

PHASE PORTRAITS OF THE RICCATI QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE¹, BRUNO D. LOPES² AND PAULO R. DA SILVA³

ABSTRACT. In this paper we characterize the phase portrait of the Riccati quadratic polynomial differential systems

$$\dot{x} = \alpha_2(x), \quad \dot{y} = ky^2 + \beta_1(x)y + \gamma_2(x),$$

with $(x, y) \in \mathbb{R}^2$, $\gamma_2(x)$ non-zero (otherwise the system is a Bernoulli differential system), $k \neq 0$ (otherwise the system is a Lienard differential system), $\beta_1(x)$ a polynomial of degree at most 1, $\alpha_2(x)$ and $\gamma_2(x)$ polynomials of degree at most 2, and the maximum of the degrees of $\alpha_2(x)$ and $ky^2 + \beta_1(x)y + \gamma_2(x)$ is 2. We give the complete description of their phase portraits in the Poincaré disk (i.e. in the compactification of \mathbb{R}^2 adding the circle \mathbb{S}^1 of the infinity) modulo topological equivalence.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Numerous problems of applied mathematics are modeled by quadratic polynomial differential systems, see for instance [9]. Excluding linear systems, such systems are the ones with the lowest degree of complexity, and the large bibliography on the subject proves its relevance. We refer for example to the books of Ye Yanqian et al. [12], Reyn [10], and Artes, Llibre, Schlomiuk, Vulpe [1], and the surveys of Coppel [3], and Chicone and Jinghuang [2] are excellent introductory readings to the quadratic polynomial differential systems.

2010 *Mathematics Subject Classification.* 37G15, 37D45.

Key words and phrases. Riccati system, Poincaré compactification, dynamics at infinity.

* The first author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is supported by PNPd/CAPES-IMECC/UNICAMP. The third author was partially supported by CAPES, CNPq and FAPESP. All the authors are supported by FP7-PEOPLE-2012-IRSES-316338.

In this paper we characterize the phase portraits of the Riccati quadratic differential systems

$$(1) \quad \dot{x} = \alpha_2(x), \quad \dot{y} = ky^2 + \beta_1(x)y + \gamma_2(x),$$

with $(x, y) \in \mathbb{R}^2$, $\gamma_2(x)$ non-zero (otherwise the system is a Bernoulli differential system), $k \neq 0$ (otherwise the system is a Lienard differential system), $\beta_1(x)$ a polynomial of degree at most 1, $\alpha_2(x)$ and $\gamma_2(x)$ polynomials of degree at most 2, and the maximum of the degrees of $\alpha_2(x)$ and $ky^2 + \beta_1(x)y + \gamma_2(x)$ is 2. In (1) the dot denotes derivative with respect to the time.

Proposition 1. *A Riccati quadratic differential system (1) is topologically equivalent to one of the following systems:*

$$\begin{aligned} (i) \quad & \dot{x} = x(x+1), & \dot{y} &= y^2 + (ax+b)y + cx^2 + dx + e; \\ (ii) \quad & \dot{x} = x^2, & \dot{y} &= y^2 + (ax+b)y + cx^2 + dx + e; \\ (iii) \quad & \dot{x} = x, & \dot{y} &= y^2 + (ax+b)y + cx^2 + dx + e; \\ (iv) \quad & \dot{x} = 1, & \dot{y} &= y^2 + (ax+b)y + cx^2 + dx + e; \\ (v) \quad & \dot{x} = x^2 + 1, & \dot{y} &= y^2 + (ax+b)y + cx^2 + dx + e. \end{aligned}$$

with $c^2 + d^2 + e^2 \neq 0$ in all these systems.

We note that the Riccati systems have no periodic orbits. In fact, the equilibrium points of systems (i), (ii) and (iii) are on invariant straight lines and systems (iv) and (v) do not have equilibrium points, and consequently they do not have limit cycles, because it is well known that a periodic orbit in the plane must surround at least one equilibrium point.

The objective of this work is to classify the phase portraits of the Riccati quadratic polynomial differential systems (1) in the Poincaré disk modulo topological equivalence. As any polynomial differential system, system (1) can be extended to an analytic system on a closed disk of radius one, whose interior is diffeomorphic to \mathbb{R}^2 and its boundary, the circle \mathbb{S}^1 , plays the role of the infinity. This closed disk is denoted by \mathbb{D}^2 and called the *Poincaré disk*, because the technique for doing such an extension is precisely the *Poincaré compactification* for a polynomial differential system in \mathbb{R}^2 , which is described in details in chapter 5 of [4]. In this paper we shall use the notation of that chapter. By using this compactification technique the dynamics of system (1) in a neighborhood of the infinity can be studied and we have the following result.

Theorem 2. *The phase portraits of the Riccati system (1) in the Poincaré disk are topologically equivalent to one of the 74 phase portraits presented in Figures 1, 2 and 3. The phase portraits of the systems of Proposition 1 are provided in Tables 1, 2, 3, 4 and 5 where*

$$(2) \quad \begin{aligned} \Delta_{F_1} &= b^2 - 4e, & \Delta_{F_2} &= (b - a)^2 - 4(c - d + e), \\ \Delta_{I_1} &= (a - 1)^2 - 4c, & \Delta_{I_2} &= a^2 - 4c. \end{aligned}$$

Three papers on generalizations of Riccati differential equations can be found in [5, 8, 11].

This paper is organized as follows. In section 2 we prove Proposition 1, and study the finite equilibria. In section 3 we study the infinite equilibria. Finally in section 4 we prove Theorem 2.

Phase Portraits of systems (i)	conditions
$P1, P2, P3, P4, P5$	$\Delta_{I_1} > 0, \Delta_{F_1} > 0, \Delta_{F_2} > 0$
$P6, P7, P8, P9$	$\Delta_{I_1} > 0, \Delta_{F_1} > 0, \Delta_{F_2} = 0$
$P10$	$\Delta_{I_1} > 0, \Delta_{F_1} > 0, \Delta_{F_2} < 0$
$P11, P12, P13, P14$	$\Delta_{I_1} > 0, \Delta_{F_1} = 0, \Delta_{F_2} > 0$
$P15, P16$	$\Delta_{I_1} > 0, \Delta_{F_1} = 0, \Delta_{F_2} = 0$
$P17$	$\Delta_{I_1} > 0, \Delta_{F_1} = 0, \Delta_{F_2} < 0$
$P18$	$\Delta_{I_1} > 0, \Delta_{F_1} < 0, \Delta_{F_2} > 0$
$P19$	$\Delta_{I_1} > 0, \Delta_{F_1} < 0, \Delta_{F_2} = 0$
$P20$	$\Delta_{I_1} > 0, \Delta_{F_1} < 0, \Delta_{F_2} < 0$
$P21, P22, P23$	$\Delta_{I_1} = 0, \Delta_{F_1} > 0, \Delta_{F_2} > 0$
$P24, P25$	$\Delta_{I_1} = 0, \Delta_{F_1} > 0, \Delta_{F_2} = 0$
$P26$	$\Delta_{I_1} = 0, \Delta_{F_1} > 0, \Delta_{F_2} < 0$
$P27$	$\Delta_{I_1} = 0, \Delta_{F_1} = 0, \Delta_{F_2} > 0$
$P28$	$\Delta_{I_1} = 0, \Delta_{F_1} = 0, \Delta_{F_2} = 0$
$P29$	$\Delta_{I_1} = 0, \Delta_{F_1} = 0, \Delta_{F_2} < 0$
$P30$	$\Delta_{I_1} = 0, \Delta_{F_1} < 0, \Delta_{F_2} > 0$
$P31$	$\Delta_{I_1} = 0, \Delta_{F_1} < 0, \Delta_{F_2} = 0$
$P32$	$\Delta_{I_1} = 0, \Delta_{F_1} < 0, \Delta_{F_2} < 0$
$P33$	$\Delta_{I_1} < 0, \Delta_{F_1} > 0, \Delta_{F_2} > 0$
$P34$	$\Delta_{I_1} < 0, \Delta_{F_1} > 0, \Delta_{F_2} = 0$
$P35$	$\Delta_{I_1} < 0, \Delta_{F_1} > 0, \Delta_{F_2} < 0$
$P36$	$\Delta_{I_1} < 0, \Delta_{F_1} = 0, \Delta_{F_2} > 0$
$P37$	$\Delta_{I_1} < 0, \Delta_{F_1} = 0, \Delta_{F_2} = 0$
$P38$	$\Delta_{I_1} < 0, \Delta_{F_2} = 0, \Delta_{I_1} < 0$
$P39$	$\Delta_{I_1} < 0, \Delta_{F_1} < 0, \Delta_{F_2} > 0$
$P40$	$\Delta_{I_1} < 0, \Delta_{F_1} < 0, \Delta_{F_2} = 0$
$P41$	$\Delta_{I_1} < 0, \Delta_{F_1} < 0, \Delta_{F_2} < 0$

