(a)=0$ and $d_B(f,a)\neq 0$.*

Then for $|\varepsilon|>0$ sufficiently small, there exists a T-periodic solution $\phi(t,\varepsilon)$ of the system such that $\phi(0,\varepsilon)\to a$ when $\varepsilon\to 0$. The kind of stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given by the eigenvalues of the Jacobian matrix $D_2(f_1(z)+\varepsilon f_2(z))|_{z=a}$.

Note that a sufficient condition for the showing that the Brouwer degree of a function f at a fixed point a is nonzero, is that the Jacobian of the function f at a (when it is defined, it is not zero), see [5].

When the first part of assumption (b) holds Theorem 5 provides the averaging theory of first order and it provides the averaging theory of second order when the second part of assumption (b) holds.

Appendix B. notations

In this section we introduce the notation introduced in the statement and proof of Theorem 3.

$$C=(a_3b_2c_{11}-a_2b_3c_{11}-a_3b_1c_{21}+a_1b_3c_{21}+a_2b_1c_{31}-a_1b_2c_{31}+c_{31}\omega^2)/(2\omega^3),$$

$D=(2a_1^4b_3(a_9b_2^2+b_3(-a_8b_2+a_7b_3))-2a_3^4b_1(b_2^2c_4-b_1b_2c_5+b_1^2c_7)+a_2^2b_3(2a_2(a_9b_1^3-a_6b_1^2b_3+a_4b_1b_3^2+b_2b_3^2b_4-b_1b_2b_3b_6+b_1^2b_2b_9+b_3^3c_4-b_1b_3^2c_6+b_1^2b_3c_9)+(2a_9b_1^2-b_3(a_6b_1+b_3(b_5+c_6)+b_1(b_2-b_8-2c_9)))\omega^2)+a_3^3(-2a_7b_1^3b_2-2a_4b_1b_2^3-2a_2b_1b_2^2b_4-2b_2^4b_4+2a_2b_1^2b_2b_5+2b_1b_2^3b_5-2a_2b_1^3b_7-2b_1^2b_2^2b_7+4a_2b_1b_2b_3c_4-2b_2^3b_3c_4-2a_2b_1^2b_3c_5+2b_1b_2^2b_3c_5-2a_2b_1^2b_2c_6-2b_1^2b_2b_3c_7+2a_2b_1^3c_8-2a_4b_1b_2\omega^2+b_1^2b_2\omega^2-2b_2^2b_4\omega^2+b_1b_2b_5\omega^2-b_1b_2c_6\omega^2+b_1^2c_8\omega^2+a_5b_1^2(2b_2^2+\omega^2))-a_2a_3(2a_2^2b_1(b_3^2b_4-b_1b_3b_6+b_1^2b_9)+b_3(2a_8b_1^2-a_6b_1b_2-b_1b_2^2-a_5b_1b_3+b_1b_2b_3-2b_2b_3b_5+2b_1b_3b_7+b_1b_2b_8-2b_2b_3c_6+b_1b_3c_8+2b_1b_2c_9)\omega^2+2a_2(