PERIODIC SOLUTIONS FOR A CLASS OF NON–AUTONOMOUS NEWTON DIFFERENTIAL EQUATIONS

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Abstract. We provide sufficient conditions for the existence of periodic solutions of the second–order non–autonomous differential equation

$$\ddot{x} = -\nabla_x V(t,x),$$

in $\mathbb{R}^n$, where $V(t,x) = \dfrac{\|x\|^2}{2} + \varepsilon W(t,x)$ with $W(t,x)$ a 2π–periodic function in the variable t, ε is a small parameter, $x \in \mathbb{R}^n$ and

$$\nabla_x V(t,x) = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n} \right).$$

Note that this is a particular class of non–autonomous Newton differential equations. Moreover we provide some applications.

1. Introduction and statement of the main results

In this paper we shall study the existence of periodic solutions of the second–order non–autonomous differential equation in $\mathbb{R}^n$ of the form

$$\ddot{x} = -\nabla_x V(t,x), \tag{1}$$

where $V(t,x) = \dfrac{\|x\|^2}{2} + \varepsilon W(t,x)$ with $W(t,x)$ a 2π–periodic function in the variable t, ε is a small parameter, $x = (x_1, \ldots, x_n)$ and

$$\nabla_x V(t,x) = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n} \right).$$

Here $\|\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Note that this is a particular class of non–autonomous Newton differential equations. The dot denotes derivative with respect to the variable t.

Many authors have studied this system under various additional conditions, see for instance [5] and the references quoted therein.

To obtain analytically periodic solutions is in general a very difficult work, usually impossible. The averaging theory reduces this difficult problem for the differential equation (1) to find the zeros of a nonlinear function. It is known that in general the averaging theory for finding periodic solutions does not provide all the periodic solutions of the system. For more information about the averaging theory see section 2 and the references quoted there.

Our main result on the periodic solutions of the second–order non–autonomous differential equation (1) is the following .

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Theorem 1. *We define the functions*

$$\begin{aligned}
\mathcal{F}_k &= \frac{1}{2\pi}\int_0^{2\pi} \sin t\, \frac{\partial W}{\partial x_k}(t, x_{10}\cos t + y_{10}\sin t, ..., x_{n0}\cos t + y_{n0}\sin t)dt,\\
\mathcal{F}_{2k} &= -\frac{1}{2\pi}\int_0^{2\pi} \cos t\, \frac{\partial W}{\partial x_k}(t, x_{10}\cos t + y_{10}\sin t, ..., x_{n0}\cos t + y_{n0}\sin t)dt,
\end{aligned}$$

for $k = 1, \dots, n$. If the function $W(t, x_1, \dots, x_n)$ is 2π-periodic in the variable t, then for every $\varepsilon \neq 0$ sufficiently small and for every $(x_{10}^*, \dots, x_{n0}^*, y_{10}^*, \dots, y_{n0}^*)$ solution of the system

$$\begin{aligned}
&\mathcal{F}_k(x_{10}, \dots, x_{n0}, y_{10}, \dots, y_{n0}) = 0,\\
&\mathcal{F}_{2k}(x_{10}, \dots, x_{n0}, y_{10}, \dots, y_{n0}) = 0, \quad \text{for } k = 1, \dots, n,
\end{aligned} \tag{2}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_1, \dots, \mathcal{F}_{2n})}{\partial(x_{10}, \dots, x_{n0}, y_{10}, \dots, y_{n0})}\Bigg|_{(x_{10},\dots,x_{n0},y_{10},\dots,y_{n0})=(x_{10}^*,\dots,x_{n0}^*,y_{10}^*,\dots,y_{n0}^*)} \right) \neq 0, \tag{3}$$

the non–autonomous differential equation (1) *has a 2π–periodic solution $x(t,\varepsilon)$ which tends to the 2π–periodic solution $(x_{10}^*\cos t + y_{10}^*\sin t, ..., x_{n0}^*\cos t + y_{n0}^*\sin t)$, of $\ddot{x} + x = 0$ when $\varepsilon \to 0$.*

Theorem 1 is proved in section 3. Its proof is based in the averaging theory for computing periodic solutions, see section 2. For others applications of the averaging theory to the study periodic solutions, see [4] and [6].

Applications of Theorem 1 are the following.

Corollary 2. *Consider the non–autonomous Newton differential equation* (1) *in $\mathbb{R}$ with the potential $V(t,x) = \dfrac{x^2}{2} + \varepsilon W(t,x)$, where $W(t,x) = (ax + bx^3)\sin t$. If $ab < 0$, then for $\varepsilon \neq 0$ sufficiently small this differential equation has two periodic solutions $x_k(t,\varepsilon)$ for $k = 1, 2$, tending to the periodic solutions $2\sqrt{-\dfrac{a}{3b}}\cos t$ and $\dfrac{2}{3}\sqrt{-\dfrac{a}{b}}\sin t$ of $\ddot{x} + x = 0$, when $\varepsilon \to 0$.*

Corollary 2 is proved in section 5.