TABLE 1. The phase portraits of systems (i).

Phase Portraits of systems (ii)	conditions
$P42, P43, P44$	$\Delta_{I_1} > 0, \Delta_{F_1} > 0$
$P45, P46, P47$	$\Delta_{I_1} > 0, \Delta_{F_1} = 0$
$P48$	$\Delta_{I_1} > 0, \Delta_{F_1} < 0$
$P49, P50, P51$	$\Delta_{I_1} = 0, \Delta_{F_1} > 0$
$P52, P53, P54$	$\Delta_{I_1} = 0, \Delta_{F_1} = 0$
$P55$	$\Delta_{I_1} = 0, \Delta_{F_1} < 0$
$P56$	$\Delta_{I_1} < 0, \Delta_{F_1} > 0$
$P57, P58$	$\Delta_{I_1} < 0, \Delta_{F_1} = 0$
$P41$	$\Delta_{I_1} < 0, \Delta_{F_1} < 0$

TABLE 2. The phase portraits of systems (ii).

Phase Portraits of systems (iii)	conditions
$P59, P60, P61$	$\Delta_{I_2} > 0, \Delta_{F_1} > 0$
$P62, P63, P64$	$\Delta_{I_2} > 0, \Delta_{F_1} = 0$
$P65$	$\Delta_{I_2} > 0, \Delta_{F_1} < 0$
$P66, P67$	$\Delta_{I_2} = 0, \Delta_{F_1} > 0$
$P68, P69$	$\Delta_{I_2} = 0, \Delta_{F_1} = 0$
$P32$	$\Delta_{I_2} = 0, \Delta_{F_1} < 0$
$P35$	$\Delta_{I_2} < 0, \Delta_{F_1} > 0$
$P38$	$\Delta_{I_2} < 0, \Delta_{F_1} = 0$
$P41$	$\Delta_{I_2} < 0, \Delta_{F_1} < 0$

TABLE 3. The phase portraits of systems (iii).

Phase Portrait of systems (iv)	conditions
$P70, P71$	$\Delta_{I_2} > 0$
$P72, P73, P74$	$\Delta_{I_2} = 0$
$P41$	$\Delta_{I_2} < 0$

TABLE 4. The phase portraits of systems (iv).

Phase Portraits of systems (v)	conditions
$P70, P71$	$\Delta_{I_1} > 0$
$P72, P73, P74$	$\Delta_{I_1} = 0$
$P41$	$\Delta_{I_1} < 0$

TABLE 5. The phase portraits of systems (v).

2. FINITE EQUILIBRIUM POINTS

We start this section with the proof of Proposition 1.

Proof of Proposition 1. Since $\alpha_2(x)$ is a polynomial of degree at most 2, we have, using a rescaling of the time if necessary,

$$\begin{aligned}\dot{x} &= (x-r)(x-s) \quad \text{with } r \neq s, \\ \dot{x} &= (x-r)^2, \\ \dot{x} &= (x-r), \\ \dot{x} &= 1, \\ \dot{x} &= (x-r)^2 + s^2 \quad \text{with } s \neq 0.\end{aligned}$$

If $\dot{x} = (x-r)(x-s)$, $r \neq s$, considering the change of coordinates

$$x_1 = \frac{x-r}{r-s}, \quad y_1 = cy \quad \text{and} \quad T = (r-s)t,$$

we get a system (i). If $\dot{x} = (x-r)^n$, $n = 1, 2$, considering the change of coordinates $x_1 = x-r$, $y_1 = cy$, we get systems (ii) for $n = 2$ and systems (iii) for $n = 1$. If $\dot{x} = 1$, considering the change of coordinates $x_1 = x$ and $y_1 = cy$, we get a systems (iv). If $\dot{x} = (x-r)^2 + s^2$, considering the change of coordinates $x_1 = (x-r)/s$, $y_1 = cy$ and $T = st$, we get a system (v). ■

Proposition 3. *The finite equilibrium points of the Riccati quadratic polynomial differential system (1) are described below.*

- (a) *Systems (i) have at most 4 equilibria which can be either a saddle, or a stable or an unstable node, or a saddle-node.*
- (b) *Systems (ii) have at most 2 equilibria which can be either a saddle-node either semi-hyperbolic or nilpotent.*
- (c) *Systems (iii) have at most 2 equilibria which can be either a saddle or an unstable node, or a saddle-node.*
- (d) *Systems (iv) and (v) have no finite equilibria.*

Proof. Systems (i): Consider the Riccati quadratic polynomial differential systems

$$(3) \quad \dot{x} = x(x+1), \quad \dot{y} = y^2 + (ax+b)y + cx^2 + dx + e.$$

The equilibrium points of system (3) are

$$(x_1, y_1) = \left(0, -\frac{b + \sqrt{\Delta_{F_1}}}{2}\right), \quad (x_2, y_2) = \left(0, -\frac{b - \sqrt{\Delta_{F_1}}}{2}\right),$$