a_9b_1^3b_2+a_8b_1^3b_3-2a_6b_1^2b_2b_3-a_5b_1^2b_3^2+3a_4b_1b_2b_3^2+3b_2^2b_3^2b_4-b_1b_2b_3^2b_5-2b_1b_2^2b_3b_6+b_1^2b_2b_3b_8+b_1^2b_2^2b_9+3b_2b_3^3c_4-b_1b_3^3c_5-2b_1b_2b_3^2c_6+b_1^2b_3^2c_8+b_1^2b_2b_3c_9+b_3^2b_4\omega^2-b_1b_3b_6\omega^2+b_1^2b_9\omega^2))+a_3^2(2a_2^2b_1(b_2(2b_3b_4-b_1b_6)-b_3^2c_4+b_1b_3(-b_5+c_6)+b_1^2(b_8-c_9))+b_3(2a_7b_1^2+b_2(-a_5b_1-b_2(b_5+c_6)+b_1(b_2+2b_7+c_8)))\omega^2+$

$$a_2(2a_8b_1^3b_2+2a_7b_1^3b_3-4a_5b_1^2b_2b_3+6a_4b_1b_2^2b_3+6b_2^3b_3b_4-4b_1b_2^2b_3b_5-2b_1b_2^3b_6+$$
$$2b_1^2b_2b_3b_7+2b_1^2b_2^2b_8+6b_2^2b_3^2c_4-4b_1b_2b_3^2c_5-2b_1b_2^2b_3c_6+2b_1^2b_3^2c_7+2b_1^2b_2b_3c_8+b_1^2b_2\omega^2+$$
$$2a_4b_1b_3\omega^2+4b_2b_3b_4\omega^2-b_1b_3b_5\omega^2-2b_1b_2b_6\omega^2+b_1^2b_8\omega^2+b_1b_3c_6\omega^2-2b_1^2c_9\omega^2-$$
$$a_6b_1^2(2b_2^2+\omega^2)))+a_1^3(2a_2b_3(-2a_9b_1b_2+a_8b_1b_3+a_6b_2b_3-a_5b_3^2+b_3^2b_7-b_2b_3b_8+$$
$$b_2^2b_9)+a_3(-2a_9b_1b_2^2+b_3(4a_8b_1b_2-2a_6b_2^2-6a_7b_1b_3+2a_5b_2b_3+2b_3^2c_7-2b_2b_3c_8+$$
$$2b_2^2c_9+b_2\omega^2)))+a_1^2(a_3^2(-2a_8b_1^2b_2+2a_6b_1b_2^2+6a_7b_1^2b_3-4a_5b_1b_2b_3+2a_4b_2^2b_3+$$
$$2b_2b_3^2c_5-2b_2^2b_3c_6-6b_1b_3^2c_7+4b_1b_2b_3c_8-2b_1b_2^2c_9-b_1b_2\omega^2+b_2b_3\omega^2)+b_3(2a_2^2(a_9b_1^2-$$
$$a_6b_1b_3+a_4b_3^2-b_3^2b_5+b_2b_3b_6+b_1b_3b_8-2b_1b_2b_9)+2(a_9b_2^2-a_8b_2b_3+a_7b_3^2)\omega^2+$$
$$a_2(2a_9b_1b_2^2-2a_8b_1b_2b_3+2a_7b_1b_3^2+2b_2b_3^2b_7-2b_2^2b_3b_8+2b_2^3b_9+2b_3^3c_7-2b_2b_3^2c_8+$$
$$2b_2^2b_3c_9+b_2b_3\omega^2))+a_3(-2a_9b_1b_2^3+2a_8b_1b_2^2b_3-2a_7b_1b_2b_3^2-2b_2^2b_3^2b_7+2b_2^3b_3b_8-2b_2^4b_9-$$
$$2b_2b_3^3c_7+2b_2^2b_3^2c_8-2b_2^3b_3c_9-a_6b_2b_3\omega^2+a_5b_3^2\omega^2+b_2b_3^2\omega^2+b_2b_3b_8\omega^2-2b_2^2b_9\omega^2+$$
$$b_3^2c_8\omega^2-2b_2b_3c_9\omega^2+a_2(4a_9b_1^2b_2-4a_8b_1^2b_3+4a_5b_1b_3^2-4a_4b_2b_3^2+2b_2b_3^2b_5-2b_2^2b_3b_6-$$
