ON THE ZERO–HOPF BIFURCATiON OF THE LOTKA–VOLTERRA SYSTEMS IN $\mathbb{R}^3$

MAOAN HAN*, JAUME LLIBRE**[1] AND YUN TIAN*

* Department of Mathematics, Shanghai Normal University, 100 Guilin Roda, Xuhui District, 200234, Shanghai, China.

** Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.

ABSTRACT. Here we study the Lotka-Volterra systems in $\mathbb{R}^3$, i.e. the differential systems of the form

$$\frac{dx_i}{dt} = x_i\big(r_i - \sum_{j=1}^{3} a_{ij}x_j\big), \quad i = 1, 2, 3.$$

It is known that some of these differential systems can have at least four periodic orbits bifurcating from one of their equilibrium points. Here we prove that there are some of these differential systems exhibiting at least six periodic orbits bifurcating from one of their equilibrium points. The tool for proving this result is the averaging theory of third order.

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[1]Jaume Llibre is the corresponding author

1. Introduction and statement of results

An equilibrium point of a 3-dimensional autonomous differential system having a pair of purely imaginary eigenvalues and a zero eigenvalue is a *zero-Hopf equilibrium.*

A 2-parameter unfolding of a 3-dimensional autonomous differential system with a zero-Hopf equilibrium is a *zero-Hopf bifurcation.* More precisely, when the two parameters of the unfolding are zero we have an isolated zero-Hopf equilibrium, and the dynamics of the unfolding is complex and sometimes chaotic in a small neighborhood of this isolated equilibrium when we vary the two parameters in a small neighborhood of the origin, see for more details [4, 7, 8, 17, 26] and references quoted there.

A *Lotka-Volterra system* in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) is a quadratic polynomial differential system of the form

$$\frac{dx_i}{dt} = x_i\big(r_i - \sum_{j=1}^{3} a_{ij}x_j\big), \quad i = 1, 2, 3, \tag{1}$$

where the dot denotes derivative with respect to the independent variable t, usually called the time, and the r_i's and the a_j's are parameters.

Many natural phenomena can be modeled by the Lotka–Volterra systems, starting in biology with the time evolution of conflicting species that now continuing being studied intensively see [9, 10, 11, 12, 13, 14, 15, 22, 25, 27, 28, 31], later on problems of plasma physics [18], or problems in hydrodynamics [3], ...

It is known that Lotka-Volterra systems can exhibit zero-Hopf equlibria, see for instance [20]. Then a natural question is if we perturbed a Lotka-Volterra system (1) having a zero-Hopf equilibrium point inside the class of all Lotka-Volterra systems how many periodic orbits can bifurcate from such an equilibrium?

Note that the unfolding of Lotka-Volterra system (1) with a zero-Hopf equilibrium needs at least a 3-parameter family. Arnold [1] in 1973 proposed to investigate bifurcations of 3-parameter families with a zero–Hopf equilibrium.

As far as we know the number of periodic orbits which can bifurcate from a zero-Hopf equilibrium point when this is perturbed inside the class of all Lotka-Volterra systems only has been studied partially in the paper [20] using averaging theory of second order. There the authors provided explicit conditions for the existence of one or two periodic orbits bifurcating from one of these equilibria.

Here we shall use the averaging theory of third order for studying the number of periodic orbits which can bifurcate from a zero-Hopf equilibrium point of a Lotka-Volterra system (1). Previous results in this direction are the following. First we say that an equilibrium point of a 3-dimensional autonomous differential system having a pair of purely imaginary eigenvalues and a non-zero eigenvalue is a *Hopf equilibrium.* The bifurcation of periodic orbits in a Hopf equilibrium of a Lotka-Volterra system (1) have been studied by many authors. Thus in the papers [16, 23, 30] the authors proved that two periodic orbits can bifurcate from a Hopf equilibrium of system (1). While in [5, 6, 24] it is shown that three periodic orbits can bifurcate from a Hopf equilibrium. Recently in [29] it is proved that four periodic orbits can bifurcate from a Hopf equilibrium of system (1). All these previous results on the number of periodic orbits bifurcating from a Hopf equilibrium are when system (1) has all its coefficients a_{ij} and r_i positive, and under this assumption in [5] it is conjectured that at least five periodic orbits can bifurcate from a such Hopf equilibrium, but this conjecture remains open.