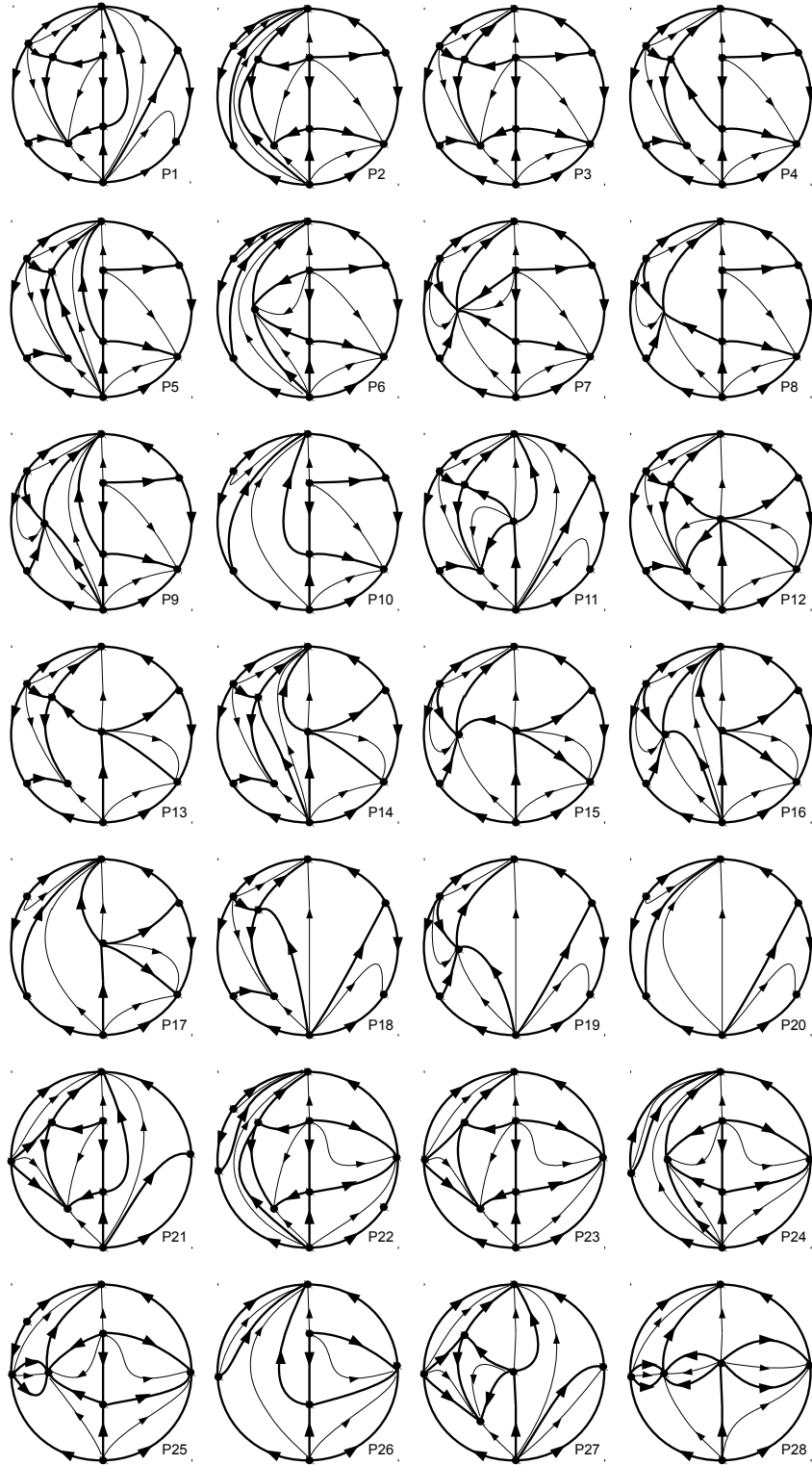


FIGURE 1. Phase portraits of systems (1) the Poincaré disk.

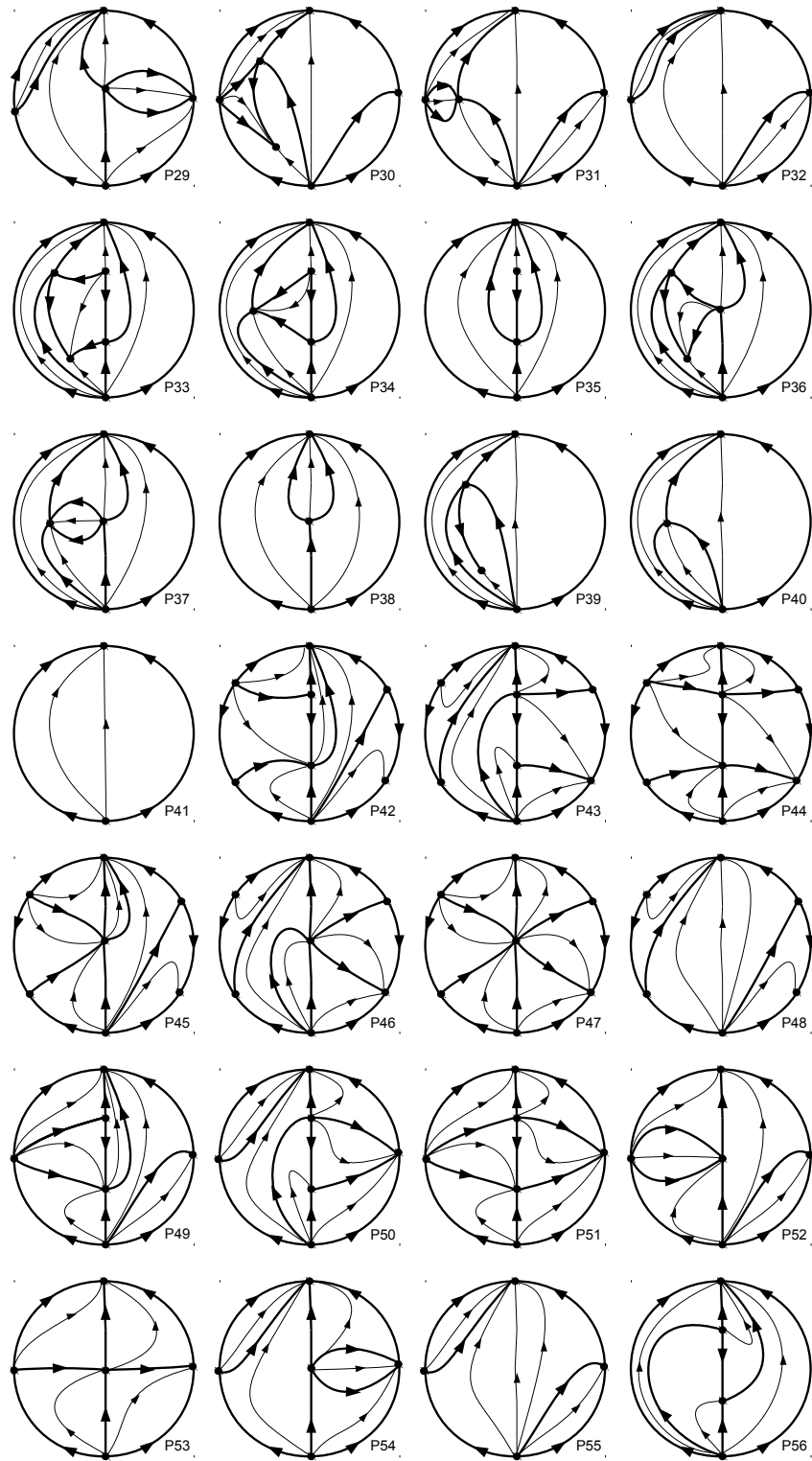


FIGURE 2. Continuation of phase portraits of systems (1) in the Poincaré disk.