Corollary 3. *Consider the non–autonomous Newton differential equation* (1) *in $\mathbb{R}^2$ with the potential $V(t, x_1, x_2) = \dfrac{x_1^2 + x_2^2}{2} + \varepsilon W(t, x_1, x_2)$, where $W(t, x_1, x_2) = (ax_1 + bx_2 + cx_1^3 + dx_2^3)\sin t$. If $ac < 0$ and $bd < 0$ then for $\varepsilon \neq 0$ sufficiently small this differential equation has four periodic solutions $x^k(t,\varepsilon)$ for $k = 1, \dots, 4$,* *tending to the periodic solutions*

$$\left(-2\sqrt{-\frac{a}{3c}}\cos t, \frac{2}{3}\sqrt{-\frac{b}{d}}\cos t\right),$$
$$\left(2\sqrt{-\frac{a}{3c}}\cos t, \frac{2}{3}\sqrt{-\frac{b}{d}}\sin t\right),$$
$$\left(2\sqrt{-\frac{b}{3d}}\cos t, \frac{2}{3}\sqrt{-\frac{a}{c}}\sin t\right),$$
$$\left(\frac{2}{3}\sqrt{-\frac{a}{c}}\sin t, \frac{2}{3}\sqrt{-\frac{b}{d}}\sin t\right),$$

of $\ddot{x}+x=0$, *when* $\varepsilon \to 0$.

Corollary 3 is proved in section 5.

2. Basic results on averaging theory

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\mathbf{x}' = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{4}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\mathbf{x}' = F_0(t,\mathbf{x}), \tag{5}$$

has a submanifold of dimension n of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the system (5) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (6).

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (4); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 4 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t,\mathbf{x}(t,\mathbf{z},0))dt. \tag{7}$$

If there exists $\alpha \in V$ *with* $\mathcal{F}(\alpha) = 0$ *and* $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(\alpha)\right) \neq 0$*, then there exists a* T*–periodic solution* $\varphi(t,\varepsilon)$ *of system* (4) *such that* $\varphi(0,\varepsilon) \to \alpha$ *as* $\varepsilon \to 0$*.*

Theorem 4 goes back to Malkin [2] and Roseau [**?**], for a shorter proof see [1].

3. Proof of Theorem 1

If $y_i = \dot{x}_i$, for $i = 1, \ldots, n$, then the second–order non-autonomous differential equation (1) can be written as the following first–order differential system in $\mathbb{R}^{2n}$

$$
\begin{aligned}
\dot{x}_i &= y_i, \\
\dot{y}_i &= -x_i - \varepsilon \frac{\partial W(t,x)}{\partial x_i},
\end{aligned} \tag{8}
$$

for $i = 1, \ldots, n$. For $\varepsilon = 0$ it follows that $(x, y) = (0, 0)$ is the unique singular point of system (8). The eigenvalues of the linearized system at this singular point are all pure imaginary $\pm i, \ldots, \pm i$. The solution $(x(t), y(t))$ of the unperturbed system (i.e. system (8) with $\varepsilon = 0$) such that $(x(0), y(0)) = (x_0, y_0)$ is

$$
\begin{aligned}
x_i(t) &= x_{i0}\cos t + y_{i0}\sin t, \\
y_i(t) &= y_{i0}\cos t - x_{i0}\sin t,
\end{aligned} \tag{9}
$$

for $i = 1, \ldots, n$. Note that all these periodic orbits have period 2π.

Using the notation introduced in section 2, we have that $\mathbf{x} = (x, y)$, $\mathbf{z} = (x_0, y_0)$, $F_0(\mathbf{x}, t) = (y, -x)$, $F_1(\mathbf{x}, t) = (0, -\nabla_x W(t, x))$ and $F_2(\mathbf{x}, t, \varepsilon) = (0, 0)$. The fundamental matrix solution $M_{\mathbf{z}}(t)$ is independent of $\mathbf{z}$ and we shall denote it by $M(t)$. An easy computation shows that

$$
M(t) = \begin{pmatrix}
\cos t & 0 & \ldots & 0 & \sin t & 0 & \ldots & 0 \\
0 & \cos t & \ldots & 0 & 0 & \sin t & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \cos t & 0 & 0 & \ldots & \sin t \\
-\sin t & 0 & \ldots & 0 & \cos t & 0 & \ldots & 0 \\
0 & -\sin t & \ldots & 0 & 0 & \cos t & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & -\sin t & 0 & 0 & \ldots & \cos t
\end{pmatrix},
$$

and

$$
M^{-1}(t) = \begin{pmatrix}
\cos t & 0 & \ldots & 0 & -\sin t & 0 & \ldots & 0 \\
0 & \cos t & \ldots & 0 & 0 & -\sin t & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \cos t & 0 & 0 & \ldots & -\sin t \\
\sin t & 0 & \ldots & 0 & \cos t & 0 & \ldots & 0 \\
0 & \sin t & \ldots & 0 & 0 & \cos t & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \sin t & 0 & 0 & \ldots & \cos t
\end{pmatrix}.
$$

