INVARIANT ALGEBRAIC SURFACES OF POLYNOMIAL VECTOR FIELDS IN DIMENSION THREE

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Abstract. We discuss criteria for the nonexistence, existence and computation of invariant algebraic surfaces for three-dimensional complex polynomial vector fields, thus transferring a classical problem of Poincaré from dimension two to dimension three. Such surfaces are zero sets of certain polynomials which we call semi–invariants of the vector fields. The main part of the work deals with finding degree bounds for irreducible semi–invariants of a given polynomial vector field that satisfies certain properties for its stationary points at infinity. As a related topic, we investigate existence criteria and properties for algebraic Jacobi multipliers. Some results are stated and proved for polynomial vector fields in arbitrary dimension and their invariant hypersurfaces. In dimension three we obtain detailed results on possible degree bounds. Moreover by an explicit construction we show for quadratic vector fields that the conditions involving the stationary points at infinity are generic but they do not a priori preclude the existence of invariant algebraic surfaces. In an appendix we prove a result on invariant lines of homogeneous polynomial vector fields.

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1. Introduction

Consider a polynomial ordinary differential equation in $\mathbb{C}^n$

$$\dot{x} = f(x) = f^{(0)}(x) + f^{(1)}(x) + \cdots + f^{(m)}(x), \tag{1}$$

with each $f^{(i)}$ a homogeneous polynomial of degree i, $0 \leq i \leq m$, and $f^{(m)} \neq 0$. By X_f we denote the vector field associated to f (also called the Lie derivative with respect to f). A polynomial $\psi : \mathbb{C}^n \to \mathbb{C}^n$ is called a *semi–invariant* of f if ψ is nonconstant and

$$X_f(\psi) = \lambda \cdot \psi, \tag{2}$$

for some polynomial λ, called the *cofactor* of ψ. As it is well–known, a polynomial is a semi–invariant of the vector field f if and only if its vanishing set is invariant for the flow of system (1). Given a degree bound, the problem of finding semi–invariants essentially reduces to solving a linear system of equations with parameters, thus a problem of linear algebra.

The existence problem for semi–invariants is relevant for several questions, notably Darboux integrability and existence of Jacobi multipliers. From the work of Żołądek [25] it is known that generically no algebraic invariant sets exist for polynomial vector fields of a fixed degree; see also Coutinho and Pereira [7]. Even for dimension $n = 2$ the existence problem is hard when the vector field has dicritical stationary points. When there are no dicritical stationary points then work of Cerveau and Lins Neto [5] and Carnicer [3] provides degree bounds for irreducible semi–invariants; see Pereira [15] for a refinement and also note the recent work by Ferragut et al. [9]. For certain classes of planar polynomial vector fields it was shown in [24] by elementary arguments that an effective degree bound for irreducible semi–invariants exists, and strong restrictions were found for possible integrating factors. For higher dimensions Jouanolou [10] showed the existence of the general degree bound $m + 1$ for semi–invariants of system (1) that define smooth hypersurfaces in projective space, and Soares [21, 22, 23] extended and refined this result. For dimension $n \geq 3$