$$6b_1b_3^2b_7+4b_1b_2b_3b_8-2b_1b_2^2b_9-2b_3^3c_5+2b_2b_3^2c_6+2b_1b_3^2c_8-4b_1b_2b_3c_9-b_1b_3\omega^2-$$
$$b_3^2\omega^2)))-a_1(a_3^3(2a_7b_1^3-2a_5b_1^2b_2+2a_4b_1b_2^2-2b_2^2b_3c_4+4b_1b_2b_3c_5-2b_1b_2^2c_6-6b_1^2b_3c_7+$$
$$2b_1^2b_2c_8+b_1b_2\omega^2)+a_3^2(2a_8b_1^2b_2^2-2a_6b_1b_2^3-4a_7b_1^2b_2b_3+2a_5b_1b_2^2b_3+2b_2^3b_3b_5-2b_2^4b_6-$$
$$4b_1b_2^2b_3b_7+2b_1b_2^3b_8+2b_2^2b_3^2c_5-2b_2^3b_3c_6-4b_1b_2b_3^2c_7+2b_1b_2^2b_3c_8-a_6b_1b_2\omega^2+b_1b_2^2\omega^2+$$
$$2a_5b_1b_3\omega^2-2a_4b_2b_3\omega^2+2b_1b_2b_3\omega^2+b_2^2b_3\omega^2+b_2b_3b_5\omega^2-2b_2^2b_6\omega^2+b_1b_2b_8\omega^2-$$
$$b_2b_3c_6\omega^2+2b_1b_3c_8\omega^2-2b_1b_2c_9\omega^2-a_2(2a_8b_1^3-2a_6b_1^2b_2-2a_5b_1^2b_3+4a_4b_1b_2b_3+$$
$$2b_2^2b_3b_4-4b_1b_2b_3b_5+2b_1b_2^2b_6+6b_1^2b_3b_7-2b_1^2b_2b_8-4b_2b_3^2c_4+4b_1b_3^2c_5-4b_1^2b_3c_8+$$
$$4b_1^2b_2c_9+b_1^2\omega^2+b_1b_3\omega^2))+a_2b_3(-2a_2^2(b_3^2b_4-b_1b_3b_6+b_1^2b_9)+(4a_9b_1b_2-b_3(2a_8b_1+$$
$$a_6b_2+b_2^2-a_5b_3+b_2b_3+2b_3b_7-b_2b_8+b_3c_8-2b_2c_9))\omega^2+a_2(4a_9b_1^2b_2-2a_8b_1^2b_3-$$
$$2a_6b_1b_2b_3+2a_5b_1b_3^2+2b_2b_3^2b_5-2b_2^2b_3b_6-2b_1b_2b_3b_8+4b_1b_2^2b_9+2b_3^3c_5-2b_2b_3^2c_6-$$
$$2b_1b_3^2c_8+4b_1b_2b_3c_9+b_1b_3\omega^2+b_3^2\omega^2))+a_3(2a_2^2(a_9b_1^3-a_6b_1^2b_3+a_4b_1b_3^2+2b_2b_3^2b_4-$$
$$2b_1b_3^2b_5+2b_1^2b_3b_8-2b_1^2b_2b_9-b_3^3c_4+b_1b_3^2c_6-b_1^2b_3c_9)+b_3(-2a_8b_1b_2+a_6b_2^2+b_2^3+$$
$$4a_7b_1b_3-a_5b_2b_3+b_2^2b_3+2b_2b_3b_7-b_2^2b_8+b_2b_3c_8-2b_2^2c_9)\omega^2+a_2(-4a_9b_1^2b_2^2+4a_7b_1^2b_3^2-$$
$$4a_5b_1b_2b_3^2-4b_2^2b_3^2b_5+4b_2^3b_3b_6+4b_1b_2b_3^2b_7-4b_1b_2^3b_9-4b_2b_3^3c_5+4b_2^2b_3^2c_6+4b_1b_3^3c_7-$$
$$4b_1b_2^2b_3c_9+2a_4b_3^2\omega^2-2b_2b_3^2\omega^2-b_3^2b_5\omega^2+2b_2b_3b_6\omega^2+b_1b_3b_8\omega^2-4b_1b_2b_9\omega^2+b_3^2c_6\omega^2-$$