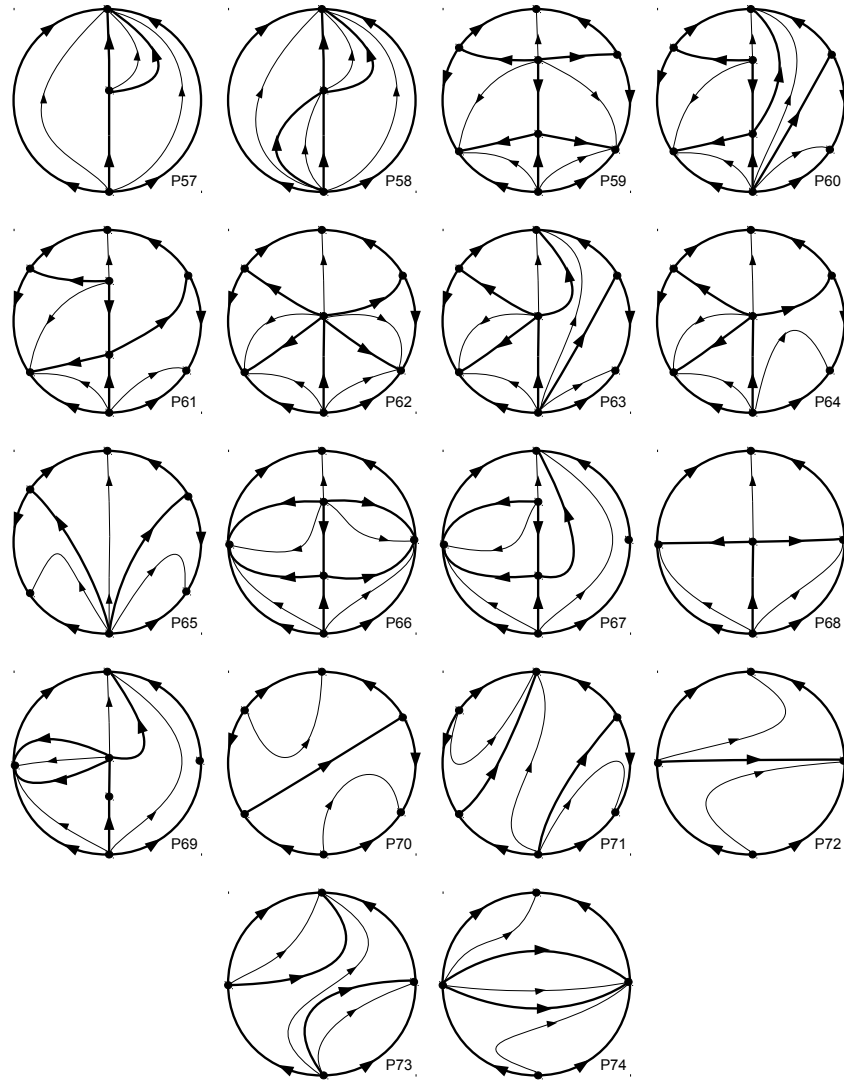


FIGURE 3. Continuation of the phase portraits of systems (1) in the Poincaré disk.

$$(x_3, y_3) = \left(-1, -\frac{-a + b + \sqrt{\Delta_{F_2}}}{2} \right),$$

$$(x_4, y_4) = \left(-1, -\frac{-a + b - \sqrt{\Delta_{F_2}}}{2} \right),$$

where Δ_{F_1} and Δ_{F_2} are given by (2).

The eigenvalues of the Jacobian matrix of system (3) evaluated at (x_i, y_i) are $(1, (-1)^i \sqrt{\Delta_{F_1}})$ for $i = 1, 2$, and $(-1, (-1)^i \sqrt{\Delta_{F_2}})$ for $i = 3, 4$, when they exist.

From the classification of the hyperbolic and semi-hyperbolic equilibrium points (see for instance Theorems 2.18 and 2.19 of [4]), we have the following (when the equilibrium point is not hyperbolic we mention this fact explicitly).

- (i) If $\Delta_{F_1} > 0$ and $\Delta_{F_2} > 0$, system (3) has two saddles, a stable node and an unstable node.
- (ii) If $\Delta_{F_1} > 0$ and $\Delta_{F_2} = 0$, system (3) has a saddle, a stable node and a semi-hyperbolic saddle-node.
- (iii) If $\Delta_{F_1} > 0$ and $\Delta_{F_2} < 0$, system (3) has a saddle and a stable node.
- (iv) If $\Delta_{F_1} = 0$ and $\Delta_{F_2} > 0$, system (3) has a saddle, an unstable node and a semi-hyperbolic saddle-node.
- (v) If $\Delta_{F_1} = 0$ and $\Delta_{F_2} = 0$, system (3) has two semi-hyperbolic saddle-nodes.
- (vi) If $\Delta_{F_1} = 0$ and $\Delta_{F_2} < 0$ system (3) has one semi-hyperbolic saddle-node.
- (vii) If $\Delta_{F_1} < 0$ and $\Delta_{F_2} > 0$, system (3) has a saddle and an unstable node.
- (viii) If $\Delta_{F_1} < 0$ and $\Delta_{F_2} = 0$, system (3) has one semi-hyperbolic saddle-node.
- (ix) If $\Delta_{F_1} < 0$ and $\Delta_{F_2} < 0$, system (3) has not equilibria.

Systems (ii): Consider the Riccati quadratic polynomial differential systems

$$(4) \quad \dot{x} = x^2, \quad \dot{y} = y^2 + (ax + b)y + cx^2 + dx + e.$$

We have that the finite equilibrium points of system (4) are

$$(5) \quad (x_1, y_1) = \left(0, -\frac{b + \sqrt{\Delta_{F_1}}}{2}\right), (x_2, y_2) = \left(0, -\frac{b - \sqrt{\Delta_{F_1}}}{2}\right),$$

where Δ_{F_1} is given by (2).

The eigenvalues of the Jacobian matrix of system (4) evaluated at (x_i, y_i) for all $i = 1, 2$ are 0 and $(-1)^i \sqrt{\Delta_{F_1}}$. Then we have

- (i) If $\Delta_{F_1} > 0$ systems (4) have two semi-hyperbolic saddle-nodes.
- (ii) If $\Delta_{F_1} = 0$ then systems (4) have one nilpotent saddle-node equilibrium point.
- (iii) If $\Delta_{F_1} < 0$ system (4) has not equilibrium points.

Systems (iii): Consider the Riccati quadratic polynomial differential systems

$$(6) \quad \dot{x} = x, \quad \dot{y} = y^2 + (ax + b)y + cx^2 + dx + e.$$

The equilibrium points of systems (6) are given by (5). Then system (6) has 0, 1 or 2 equilibrium points if Δ_{F_1} is negative, zero or positive, respectively. The eigenvalues of the Jacobian matrix of system (6) evaluated at (x_i, y_i) for $i = 1, 2$ are 1 and $(-1)^i \sqrt{\Delta_{F_1}}$. Thus we have:

- (i) If $\Delta_{F_1} > 0$ systems (6) have a saddle and an unstable node.
- (ii) If $\Delta_{F_1} = 0$ systems (6) have a semi-hyperbolic saddle-node.
- (iii) If $\Delta_{F_1} < 0$ system (6) has no equilibria.

Systems (iv) and (v): These systems are chordal quadratic systems, or quadratic system without finite singularities.

□

3. INFINITE EQUILIBRIUM POINTS

For a complete description of the Poincaré compactification method we refer to chapter 5 of [4]. In what follows we remember some formulas.

Consider a polynomial differential system in \mathbb{R}^2 with degree 2.

$$(7) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

or equivalently its associated polynomial vector field $X = (P, Q)$. As we said before, any polynomial differential system can be extended to an analytic differential system on a closed disk of radius one centered at their origin of coordinates, whose interior is diffeomorphic to \mathbb{R}^2 and its boundary, the circle \mathbb{S}^1 , plays the role of the infinity.

We consider 4 open charts covering the disk \mathbb{D} :

$$\begin{aligned} \phi_1 : \mathbb{R}^2 &\longrightarrow U_1, & \phi_1(x, y) &= (1/v, u/v), \\ \phi_2 : \mathbb{R}^2 &\longrightarrow U_2, & \phi_2(x, y) &= (u/v, 1/v) \end{aligned}$$

and

$$\psi_k : \mathbb{R}^2 \longrightarrow V_k, \quad \psi_k(x, y) = -\phi_k(x, y), \quad k = 1, 2$$

with

$$\begin{aligned} U_1 &= \{(u, v) \in \mathbb{D} : u^2 + v^2 \leq 1 \text{ and } u > 0\}, \\ U_2 &= \{(u, v) \in \mathbb{D} : u^2 + v^2 \leq 1 \text{ and } v > 0\}, \\ V_1 &= \{(u, v) \in \mathbb{D} : u^2 + v^2 \leq 1 \text{ and } u < 0\}, \\ V_2 &= \{(u, v) \in \mathbb{D} : u^2 + v^2 \leq 1 \text{ and } v < 0\}. \end{aligned}$$

The Poincaré compactification is denoted by $p(X)$. The expression of $p(X)$ in the chart U_1 is

$$(8) \quad \dot{u} = v^2(-uP + Q), \quad \dot{v} = -v^3P,$$

where P and Q are evaluated at $(1/v, u/v)$.

