mathematics

Article

On the Zero-Hopf Bifurcation of the Lotka–Volterra Systems in $\mathbb{R}^3$

Maoan Han [1], Jaume Llibre [2,*] and Yun Tian [1]

[1] Department of Mathematics, Shanghai Normal University, 100 Guilin Road, Xuhui District, Shanghai 200234, China; mahan@shnu.edu.cn (M.H.); ytian22@shnu.edu.cn (Y.T.)

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

* Correspondence: jllibre@mat.uab.cat

Received: 5 May 2020; Accepted: 5 July 2020; Published: 12 July 2020

Abstract: Here we study 3-dimensional Lotka–Volterra systems. It is known that some of these differential systems can have at least four periodic orbits bifurcating from one of their equilibrium points. Here we prove that there are some of these differential systems exhibiting at least six periodic orbits bifurcating from one of their equilibrium points. We remark that these systems with such six periodic orbits are non-competitive Lotka–Volterra systems. The proof is done using the algorithm that we provide for computing the periodic solutions that bifurcate from a zero-Hopf equilibrium based in the averaging theory of third order. This algorithm can be applied to any differential system having a zero-Hopf equilibrium.

Keywords: Lotka–Volterra differential systems; periodic orbit; Hopf bifurcation; averaging theory

MSC: Primary 34C07; 34C08; 37G15

---

1. Introduction and Statement of Results

An equilibrium point of a 3-dimensional autonomous differential system having a pair of purely imaginary eigenvalues and a zero eigenvalue is a zero-Hopf equilibrium.

A 2-parameter unfolding of a 3-dimensional autonomous differential system with a zero-Hopf equilibrium is a zero-Hopf bifurcation. More precisely, when the two parameters of the unfolding are zero we have an isolated zero-Hopf equilibrium, and the dynamics of the unfolding is complex and sometimes chaotic in a small neighborhood of this isolated equilibrium when we vary the two parameters in a small neighborhood of the origin, see for more details [1–8] and references quoted there.

A Lotka–Volterra system in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) is a quadratic polynomial differential system of the form

$$\frac{dx_i}{dt} = x_i\big(r_i - \sum_{j=1}^{3} a_{ij}x_j\big), \quad i = 1, 2, 3, \tag{1}$$

where the dot denotes derivative with respect to the independent variable t, usually called the time, and the r_i's and the a_{ij}'s are parameters.

Many natural phenomena can be modeled by the Lotka–Volterra systems, starting in biology with the time evolution of conflicting species that now continuing being studied intensively see [9–20], later on problems of plasma physics [21], or problems in hydrodynamics [22], $\cdots$.

It is known that Lotka–Volterra systems can exhibit zero-Hopf equlibria, see for instance [23]. Then a natural question is if we perturbed a Lotka–Volterra system (1) having a zero-Hopf equilibrium point inside the class of all Lotka–Volterra systems (1) how many periodic orbits can bifurcate from such an equilibrium?