$$2b_1b_3c_9\omega^2+a_6b_1b_3(4b_2^2-\omega^2))))))/(2(a_3^2b_1+a_3(-a_1+b_2)b_3-a_2b_3^2)\omega^5),$$

$$E=((a_2b_1-a_1b_2)^2(a_3^4b_1(b_2^2c_4-b_1b_2c_5+b_1^2c_7+c_4\omega^2)+a_3^3(a_7b_1^3b_2-a_5b_1^2b_2^2+a_4b_1b_2^3+$$
$$a_2b_1b_2^2b_4+b_2^4b_4-a_2b_1^2b_2b_5-b_1b_2^3b_5+a_2b_1^3b_7+b_1^2b_2^2b_7-2a_2b_1b_2b_3c_4+b_2^3b_3c_4+a_2b_1^2b_3c_5-$$
$$b_1b_2^2b_3c_5+a_2b_1^2b_2c_6+b_1^2b_2b_3c_7-a_2b_1^3c_8+a_4b_1b_2\omega^2+a_2b_1b_4\omega^2+2b_2^2b_4\omega^2-b_1b_2b_5\omega^2+$$
$$b_1^2b_7\omega^2+b_2b_3c_4\omega^2+b_1b_3c_5\omega^2-b_1^2c_8\omega^2+b_4\omega^4)-a_1^4b_3(b_3(-a_8b_2+a_7b_3)+a_9(b_2^2+\omega^2))-$$
$$a_3^2(a_2^2b_1(b_2(2b_3b_4-b_1b_6)-b_3^2c_4+b_1b_3(-b_5+c_6)+b_1^2(b_8-c_9))+\omega^2(a_8b_1^2b_2+a_7b_1^2b_3-$$
$$2a_5b_1b_2b_3+a_4b_2^2b_3-b_2^2b_3b_5+b_1b_2^2b_8-b_2b_3^2c_5-b_1b_3^2c_7+2b_1b_2b_3c_8-b_1b_2^2c_9+a_4b_3\omega^2-$$
$$b_3b_5\omega^2+b_1b_8\omega^2-b_1c_9\omega^2)+a_2(a_8b_1^3b_2-a_6b_1^2b_2^2+a_7b_1^3b_3-2a_5b_1^2b_2b_3+3a_4b_1b_2^2b_3+$$
$$3b_2^3b_3b_4-2b_1b_2^2b_3b_5-b_1b_2^3b_6+b_1^2b_2b_3b_7+b_1^2b_2^2b_8+3b_2^2b_3^2c_4-2b_1b_2b_3^2c_5-b_1b_2^2b_3c_6+$$
$$b_1^2b_3^2c_7+b_1^2b_2b_3c_8+a_4b_1b_3\omega^2+3b_2b_3b_4\omega^2-2b_1b_3b_5\omega^2-b_1b_2b_6\omega^2+2b_1^2b_8\omega^2+b_3^2c_4\omega^2+$$
$$b_1b_3c_6\omega^2-2b_1^2c_9\omega^2))+a_3(a_2^3b_1(b_3^2b_4-b_1b_3b_6+b_1^2b_9)+a_2\omega^2(2a_9b_1^2b_2+2a_8b_1^2b_3-$$
$$2a_6b_1b_2b_3-2a_5b_1b_3^2+2a_4b_2b_3^2-b_2b_3^2b_5-b_2^2b_3b_6+b_1b_3^2b_7+3b_1b_2^2b_9-b_3^3c_5-b_2b_3^2c_6+$$
$$b_1b_3^2c_8+2b_1b_2b_3c_9-b_3b_6\omega^2+3b_1b_9\omega^2)+a_2^2(a_9b_1^3b_2+a_8b_1^3b_3-2a_6b_1^2b_2b_3-a_5b_1^2b_3^2+$$
$$3a_4b_1b_2b_3^2+3b_2^2b_3^2b_4-b_1b_2b_3^2b_5-2b_1b_2^2b_3b_6+b_1^2b_2b_3b_8+b_1^2b_2^2b_9+3b_2b_3^3c_4-b_1b_3^3c_5-$$
$$2b_1b_2b_3^2c_6+b_1^2b_3^2c_8+b_1^2b_2b_3c_9+b_3^2b_4\omega^2-2b_1b_3b_6\omega^2+3b_1^2b_9\omega^2)+\omega^2(a_7b_1b_2b_3^2+b_2^2b_3^2b_7-$$