The expression of $p(X)$ in the chart U_2 is

$$(9) \quad \dot{u} = v^2(P - uQ), \quad \dot{v} = -v^3Q,$$

where P and Q are evaluated at $(u/v, 1/v)$. Moreover in all these local charts the points (u, v) of the infinity have its coordinate $v = 0$.

The expression for the extend differential system in the local chart V_i , $i = 1, 2$ is the same as in U_i multiplied by -1 .

Proposition 4. *On the circle of the infinity, for any systems of Proposition 1 the origin of U_2 , denoted by n , is an attracting node and the origin of V_2 , denoted by s , is a repelling node of the Riccati quadratic polynomial differential system (1). Moreover, the remaining infinite equilibrium points are described below.*

- (a) *For systems (i), (ii) and (v) three situations can occur.*
 - 4 equilibrium points being 2 saddles, 1 attracting node and 1 repelling node;
 - 2 equilibrium points being 2 saddle-nodes;
 - The only equilibria are n and s .
- (b) *For systems (iii) three situations can occur.*
 - 4 equilibrium points being 4 nilpotent saddle-nodes;
 - 2 equilibrium points being 2 semi-hyperbolic saddle-nodes;
 - The only equilibria are n and s .
- (c) *For systems (iv) three situations can occur.*
 - 4 equilibrium points being 2 semi-hyperbolic saddles, 1 semi-hyperbolic attracting node and 1 semi-hyperbolic repelling node;
 - 2 equilibrium points being 2 semi-hyperbolic saddle-nodes;
 - The only equilibria are n and s .

Proof. Systems (i): First we analyze the phase portrait in the local chart U_1 . The expression of the system in this chart is

$$(10) \quad \dot{u} = v((b-1)u + d) + ev^2 + p(u), \quad \dot{v} = -(v + v^2),$$

where $p(u) = u^2 + (a-1)u + c$.

Note that $(u_0, 0)$ is an infinite equilibrium point of (10) if, and only if, $p(u_0) = 0$. System (10) has 0, 1 or 2 two infinite equilibrium points:

$$(11) \quad S_i = \left(\frac{1 - a + (-1)^i \sqrt{\Delta_{I_1}}}{2}, 0 \right),$$

for $i = 1, 2$, where Δ_{I_1} is given (2).

The eigenvalues of the Jacobian matrix of system (10) are -1 and $(-1)^i \sqrt{\Delta_{I_1}}$. Thus we have:

- (i) If $\Delta_{I_1} > 0$ systems (10) have a saddle and a stable node.
- (ii) If $\Delta_{I_1} = 0$ systems (10) have a semi-hyperbolic saddle-node.
- (iii) If $\Delta_{I_1} < 0$ systems (10) have no equilibrium points.

Now we analyze the phase portrait in the local chart U_2 , we need to the study the origin of U_2 , the others infinite singularity ahead, have been studied in the local chart U_1 . The expression of the system in this chart is

$$(12) \quad \begin{aligned} \dot{u} &= v(vue - u(du + b - 1)) + q(u), \\ \dot{v} &= -v(1 + au + cu^2) - v^2(b + du) - ev^3, \end{aligned}$$

where $q(u) = -u(1 + (a - 1)u + cu^2)$.

The eigenvalues of the Jacobian matrix at the origin of U_2 of system (12) are -1 and -1 . Therefore system (12) has a stable node at $(0, 0)$.

Thus, the equilibrium points of system (1), system (i), on the circle \mathbb{S}^1 are classified as follows.

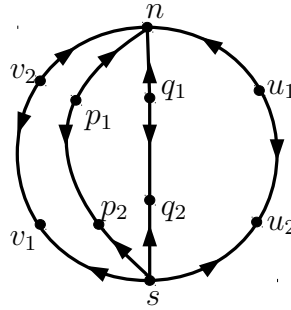


FIGURE 4. Finite and infinite equilibrium of system (1), systems (i).

- (a) If $\Delta_{I_1} > 0$ system (1) has 6 equilibrium points.
 - 2 saddles: u_1 and v_1 diametrically opposed to u_1 ;
 - 2 attracting nodes: u_2 and n the origin of U_2);
 - 2 repelling nodes: v_2 diametrically opposed to u_2 and s the origin of V_2 diametrically opposed to n .
- (b) If $\Delta_{I_1} = 0$ system (1) has 4 equilibrium points.

- 2 saddle-node: u_{12} and v_{12} (diametrically opposed to u_{12} ;
 - 1 attracting node: n ;
 - 1 repelling node: s .
- (c) If $\Delta_{I_1} < 0$ system (1) has 2 equilibrium points.
- 1 attracting node: n ;
 - 1 repelling node: s .

Systems (ii): The expression of the system in the local chart U_1 is

$$(13) \quad \dot{u} = v((d + bu) + ev) + p(u), \quad \dot{v} = -v,$$

where $p(u) = u^2 + (a - 1)u + c$, and in the local chart U_2 is

$$(14) \quad \begin{aligned} \dot{u} &= -v(u(d + bu) + vue) + q(u), \\ \dot{v} &= -v(c + du + cu^2) - v^2(b + du) - ev^3, \end{aligned}$$

where $q(u) = -u(1 + (a - 1)u + cu^2)$. The equilibrium point at infinity and their classification are exactly the same of system (i).

Systems (iii): The expression of this system in the local chart U_1 is

$$(15) \quad \begin{aligned} \dot{u} &= v(u(b - 1) + d + ev) + p(u), \\ \dot{v} &= -v^2, \end{aligned}$$

where $p(u) = u^2 + au + c$. System (15) has 0, 1 or 2 equilibrium points.

$$S_i = \left(\frac{-a + (-1)^i \sqrt{\Delta_{I_2}}}{2}, 0 \right)$$

for $i = 1, 2$, where Δ_{I_2} is given by (2). The eigenvalues of the Jacobian matrix of system (15) are 0 and $(-1)^i \sqrt{\Delta_{I_2}}$. Thus we have:

- (i) If $\Delta_{I_2} > 0$ systems (15) have two nilpotent saddle-nodes.
- (ii) If $\Delta_{I_2} = 0$ systems (15) have a saddle-node with both eigenvalues being zero.
- (iii) If $\Delta_{I_2} < 0$ systems (15) have no equilibrium points.

The expression of the system in the local chart U_2 is

$$(16) \quad \begin{aligned} \dot{u} &= v(-v(eu) - u(-1 + b + du)) + q(u), \\ \dot{v} &= -v(1 + au + cu^2) - v^2(b + du) - ev^3, \end{aligned}$$

where $q(u) = -u(1 + (a - 1)u + cu^2)$. The equilibrium points at infinity and their classification are exactly the same of systems (i).

In summary, the equilibrium points of system (1), system (iii), on the circle \mathbb{S}^1 are classified as follows.