In short, until now it is known that there are Lotka-Volterra systems (1) having at least four periodic orbits bifurcating from one of their equilibrium points. Our main result is the following one.

Theorem 1. *There are Lotka-Volterra systems* (1) *having at least six periodic orbits bifurcating from a zero–Hopf equilibrium.*

We remark that those Lotka-Volterra systems (1) exhibiting a Hopf bifurcation with at least six periodic orbits do not have all the coefficients a_{ij} and r_i positive.

The proof of Theorem 1 is given in the next section.

2. Proof of Theorem 1

If system (1) has a zero-Hopf equilibrium (a, b, c) with non-zero components without loss of generality we can consider this equilibrium at the point $(1, 1, 1)$ doing the rescaling $(x, y, z) \to (x/a, y/b, z/c)$. Then every Lotka-Volterra system (1) having the equilibrium $(1, 1, 1)$ can be written as

$$\begin{aligned} \dot{x} &= x\big(a_{11}(x-1) + a_{12}(y-1) + a_{13}(z-1)\big), \\ \dot{y} &= y\big(a_{21}(x-1) + a_{22}(y-1) + a_{23}(z-1)\big), \\ \dot{z} &= z\big(a_{31}(x-1) + a_{32}(y-1) + a_{33}(z-1)\big), \end{aligned} \tag{2}$$

where now we denote the coordinates of $\mathbb{R}^3$ by (x, y, z). Since we shall use the averaging theory of third order for studying the periodic orbits of this system we take the coefficients a_{ij} as follows

$$a_{ij} = a_{ij0} + \varepsilon a_{ij1} + \varepsilon^2 a_{ij2} + \varepsilon^3 a_{ij3},$$

with i and j varying in $\{1, 2, 3\}$, being ε a small parameter. Note that in the differential system (2) there are 37 parameters. This big number of parameters produce that the computations for studying the number of periodic orbits which can bifurcate from the equilibrium $(1, 1, 1)$ are tedious and huge. All the computations of this paper has been done with the help of the algebraic manipulator *mathematica*.

First we translate the equilibrium $(1, 1, 1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned} \dot{x} = {} & (1+x)\big(a_{110}x + a_{120}y + a_{130}z + \varepsilon(a_{111}x + a_{121}y + a_{131}z) + \\ & \varepsilon^2(a_{112}x + a_{122}y + a_{132}z) + \varepsilon^3(a_{113}x + a_{123}y + a_{133}z)\big), \\ \dot{y} = {} & (1+y)\big(a_{210}x + a_{220}y + a_{230}z + \varepsilon(a_{211}x + a_{221}y + a_{231}z) + \\ & \varepsilon^2(a_{212}x + a_{222}y + a_{232}z) + \varepsilon^3(a_{213}x + a_{223}y + a_{233}z)\big), \\ \dot{z} = {} & (1+z)\big(a_{310}x + a_{320}y + a_{330}z + \varepsilon(a_{311}x + a_{321}y + a_{331}z) + \\ & \varepsilon^2(a_{312}x + a_{322}y + a_{332}z) + \varepsilon^3(a_{313}x + a_{323}y + a_{333}z)\big). \end{aligned} \tag{3}$$

Choosing the conditions

$$a_{110} = a_{120} = a_{130} = a_{210} = 0, a_{320} = -(a_{220}^2 + \omega^2)/a_{230} \text{ and } a_{330} = -a_{220}, \tag{4}$$

with $a_{230}\omega \neq 0$ it is easy to check that the linear part of system (3) at the origin has eigenvalues 0 and $\pm\omega i$. So the origin of system (3) is a zero-Hopf equilibrium, and consequently system (2) has a zero-Hopf equilibrium at the point $(1,1,1)$. We remark that there are other conditions which also provide that the point $(1,1,1)$ be a zero-Hopf equilibrium.

In what follows we shall study the periodic orbits bifurcating from the zero-Hopf equilibrium $(0,0,0)$ *of system* (3) *under conditions* (4).

As we shall see the amount of computations for studying this Hopf-bifurcation are huge due to the big number of parameters in system (3).