$$b_2^3b_3b_8+b_2^4b_9+b_2b_3^3c_7-b_2^2b_3^2c_8+b_2^3b_3c_9-a_5b_3^2\omega^2+b_3^2b_7\omega^2-b_2b_3b_8\omega^2+2b_2^2b_9\omega^2+$$
$$b_2b_3c_9\omega^2+b_9\omega^4+a_8b_1b_3(-b_2^2+\omega^2)+a_9b_1b_2(b_2^2+\omega^2)))-b_3(a_2^3(a_9b_1^3-a_6b_1^2b_3+a_4b_1b_3^2+$$
$$b_2b_3^2b_4-b_1b_2b_3b_6+b_1^2b_2b_9+b_3^3c_4-b_1b_3^2c_6+b_1^2b_3c_9)+a_2^2(3a_9b_1^2-2a_6b_1b_3+a_4b_3^2-b_2b_3b_6+$$
$$2b_1b_2b_9-b_3^2c_6+2b_1b_3c_9)\omega^2+\omega^4(b_3(-a_8b_2+a_7b_3)+a_9(b_2^2+\omega^2))+a_2\omega^2(-a_8b_1b_2b_3+$$
$$a_7b_1b_3^2+b_2b_3^2b_7-b_2^2b_3b_8+b_2^3b_9+b_3^3c_7-b_2b_3^2c_8+b_2^2b_3c_9-a_6b_3\omega^2+b_2b_9\omega^2+b_3c_9\omega^2+$$
$$a_9b_1(b_2^2+3\omega^2)))+a_1^3(a_2b_3(2a_9b_1b_2-a_8b_1b_3-a_6b_2b_3+a_5b_3^2-b_3^2b_7+b_2b_3b_8-b_2^2b_9-$$

$b_9\omega^2)+a_3(a_9b_1(b_2^2+\omega^2)-b_3(2a_8b_1b_2-3a_7b_1b_3+a_5b_2b_3+b_3^2c_7-b_2b_3c_8+b_2^2c_9+c_9\omega^2-$
$a_6(b_2^2+\omega^2))))+a_1^2(a_3^2(a_8b_1^2b_2-3a_7b_1^2b_3+2a_5b_1b_2b_3-a_4b_2^2b_3-b_2b_3^2c_5+b_2^2b_3c_6+3b_1b_3^2c_7-$
$2b_1b_2b_3c_8+b_1b_2^2c_9-a_4b_3\omega^2+b_3c_6\omega^2+b_1c_9\omega^2-a_6b_1(b_2^2+\omega^2))+a_3(-a_8b_1b_2^2b_3+$
$a_7b_1b_2b_3^2+b_2^2b_3^2b_7-b_2^3b_3b_8+b_2^4b_9+b_2b_3^3c_7-b_2^2b_3^2c_8+b_2^3b_3c_9+a_8b_1b_3\omega^2-a_5b_3^2\omega^2+$
$b_3^2b_7\omega^2-b_2b_3b_8\omega^2+2b_2^2b_9\omega^2+b_2b_3c_9\omega^2+b_9\omega^4+a_9b_1b_2(b_2^2+\omega^2)+a_2(-2a_9b_1^2b_2+$
$2a_8b_1^2b_3-2a_5b_1b_3^2+2a_4b_2b_3^2-b_2b_3^2b_5+b_2^2b_3b_6+3b_1b_3^2b_7-2b_1b_2b_3b_8+b_1b_2^2b_9+b_3^3c_5-$
$b_2b_3^2c_6-b_1b_3^2c_8+2b_1b_2b_3c_9+b_3b_6\omega^2+b_1b_9\omega^2))-b_3(a_2^2(a_9b_1^2-a_6b_1b_3+a_4b_3^2-b_3^2b_5+$
$b_2b_3b_6+b_1b_3b_8-2b_1b_2b_9)+2\omega^2(b_3(-a_8b_2+a_7b_3)+a_9(b_2^2+\omega^2))+a_2(-a_8b_1b_2b_3+$