- (a) If $\Delta_{I_2} > 0$ system (1) has 6 equilibrium points.
 - 4 saddle-nodes: u_1, v_1 diametrically opposed to u_1, u_2 and v_2 diametrically opposed to u_2 ;

- 1 attracting node: n ;
- 1 repelling node: s diametrically opposed to n .
- (b) If $\Delta_{I_2} = 0$ system (1) has 4 equilibrium points.
 - 2 saddle-node: u_{12} and v_{12} diametrically opposed to u_{12} ;
 - 1 attracting node: n ;
 - 1 repelling node: s .
- (c) If $\Delta_{I_2} < 0$ system (1) has 2 equilibrium points.
 - 1 attracting node: n ;
 - 1 repelling node: s .

Systems (iv): The expression of the system in the local chart U_1 is

$$(17) \quad \dot{u} = v(d + bu + (e - u)v) + p(u), \quad \dot{v} = -v^3,$$

where $p(u) = u^2 + au + c$. System (17) has 0, 1 or 2 equilibrium points.

$$S_i = \left(\frac{-a + (-1)^i \sqrt{\Delta_{I_2}}}{2}, 0 \right)$$

for $i = 1, 2$, where Δ_{I_2} is given by (2). The eigenvalues of the Jacobian matrix of system (17) are 0 and $(-1)^i \sqrt{\Delta_{I_2}}$. Thus we have:

- (i) If $\Delta_{I_2} > 0$ systems (17) have a semi-hyperbolic stable node and a semi-hyperbolic saddle.
- (ii) If $\Delta_{I_2} = 0$ systems (17) have a semi-hyperbolic saddle-node.
- (iii) If $\Delta_{I_2} < 0$ systems (17) have no equilibrium points.

The expression of the system in the local chart U_2 is

$$(18) \quad \begin{aligned} \dot{u} &= v(v(1 - eu) - u(b + du)) + q(u), \\ \dot{v} &= -v(1 + au + cu^2) - v^2(b + du) - ev^3, \end{aligned}$$

where $q(u) = -u(1 + (a - 1)u + cu^2)$. The equilibrium points at infinity and their classification are exactly the same of systems (i).

In short, the equilibrium points of system (1), systems (iv), on the circle \mathbb{S}^1 are classified as follows.

- (a) If $\Delta_{I_2} > 0$ system (1) has 6 equilibrium points.
 - 2 semi-hyperbolic saddles: u_1 and v_1 diametrically opposed to u_1 ;
 - 1 semi-hyperbolic attracting node: u_2 ;
 - 1 attracting node: n ;
 - 1 semi-hyperbolic repelling node: v_2 diametrically opposed to u_2 ;
 - 1 repelling node: s diametrically opposed to n .
- (b) If $\Delta_{I_2} = 0$ system (1) has 4 equilibrium points.
 - 2 semi-hyperbolic saddle-nodes: u_{12} and v_{12} diametrically opposed to u_{12} ;

- 1 attracting node: n ;
- 1 repelling node: s diametrically opposed to n .
- (c) If $\Delta_{I_2} < 0$ system (1) has 2 equilibrium points.
 - 1 attracting node: n ;
 - 1 repelling node: s .

Systems (v): The expression of the system in the local chart U_1 is

$$(19) \quad \begin{aligned} \dot{u} &= v((d + bu) + v(e - u)) + p(u) \\ \dot{v} &= -(v + v^3). \end{aligned}$$

where $p(u) = u^2 + (a - 1)u + c$. The equilibrium points at infinity and their classification are exactly the same than of systems (i).

The expression of the system in the local chart U_2 is

$$(20) \quad \begin{aligned} \dot{u} &= v(v(1 - eu) - u(b + du)) + q(u), \\ \dot{v} &= -v(1 + au + cu^2) - v^2(b + du) - ev^3, \end{aligned}$$

where $q(u) = -u(1 + (a - 1)u + cu^2 - u)$. The origin and its classification is exactly the same than of systems (i). □

4. PROOF OF THEOREM 2

We start this section considering the Tables 1, ..., 5, one for each of the possible Riccati systems. In each table, we list the conditions about the parameters and indicate the possible phase portraits.

4.1. Proof of Theorem 2. We remember the notation introduced in previous sections

$$\begin{aligned} \Delta_{F_1} &= b^2 - 4e, & \Delta_{F_2} &= (b - a)^2 - 4(c - d + e), \\ \Delta_{I_1} &= (a - 1)^2 - 4c & \text{and} & \quad \Delta_{I_2} = a^2 - 4c. \end{aligned}$$

4.1.1. Proof of Theorem 2 – System (i). We begin the proof considering the assumptions of the first row of Table 1. These systems have 4 finite equilibrium p_1, p_2, q_1, q_2 and 6 infinite equilibrium n, s, u_1, u_2, v_1, v_2 , according to sections 2 and 3, see Figure (3).

Let r_1 be the straight line joining v_1, p_1 and u_1 , and r_2 be the straight line joining v_1, q_2 and u_1 :

$$r_1 = y - u_1x - k_1 = 0, \quad r_2 = y - u_1x - k_2 = 0,$$

where

$$k_1 = \frac{1}{2}(1 - b + \sqrt{(a - 1)^2 - 4c} + \sqrt{(a - b)^2 - 4(c - d + e)})$$

and

$$k_2 = \frac{1}{2}(-b - \sqrt{b^2 - 4c}).$$

We analyze the position of q_1 with respect to r_1 and the position of p_2 with respect to r_2 . We have four possibilities.

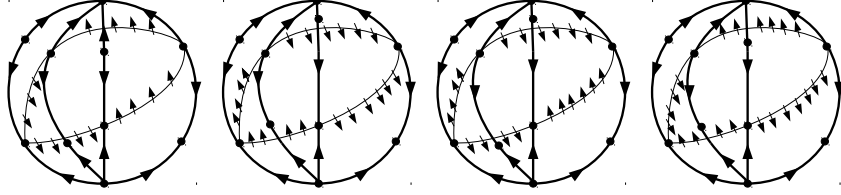


FIGURE 5. Straight lines r_1 and r_2 and the directions of the vector field $X(x, y) = (\alpha_2(x), ky^2 + \beta_1(x)y + \gamma_2(x))$.

Assume the first possibility. By Lemma 6 (see Appendix) the vector field $X(x, y) = (\alpha_2(x), ky^2 + \beta_1(x)y + \gamma_2(x))$ has only the equilibrium p_1 as a contact point with r_1 , and the equilibrium q_2 as a contact point with r_2 . Thus p_1 divides r_1 into two semi-straight lines and we have the direction of the field downward between v_2 and p_1 and upward between p_1 and u_1 . In fact this is due to the fact that the repelling node is below the line r_1 , and there is a trajectory with α -limit q_1 and ω -limit n . Similarly we concluded that q_2 divides r_2 into two semi-straight lines and we have the direction of the vector field downward between v_2 and q_2 and upward between q_2 and u_1 . Thus the only way to complete the phase portrait is shown in figure *P1*.

