Period function of planar turning points

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Abstract

This paper is devoted to the study of the period function of planar generic and non-generic turning points. In the generic case (resp. non-generic) a non-degenerate (resp. degenerate) center disappears in the limit $\varepsilon \to 0$, where $\varepsilon \geq 0$ is the singular perturbation parameter. We show that, for each $\varepsilon > 0$ and $\varepsilon \sim 0$, the period function is monotonously increasing (resp. has exactly one minimum). The result is valid in an ε-uniform neighborhood of the turning points. We also solve a part of the conjecture about a uniform upper bound for the number of critical periods inside classical Liénard systems of fixed degree, formulated by De Maesschalck and Dumortier in 2007. We use singular perturbation theory and the family blow-up.

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1 Introduction

We consider slow-fast polynomial Liénard equations of center type

$$X_{\varepsilon,\eta}: \left\{ \begin{array}{rcl} \dot{x} & = & y - \left(x^{2n} + \sum_{k=1}^{l} a_k x^{2n+2k}\right) \\ \dot{y} & = & \varepsilon^{2n}\left(-x^{2n-1} + \sum_{k=1}^{m} b_k x^{2n+2k-1}\right), \end{array} \right. \tag{1}$$

where $l, m, n \geq 1$, $\eta := (a_1, \ldots, a_l, b_1, \ldots, b_m)$ is kept in a compact set K of $\mathbb{R}^{l+m}$ and $\varepsilon \geq 0$ is the singular perturbation parameter kept small. System $X_{\varepsilon,\eta}$ is invariant under the symmetry $(x, t) \to (-x, -t)$ and has a center at the origin for all $\varepsilon > 0$, $\varepsilon \sim 0$, and for all $\eta \in K$. The center is non-degenerate when $n = 1$ or nilpotent when $n > 1$. In the limit $\varepsilon = 0$, we encounter drastic changes in the dynamics of (1): the system has a curve of singular points, given by $\{y = x^{2n} + \sum_{k=1}^{l} a_k x^{2n+2k}\}$, passing through the origin, and horizontal regular orbits (see Figure 1). A portion of the curve of singularities near the origin consists of the normally attracting part $\{x > 0\}$, the normally repelling