$a_7b_1b_3^2+b_2b_3^2b_7-b_2^2b_3b_8+b_2^3b_9+b_3^3c_7-b_2b_3^2c_8+b_2^2b_3c_9-a_6b_3\omega^2+b_2b_9\omega^2+b_3c_9\omega^2+$
$a_9b_1(b_2^2+3\omega^2))))+a_1(a_3^3(a_7b_1^3-a_5b_1^2b_2+a_4b_1b_2^2-b_2^2b_3c_4+2b_1b_2b_3c_5-b_1b_2^2c_6-$
$3b_1^2b_3c_7+b_1^2b_2c_8+a_4b_1\omega^2-b_3c_4\omega^2-b_1c_6\omega^2)-a_3^2(a_6b_1b_2^3+2a_7b_1^2b_2b_3-a_5b_1b_2^2b_3-$
$b_2^3b_3b_5+b_2^4b_6+2b_1b_2^2b_3b_7-b_1b_2^3b_8-b_2^2b_3^2c_5+b_2^3b_3c_6+2b_1b_2b_3^2c_7-b_1b_2^2b_3c_8+a_6b_1b_2\omega^2-$
$a_5b_1b_3\omega^2-b_2b_3b_5\omega^2+2b_2^2b_6\omega^2+2b_1b_3b_7\omega^2-b_1b_2b_8\omega^2+b_3^2c_5\omega^2+b_2b_3c_6\omega^2-b_1b_3c_8\omega^2+$
$b_6\omega^4+a_8b_1^2(-b_2^2+\omega^2)+a_2(a_8b_1^3-a_6b_1^2b_2-a_5b_1^2b_3+2a_4b_1b_2b_3+b_2^2b_3b_4-2b_1b_2b_3b_5+$
$b_1b_2^2b_6+3b_1^2b_3b_7-b_1^2b_2b_8-2b_2b_3^2c_4+2b_1b_3^2c_5-2b_1^2b_3c_8+2b_1^2b_2c_9+b_3b_4\omega^2+b_1b_6\omega^2))-$
$a_2b_3(a_2^2(b_3^2b_4-b_1b_3b_6+b_1^2b_9)+\omega^2(-2a_9b_1b_2+a_8b_1b_3+a_6b_2b_3-a_5b_3^2+b_3^2b_7-b_2b_3b_8+$
$b_2^2b_9+b_9\omega^2)+a_2(-2a_9b_1^2b_2+a_8b_1^2b_3+a_6b_1b_2b_3-a_5b_1b_3^2-b_2b_3^2b_5+b_2^2b_3b_6+b_1b_2b_3b_8-$
$2b_1b_2^2b_9-b_3^3c_5+b_2b_3^2c_6+b_1b_3^2c_8-2b_1b_2b_3c_9-b_3b_6\omega^2+2b_1b_9\omega^2))+a_3(a_2^2(a_9b_1^3-a_6b_1^2b_3+$
$a_4b_1b_3^2+2b_2b_3^2b_4-2b_1b_3^2b_5+2b_1^2b_3b_8-2b_1^2b_2b_9-b_3^3c_4+b_1b_3^2c_6-b_1^2b_3c_9)+2a_2(a_6b_1b_2^2b_3+$
$a_7b_1^2b_3^2-a_5b_1b_2b_3^2-b_2^2b_3^2b_5+b_2^3b_3b_6+b_1b_2b_3^2b_7-b_1b_2^3b_9-b_2b_3^3c_5+b_2^2b_3^2c_6+b_1b_3^3c_7-$
$b_1b_2^2b_3c_9-b_3^2b_5\omega^2+b_2b_3b_6\omega^2+b_1b_3b_8\omega^2-b_1b_2b_9\omega^2+b_3^2c_6\omega^2-b_1b_3c_9\omega^2+a_9b_1^2(-b_2^2+$
$\omega^2))+\omega^2(a_9b_1(b_2^2+\omega^2)-b_3(2a_8b_1b_2-3a_7b_1b_3+a_5b_2b_3+b_3^2c_7-b_2b_3c_8+b_2^2c_9+c_9\omega^2-$
$a_6(b_2^2+\omega^2)))))))/(2(a_3^2b_1+a_3(-a_1+b_2)b_3-a_2b_3^2)\omega^3(a_3^2b_2^2(b_2^2+\omega^2)+2a_2a_3b_2(a_3b_1b_2-$