In the second case and in an analogous way, we conclude that the phase portrait is shown in figure *P2*. The third case does not occur, because the conditions $r_1(q_1) > 0$ and $r_2(p_2) < 0$ will never be satisfied at the same time. In the fourth case we concluded that p_1 divides r_1 into two semi-straight lines and the direction of the field is downward between v_2 and p_1 and upward between p_1 and u_1 . Moreover q_2 divides r_2 into two semi-straight lines and the direction of the field is upward between v_2 and q_2 and downward between q_2 and u_1 . There are three possibilities to complete the phase portrait. To analyze this case we consider the straight line $S : y = mx + n$ joining p_1 and q_2 . The coefficients are

$$m = \frac{\pi_2(q_2) - \pi_2(p_1)}{\pi_1(q_2) - \pi_1(p_1)} = (-a - \sqrt{\Delta_{F_1}} - \sqrt{\Delta_{F_2}})/2 \quad \text{and} \quad n = (-b - \sqrt{\Delta_{I_1}})/2$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. We analyze how the straight line S reaches the infinite. If $-a - \sqrt{\Delta_{F_1}} - \sqrt{\Delta_{F_2}} < 1 - a - \sqrt{\Delta_{I_1}}$, then u_2 is above S , and the only possibility to complete the phase portrait is shown in figure *P3*. If $-a - \sqrt{\Delta_{F_1}} - \sqrt{\Delta_{F_2}} > 1 - a - \sqrt{\Delta_{I_1}}$ then

u_2 is below S , and the phase portrait is shown in figure $P5$. Finally, if $-a - \sqrt{\Delta_{F_1}} - \sqrt{\Delta_{F_2}} = 1 - a - \sqrt{\Delta_{I_1}}$ then u_2 belong to S , the phase portrait is shown in figure $P4$.

Now we explicit the parameter values for each phase portrait.

- $P_1: (a, b, c, d, e) = (0, 0, 0, 3.75, -0.25)$.
- $P_2: (a, b, c, d, e) = (0, 0, 0, -3.75, -4)$.
- $P_3: (a, b, c, d, e) = (0, 0, -0, 75, -0.75, -0.25)$.
- $P_4: (a, b, c, d, e) = (0, 0, -2, -2, -0.25)$.
- $P_5: (a, b, c, d, e) = (0, 0, -3.75, -3.75, -0.25)$.

Assume the conditions in the second row of Table 1. systems (i) have 3 finite equilibria $p_{1,2}, q_1, q_2$ and 6 infinite equilibria n, s, u_1, u_2, v_1, v_2 . Note that $p_{1,2}$ comes from the collision of p_1 and p_2 (these equilibria exist when we assume the conditions of the first row of Table 1) when $\Delta_{F_2} \rightarrow 0$. Consequently systems (i) have at most five phase portraits which are obtained from the 5 possible phase portraits of row 1 of Table 1. Applying Lemma 7, we can see that effectively only the 4 phase portraits listed in row 2 of Table 1 occur. Next we explicit the parameter values for each phase portrait.

- $P_6: (a, b, c, d, e) = (0, 0, 0, -3, -3)$.
- $P_7: (a, b, c, d, e) = (0, 0, -1, -2, -1)$.
- $P_9: (a, b, c, d, e) = (0, 0, -29, -30, -1)$.
- P_8 : We cannot explicit a choice of (a, b, c, d, e) . However its existence follows from continuity when we pass from the phase portraits P_7 to P_9 .

The analysis of the phase portraits for the conditions listed in the other rows of Table 1 is analogous to the one that we did above. We will only give an example for each phase portrait.

- $P_{10}: (a, b, c, d, e) = (0, 0, 0, -1, -0.25)$.
- $P_{11}: (a, b, c, d, e) = (0, 0, -1, 10, 0)$.
- $P_{12}: (a, b, c, d, e) = (2, 0, -1, -1, 0)$.
- $P_{13}: (a, b, c, d, e) = (1, 0, -1, -1, 0)$.
- $P_{14}: (a, b, c, d, e) = (-2, 0, -1, -1, 0)$.
- $P_{15}: (a, b, c, d, e) = (4, 0, 2, -2, 0)$.
- $P_{16}: (a, b, c, d, e) = (1, 0, -1, -1.25, 0)$.
- $P_{17}: (a, b, c, d, e) = (0, 0, -1, -10, 0)$.
- $P_{18}: (a, b, c, d, e) = (0, 1, 0, 1, 0.75)$.
- $P_{19}: (a, b, c, d, e) = (0, 0, -0.75, 0.25, 1)$.
- $P_{20}: (a, b, c, d, e) = (1, 0, -1, -1.25, 1)$.
- $P_{21}: (a, b, c, d, e) = (1, 2, 0, 4.75, 0)$.
- $P_{22}: (a, b, c, d, e) = (2, 0, 0.25, -9.75, -10)$.
- $P_{23}: (a, b, c, d, e) = (2, 0, 0.25, -0.75, -1)$.

- P_{24} : $(a, b, c, d, e) = (0, 0, 0.25, -0.75, -1)$.
- P_{25} : $(a, b, c, d, e) = (1, 1, 0, 0.2, 0.2)$.
- P_{26} : $(a, b, c, d, e) = (0, 0, 0.25, -1.75, -1)$.
- P_{27} : $(a, b, c, d, e) = (0, 0, 0.25, 1.25, 0)$.
- P_{28} : $(a, b, c, d, e) = (1, 1, 0, 0.25, 0.25)$.
- P_{29} : $(a, b, c, d, e) = (1, 0, 0, -1, 0)$.
- P_{30} : $(a, b, c, d, e) = (0, 0, 0.25, 2.25, 1)$.
- P_{31} : $(a, b, c, d, e) = (0, 0, 0.25, 1.25, 1)$.
- P_{32} : $(a, b, c, d, e) = (0, 0, 0.25, 0.25, 1)$.
- P_{33} : $(a, b, c, d, e) = (0, 0, 1.25, 1.25, -1)$.
- P_{34} : $(a, b, c, d, e) = (1, 1, 1, 1.2, 0.2)$.
- P_{35} : $(a, b, c, d, e) = (1, 1, 2, 0, 0.2)$.
- P_{36} : $(a, b, c, d, e) = (1, 2, 1, 2, 1)$.
- P_{37} : $(a, b, c, d, e) = (2, 0, 2, 1, 0)$.
- P_{38} : $(a, b, c, d, e) = (1, 0, 1, -1, 0)$.
- P_{39} : $(a, b, c, d, e) = (1, 0, 1, 2, 1)$.
- P_{40} : $(a, b, c, d, e) = (1, 0, 1, 1.75, 1)$.
- P_{41} : $(a, b, c, d, e) = (0, 0, 1.25, 1.25, 1)$.

4.1.2. *Proof of Theorem 2 -System (ii)*. The phase portraits listed in row 1 of Table 2 are obtained from row 1 of Table 1. Note that system (ii) has only $x = 0$ as an invariant vertical line, which comes when the two straight lines $x = 0, x = -1$ of system (i) collide at $x = 0$. Thus we consider the phase portraits represented in the figures $P1, P2, P3, P4$ and $P5$, excluding what occurs in the strip $-1 \leq x \leq 0$. This reduces the possible phase portraits to $P42, P43$ and $P44$ obtained from $P1, P2$ and $P3$ respectively. Note that no new configurations can be obtained from $P4$ and $P5$ because the phase portraits are equal in the complement of the strip $-1 \leq x \leq 0$. The possibilities listed in the other rows of Table 2 are obtained in a similar way. Below we list values of the parameters that realize each one of the possible phase portraits.

- P_{42} : $(a, b, c, d, e) = (1, 1, -1, 4, -1)$.
- P_{43} : $(a, b, c, d, e) = (1, 1, -1, -4, -1)$.
- P_{44} : $(a, b, c, d, e) = (1, 1, -1, 0, -1)$.
- P_{45} : $(a, b, c, d, e) = (1, 1, -1, 4, 0.25)$.
- P_{46} : $(a, b, c, d, e) = (1, 1, -1, 0, 0.25)$.
- P_{47} : $(a, b, c, d, e) = (2, 1, -1, 1, 0.25)$.
- P_{48} : $(a, b, c, d, e) = (1, 2, -1, 0, 2)$.
- P_{49} : $(a, b, c, d, e) = (1, 1, 0, 4, -1)$.
- P_{50} : $(a, b, c, d, e) = (1, 1, 0, -2, -1)$.
- P_{51} : $(a, b, c, d, e) = (1, 1, 0, 0, -1)$.

