THE PLANAR DISCONTINUOUS PIECEWISE LINEAR REFRACTING SYSTEMS HAVE AT MOST ONE LIMIT CYCLE

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ABSTRACT. In this paper we investigate the limit cycles of planar piecewise linear differential systems with two zones separated by a straight line. It is well known that when these systems are continuous they can exhibit at most one limit cycle, while when they are discontinuous the maximum number of limit cycles that they can exhibit is still open. For these last systems there are examples exhibiting three limit cycles.

The aim of this paper is to study the number of limit cycles for a special kind of planar discontinuous piecewise linear differential systems with two zones separated by a straight line which are known as refracting systems. First we obtain the existence and uniqueness of limit cycles for refracting systems of focus-node type. Second we prove that refracting systems of focus-focus type have at most one limit cycle, thus we give a positive answer to a conjecture on the uniqueness of limit cycle stated by Freire, Ponce and Torres in [10]. These two results complete the proof that any refracting system has at most one limit cycle.

1. Introduction

In the qualitative theory of the differential systems in the plane one of the most important problems is the determination and distribution of limit cycles, which is known as the famous Hilbert's 16-th problem [18, 28] and its weak form [4, 5, 12, 29].

Since many real world differential systems involve a discontinuity or a sudden change [2], in recent years there is a growing interest in the following planar piecewise smooth vector fields

$$\mathcal{X}(q) = \begin{cases} X^-(q) & \text{if } h(q) < 0, \\ X^+(q) & \text{if } h(q) > 0, \end{cases} \tag{1}$$

where the discontinuity boundary $\Sigma = \{q \in \mathbb{R}^2 : h(q) = 0\}$ divides the plane $\mathbb{R}^2$ into two regions $\Sigma^\pm = \{q \in \mathbb{R}^2 : \pm h(q) > 0\}$. The singularities $p^\pm$ of $X^\pm$ are called *visible* or *invisible* if $p^\pm \in \Sigma^\pm$ or $p^\pm \in \Sigma^\mp$, respectively.

Clearly the orbits are well defined in both zones $\Sigma^\pm$. While if an orbit arrives to the discontinuous boundary Σ, different things can occur.

Definition 1. *Let $X^\pm h(q) = \langle \nabla h(q), X^\pm(q)\rangle$. Then we can classify Σ into the following three open regions:*

(i) *crossing region $\Sigma^c = \{q \in \Sigma : X^+h(q)X^-h(q) > 0\}$, see Fig.1.1.*

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