$b_3(a_1b_2+b_2^2+\omega^2))+a_2^2(a_3^2b_1^2-2a_3b_1(a_1+b_2)b_3+b_3^2(a_1^2+2a_1b_2+b_2^2+\omega^2))))$

and

$F=(-a_1^4b_3(a_9b_2^2+b_3(-a_8b_2+a_7b_3))+a_3^4b_1(b_2^2c_4-b_1b_2c_5+b_1^2c_7)+a_1^3(a_2b_3(2a_9b_1b_2+$
$b_3(-a_8b_1-a_6b_2+a_5b_3-b_3b_7+b_2b_8)-b_2^2b_9)+a_3(a_9b_1b_2^2-b_3(2a_8b_1b_2-a_6b_2^2-$
$3a_7b_1b_3+b_3(a_5b_2+b_3c_7-b_2c_8)+b_2^2c_9)))+a_2^2b_3(-a_2(a_9b_1^3-a_6b_1^2b_3+a_4b_1b_3^2+$
$b_2b_3^2b_4-b_1b_2b_3b_6+b_1^2b_2b_9+b_3^3c_4-b_1b_3^2c_6+b_1^2b_3c_9)-(a_9b_1^2+b_3(-a_6b_1+a_4b_3))\omega^2)+$
$a_3^3(a_7b_1^3b_2-a_5b_1^2b_2^2+a_4b_1b_2^3+a_2b_1b_2^2b_4+b_2^4b_4-a_2b_1^2b_2b_5-b_1b_2^3b_5+a_2b_1^3b_7+b_1^2b_2^2b_7-$
$2a_2b_1b_2b_3c_4+b_2^3b_3c_4+a_2b_1^2b_3c_5-b_1b_2^2b_3c_5+a_2b_1^2b_2c_6+b_1^2b_2b_3c_7-a_2b_1^3c_8+(b_2^2b_4-$
$b_1b_2b_5+b_1^2b_7)\omega^2)-a_3^2(a_2^2b_1(b_2(2b_3b_4-b_1b_6)-b_3^2c_4+b_1b_3(-b_5+c_6)+b_1^2(b_8-c_9))+$
$(a_7b_1^2+b_2(-a_5b_1+a_4b_2))b_3\omega^2+a_2(a_8b_1^3b_2-a_6b_1^2b_2^2+a_7b_1^3b_3-2a_5b_1^2b_2b_3+3a_4b_1b_2^2b_3+$
$3b_2^3b_3b_4-2b_1b_2^2b_3b_5-b_1b_2^3b_6+b_1^2b_2b_3b_7+b_1^2b_2^2b_8+3b_2^2b_3^2c_4-2b_1b_2b_3^2c_5-b_1b_2^2b_3c_6+$
$b_1^2b_3^2c_7+b_1^2b_2b_3c_8+(2b_2b_3b_4-b_1b_3b_5-b_1b_2b_6+b_1^2b_8)\omega^2))+a_2a_3(a_2^2b_1(b_3^2b_4-b_1b_3b_6+$
$b_1^2b_9)+b_3(a_8b_1^2-a_6b_1b_2-a_5b_1b_3+2a_4b_2b_3)\omega^2+a_2(a_9b_1^3b_2+a_8b_1^3b_3-2a_6b_1^2b_2b_3-$
$a_5b_1^2b_3^2+3a_4b_1b_2b_3^2+3b_2^2b_3^2b_4-b_1b_2b_3^2b_5-2b_1b_2^2b_3b_6+b_1^2b_2b_3b_8+b_1^2b_2^2b_9+3b_2b_3^3c_4-$
$b_1b_3^3c_5-2b_1b_2b_3^2c_6+b_1^2b_3^2c_8+b_1^2b_2b_3c_9+(b_3^2b_4-b_1b_3b_6+b_1^2b_9)\omega^2))+a_1(a_3^3(a_7b_1^3-$
$a_5b_1^2b_2+a_4b_1b_2^2-b_2^2b_3c_4+2b_1b_2b_3c_5-b_1b_2^2c_6-3b_1^2b_3c_7+b_1^2b_2c_8)+a_2b_3(-a_2(-2a_9b_1^2b_2+$
$a_8b_1^2b_3+a_6b_1b_2b_3-a_5b_1b_3^2+a_2b_3^2b_4-b_2b_3^2b_5-a_2b_1b_3b_6+b_2^2b_3b_6+b_1b_2b_3b_8+a_2b_1^2b_9-$