According to Theorem 4 we study the zeros $\alpha = (x_0, y_0)$ of the 2n components of the function $\mathcal{F}(\alpha)$ given in (7). More precisely we have $\mathcal{F}(\alpha) = (\mathcal{F}_1(\alpha), \ldots, \mathcal{F}_{2n}(\alpha))$, such that

$$\mathcal{F}_k(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \sin t\, \frac{\partial W}{\partial x_k}(t, x_{10}\cos t + y_{10}\sin t, ..., x_{n0}\cos t + y_{n0}\sin t)dt,$$
$$\mathcal{F}_{2k}(\alpha) = -\frac{1}{2\pi}\int_0^{2\pi} \cos t\, \frac{\partial W}{\partial x_k}(t, x_{10}\cos t + y_{10}\sin t, ..., x_{n0}\cos t + y_{n0}\sin t)dt,$$

for $k = 1, \ldots, n$. Now the rest of the proof of Theorem 1 follows directly from the statement of Theorem 4.

4. Proof of Corollary 2

We must apply Theorem 1 with $W(t, x) = (ax + bx^3)\sin t$. After computations the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ of Theorem 1 are

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{8}(4a + 3b(x_0^2 + 3y_0^2)),$$
$$\mathcal{F}_2(x_0, y_0) = -\frac{3}{4}bx_0y_0.$$

If $ab < 0$ then system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ has four solutions (x_0^*, y_0^*) given by $\left(\pm 2\sqrt{-\dfrac{a}{3b}}, 0\right)$ and $\left(0, \pm\dfrac{2}{3}\sqrt{-\dfrac{a}{b}}\right)$. Since the Jacobian

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\bigg|_{(x_0,y_0)=(x_0^*,y_0^*)}\right)$$

for the first two solutions (respectively for the last two solutions) is $\dfrac{4a}{3b}$ (respectively $-\dfrac{3ab}{4}$), we obtain using Theorem 1 only the two periodic solutions given in the statement of the corollary,because two of them correspond to the same periodic solution with different initial conditions.

5. Proof of Corollary 3

We must apply Theorem 1 with $W(t, x) = (ax_1 + bx_2 + cx_1^3 + dx_2^3)\sin t$. After computations the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ of Theorem 1 are

$$\mathcal{F}_1(x_{10}, x_{20}, y_{10}, y_{20}) = \frac{1}{8}(4a + 3c(x_{10}^2 + 3y_{10}^2)),$$
$$\mathcal{F}_2(x_{10}, x_{20}, y_{10}, y_{20}) = \frac{1}{8}(4b + 3d(x_{20}^2 + 3y_{20}^2)),$$
$$\mathcal{F}_2(x_{10}, x_{20}, y_{10}, y_{20}) = -\frac{3}{4}cx_{10}y_{10},$$
$$\mathcal{F}_2(x_{10}, x_{20}, y_{10}, y_{20}) = -\frac{3}{4}dx_{20}y_{20}.$$

If $ac < 0$ and $bd < 0$, then system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$ has 16 solutions $(x_{10}^*, x_{20}^*, y_{10}^*, y_{20}^*))$ given by

$$\left(\pm 2\sqrt{-\frac{a}{3c}}, \pm\frac{2}{3}\sqrt{-\frac{b}{d}}, 0, 0\right),$$
$$\left(\pm 2\sqrt{-\frac{a}{3c}}, 0, 0, \pm\frac{2}{3}\sqrt{-\frac{b}{d}}\right),$$
$$\left(0, \pm 2\sqrt{\frac{-b}{3d}}, \pm\frac{2}{3}\sqrt{\frac{-a}{c}}, 0\right),$$
$$\left(0, 0, \pm\frac{2}{3}\sqrt{-\frac{a}{c}}, \pm\frac{2}{3}\sqrt{-\frac{b}{d}}\right).$$

Since the Jacobian

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(x_{10}, x_{20}, y_{10}, y_{20})}\right|_{(x_{10},x_{20},y_{10},y_{20})=(x_{10}^*,x_{20}^*,y_{10}^*,y_{20}^*)}\right)$$

for these four set of solutions is respectively $\frac{9}{16}abcd$, $-\frac{9}{16}abcd$, -$\frac{9}{16}abcd$ and $\frac{9}{16}abcd$ we obtain using Theorem 1 the four periodic solutions given in the statement of the corollary, because every set of solutions provides different initial conditions of the same periodic orbit.

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