- P_{52} : $(a, b, c, d, e) = (1, 1, 0, 4, 0.25)$.
- P_{53} : $(a, b, c, d, e) = (2, 1, 0.25, 1, 0.25)$.
- P_{54} : $(a, b, c, d, e) = (1, 1, 0, -2, 0.25)$.
- P_{55} : $(a, b, c, d, e) = (1, 1, 0, 0, 1)$.
- P_{56} : $(a, b, c, d, e) = (1, 1, 1, 0, 0.2)$.
- P_{57} : $(a, b, c, d, e) = (1, 1, 1, 0, 0.25)$.
- P_{58} : $(a, b, c, d, e) = (2, 1, 0.3, 1, 0.25)$.

4.1.3. *Proof of Theorem 2 –System (iii).* If $\Delta_{I_2} > 0$ and $\Delta_{F_I} > 0$, corresponding to the case considered in the first row of Table 3, systems (iii) have 2 finite equilibria q_1, q_2 and 6 infinite equilibria n, s, u_1, u_2, v_1, v_2 , according to sections 2 and 3.

We consider the straight line r joining v_2, q_1 and u_2 . Applying Lemma (6) we can prove that the following configurations cannot occur:

- (a) both unstable separatrix of q_2 have ω -limit n ;
- (b) the left hand side of unstable separatrix of q_2 has ω -limit n and the right hand side separatrix of q_2 has ω -limit u_1 ;
- (c) the left hand side of unstable separatrix of q_2 has ω -limit v_2 and the right hand side separatrix of q_2 has ω -limit n ;
- (d) the left hand side of unstable separatrix of q_2 has ω -limit v_2 and the right hand side separatrix of q_2 has ω -limit u_1 ;
- (e) the left hand side of unstable separatrix of q_2 has ω -limit u_1 and the right hand side separatrix of q_2 has ω -limit v_2 .

Taking into account this previous informative the only possible phase portraits are P_{59}, P_{60} and P_{60} remain. The other lines of Table 3 are similarly analyzed. Below we list the parameter values that realize each one of the possible phase portraits.

- P_{59} : $(a, b, c, d, e) = (1, 2, 0.2, 1, 0.2)$.
- P_{60} : $(a, b, c, d, e) = (1, 1, 0.2, 2, 0.2)$.
- P_{61} : $(a, b, c, d, e) = (1, 1, 0.2, **, 0.2)$.
- P_{62} : $(a, b, c, d, e) = (2, 2, 0.2, 1, 1)$.
- P_{63} : $(a, b, c, d, e) = (1, 2, 0.2, 1, 1)$.
- P_{64} : $(a, b, c, d, e) = (**, 2, 0.2, 1, 1)$.
- P_{65} : $(a, b, c, d, e) = (8, 2, 2, 1, 5)$.
- P_{66} : $(a, b, c, d, e) = (2, 2, 1, 1, 0.2)$.
- P_{67} : $(a, b, c, d, e) = (2, 1, 1, 1, 0.2)$.
- P_{68} : $(a, b, c, d, e) = (2, 2, 1, 1, 1)$.
- P_{69} : $(a, b, c, d, e) = (1, 2, 0.25, 1, 1)$.

4.1.4. *Proof of Theorem 2 –Systems (iv) and (v).* The classification given in Tables 4 and 5 follows directly from the analysis of singularities at infinity. We list a parameter value that realize each phase portrait.

- P_{70} : $(a, b, c, d, e) = (1, 1, 0, 0, 0)$.
- P_{71} : $(a, b, c, d, e) = (1, 1, 0, 0, 1)$.
- P_{72} : $(a, b, c, d, e) = (1, 1, 0, 0, 0)$.
- P_{73} : $(a, b, c, d, e) = (1, 1, 0, 0, 1)$.
- P_{74} : $(a, b, c, d, e) = (1, 1, 0, 0, -1)$.

5. APPENDIX: SEMI-HYPERBOLIC EQUILIBRIUM POINTS

The following two lemmas are very useful in the proofs and they proved in Chapter 11 of [12].

Lemma 5. *If the straight line passing through two singular points S_1 and S_2 of a quadratic system is not an integral line, then it must be formed by three open line segments without contact points $\overline{\infty S_1}$, $\overline{S_1 S_2}$ and $\overline{S_2 \infty}$. Moreover the trajectories cross $\overline{\infty S_1}$ and $\overline{S_2 \infty}$ in one direction, and cross $\overline{S_1 S_2}$ in the opposite direction.*

Lemma 6. *The straight line connecting one finite singular point and a pair of infinite singular points in a quadratic system is either formed by trajectories or it is a line with exactly one contact point. This contact point is the finite singular point. For the latter case the flow goes in different directions on each half–line.*

REFERENCES

- [1] J.C. ARTÉS, J. LLIBRE, D. SCHLOMIUK AND N. VULPE, *Geometric configurations of singularities of planar polynomial differential systems. A global classification in the quadratic case*, to appear in Springer.
- [2] C. CHICONE AND T. JINGHUANG, *On General Properties of Quadratic Systems*, The American Mathematical Monthly, **89**, No. 3 (1982), 167–178.
- [3] W.A. COPPEL, *A Survey of Quadratic Systems*, *J. Differential Equations* **2** (1966), 293–304.
- [4] F. DUMORTIER, J. LLIBRE AND J.C. ARTÉS, *Qualitative Theory of Planar Differential Systems*, Springer, 2006.
- [5] P. FIGUEROA AND M. PINTO, *Riccati equations and nonoscillatory solutions of third order differential equations*, *Dynam. Systems Appl.* **17** (2008), no. 3-4, 459–475.
- [6] A. GASULL AND J. LLIBRE, *On the nonsingular quadratic differential equations in the plane*, *Proc. Amer. Math. Soc.* **104** (1988), 793-794.
- [7] A. GASULL, SHENG LI-REN, J. LLIBRE, *Chordal quadratic systems*, *Rocky Mountain J. of Math.* **16** (1986), 751-782.
- [8] J. LLIBRE, R. OLIVEIRA, C. VALLS, *Phase portraits for \mathbb{Z}_2 -symmetric cubic Riccati polynomial differential equations*, *Topology and its Applications* **234** (2018), 220–237.

- [9] S.D. MAHARAJ, A.K. TIWARI, R. MOHANLAL, R. NARAIN , *Riccati equations for bounded radiating systems*, J. Math. Phys. **57-9** (2016), 092501, 12p.
- [10] J.W. REYN, *Phase portraits of planar quadratic systems, Mathematics and its Applications*, **583**, Springer, 2007.
- [11] A.G. WU, F.J. SUN AND Y. ZHANG, *A novel iterative algorithm for solving coupled Riccati equations*, Appl. Math. Comput. **364** (2020), 124645, 14 pp.
- [12] YE YANQIAN EL AL., *Theory of limit cycles*, Trans. of Mathematical Monographs **66**, Amer. Math. Soc., Providence, RI, 2 edition, 1984.

¹ DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN.

² IMECC–UNICAMP, CEP 13081–970, CAMPINAS, SÃO PAULO, BRAZIL.

³ DEPARTAMENTO DE MATEMÁTICA – IBILCE–UNESP, RUA C. COLOMBO, 2265, CEP 15054–000 S. J. RIO PRETO, SÃO PAULO, BRAZIL.

E-mail address: jllibre@mat.uab.cat

E-mail address: brunodomicianolopes@gmail.com

E-mail address: paulo.r.silva@unesp.br