$2b_1b_2^2b_9-b_3^3c_5+b_2b_3^2c_6+b_1b_3^2c_8-2b_1b_2b_3c_9)+(2a_9b_1b_2+b_3(-a_8b_1-a_6b_2+a_5b_3))\omega^2)+$
$a_3^2(b_2(a_8b_1^2b_2-a_6b_1b_2^2-2a_7b_1^2b_3+a_5b_1b_2b_3+b_2^2b_3b_5-b_2^3b_6-2b_1b_2b_3b_7+b_1b_2^2b_8+$
$b_2b_3^2c_5-b_2^2b_3c_6-2b_1b_3^2c_7+b_1b_2b_3c_8)+a_2(-a_8b_1^3+a_6b_1^2b_2+a_5b_1^2b_3-2a_4b_1b_2b_3-$
$b_2^2b_3b_4+2b_1b_2b_3b_5-b_1b_2^2b_6-3b_1^2b_3b_7+b_1^2b_2b_8+2b_2b_3^2c_4-2b_1b_3^2c_5+2b_1^2b_3c_8-2b_1^2b_2c_9)+$
$(b_2b_3b_5-b_2^2b_6-2b_1b_3b_7+b_1b_2b_8)\omega^2)+a_3(a_2^2(a_9b_1^3-a_6b_1^2b_3+a_4b_1b_3^2+2b_2b_3^2b_4-$
$2b_1b_3^2b_5+2b_1^2b_3b_8-2b_1^2b_2b_9-b_3^3c_4+b_1b_3^2c_6-b_1^2b_3c_9)-2a_2(a_9b_1^2b_2^2-a_6b_1b_2^2b_3-$
$a_7b_1^2b_3^2+a_5b_1b_2b_3^2+b_2^2b_3^2b_5-b_2^3b_3b_6-b_1b_2b_3^2b_7+b_1b_2^3b_9+b_2b_3^3c_5-b_2^2b_3^2c_6-b_1b_3^3c_7+$

$b_1b_2^2b_3c_9) + b_3(-a_8b_1b_2 + a_6b_2^2 + 2a_7b_1b_3 - a_5b_2b_3)\omega^2 + a_2(-b_3^2b_5 + b_2b_3b_6 + b_1b_3b_8 - 2b_1b_2b_9)\omega^2)) + a_1^2(a_3^2(a_8b_1^2b_2 - a_6b_1b_2^2 + b_3(-3a_7b_1^2 + 2a_5b_1b_2 - a_4b_2^2 - b_2b_3c_5 + b_2^2c_6 + 3b_1b_3c_7 - 2b_1b_2c_8) + b_1b_2^2c_9) - a_2b_3(a_9b_1b_2^2 - a_8b_1b_2b_3 + a_7b_1b_3^2 + b_2b_3^2b_7 - b_2^2b_3b_8 + b_2^3b_9 + a_2(a_9b_1^2 - a_6b_1b_3 + b_3(a_4b_3 - b_3b_5 + b_2b_6 + b_1b_8) - 2b_1b_2b_9) + b_3^3c_7 - b_2b_3^2c_8 + b_2^2b_3c_9) - b_3(a_9b_2^2 + b_3(-a_8b_2 + a_7b_3))\omega^2 + a_3(a_9b_1b_2^3 + a_2(-2a_9b_1^2b_2 + 2a_8b_1^2b_3 - 2a_5b_1b_3^2 + 2a_4b_2b_3^2 - b_2b_3^2b_5 + b_2^2b_3b_6 + 3b_1b_3^2b_7 - 2b_1b_2b_3b_8 + b_1b_2^2b_9 + b_3^3c_5 - b_2b_3^2c_6 - b_1b_3^2c_8 + 2b_1b_2b_3c_9) + b_2(-a_8b_1b_2b_3 + a_7b_1b_3^2 + b_2b_3^2b_7 - b_2^2b_3b_8 + b_2^3b_9 + b_3^3c_7 - b_2b_3^2c_8 + b_2^2b_3c_9) + (b_3^2b_7 - b_2b_3b_8 + b_2^2b_9)\omega^2)))/((a_3^2b_1 + a_3(-a_1 + b_2)b_3 - a_2b_3^2)\omega^5)$

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