In order to study the periodic orbits bifurcating from the zero-Hopf equilibrium at the origin of the differential system (3) using the averaging theory of third order (see the appendix), we need to introduce a small parameter and take a new independent variable in which the differential system be periodic.

The small parameter for the averaging theory will be the parameter ε, and we do the rescaling $(x,y,z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$. Then system (3) in the new variables (X,Y,Z) writes

$$
\begin{aligned}
\dot{X} = \;& \varepsilon(a_{111}X + a_{121}Y + a_{131}Z) + \varepsilon^2(a_{112}X + a_{111}X^2 + a_{122}Y + \\
& a_{121}XY + a_{132}Z + a_{131}XZ) + \varepsilon^3(a_{113}X + a_{112}X^2 + a_{123}Y + \\
& a_{122}XY + a_{133}Z + a_{132}XZ) + O(\varepsilon^4), \\
\dot{Y} = \;& a_{220}Y + a_{230}Z + \varepsilon(a_{211}X + a_{221}Y + a_{220}Y^2 + a_{231}Z + a_{230}YZ) + \\
& \varepsilon^2(a_{212}X + a_{222}Y + a_{211}XY + a_{221}Y^2 + a_{232}Z + a_{231}YZ) + \\
& \varepsilon^3(a_{213}X + a_{223}Y + a_{212}XY + a_{222}Y^2 + a_{233}Z + a_{232}YZ) + O(\varepsilon^4), \\
\dot{Z} = \;& (a_{230}a_{310}X - a_{220}^2Y - a_{220}a_{230}Z - Y\omega^2)/a_{230} + \varepsilon(a_{230}a_{311}X + \\
& a_{230}a_{321}Y + a_{230}a_{331}Z + a_{230}a_{310}XZ - a_{220}^2YZ - a_{220}a_{230}Z^2 - \\
& YZ\omega^2)/a_{230} + \varepsilon^2(a_{312}X + a_{322}Y + a_{332}Z + a_{311}XZ + a_{321}YZ + \\
& a_{331}Z^2) + \varepsilon^3(a_{313}X + a_{323}Y + a_{333}Z + a_{312}XZ + a_{322}YZ + \\
& a_{332}Z^2) + O(\varepsilon^4).
\end{aligned}
\tag{5}
$$

In order to simplify the computations of the averaging theory we shall write the linear part of the differential system (5) into its real Jordan normal form doing the linear change of variables $(X,Y,Z) \to (u,v,w)$ given by

$$
\begin{aligned}
X = \;& w, \\
Y = \;& \frac{a_{230}a_{310}w}{\omega^2} + \frac{a_{230}\omega v - a_{220}a_{230}u}{a_{220}^2 + \omega^2}, \\
Z = \;& -a_{220}a_{230}a_{310}w + a_{230}\omega^2 u.
\end{aligned}
$$

Now the differential system (5) in the new variables (u, v, w) becomes

$$
\begin{aligned}
\dot{u} = {} & -\omega v + \frac{\varepsilon}{\omega^4(a_{220}^2+\omega^2)}\big((a_{131}a_{220}^3a_{310} - a_{121}a_{220}^2a_{230}a_{310} + \\
& a_{131}a_{220}a_{310}\omega^2 - a_{220}a_{230}a_{321}\omega^2 + a_{220}^2a_{331}\omega^2 + a_{331}\omega^4)\omega^2 u + \\
& a_{230}(a_{121}a_{220}a_{310} + a_{321}\omega^2)\omega^3 v - (a_{220}^2+\omega^2)\big((a_{131}a_{220}^2a_{310}^2 - \\
& a_{121}a_{220}a_{230}a_{310}^2 - a_{111}a_{220}a_{310}\omega^2 - a_{230}a_{310}a_{321}\omega^2 + \\
& a_{220}a_{310}a_{331}\omega^2 - a_{311}\omega^4)w - \omega^5 uv + a_{220}a_{310}\omega^3 vw\big)\big) + O(\varepsilon^2), \\
\dot{v} = {} & \omega u + \frac{\varepsilon}{a_{230}\omega^3(a_{220}^2+\omega^2)}\big((-a_{220}^3a_{221}a_{230} + a_{220}^4a_{231} - \\
& a_{131}a_{220}^2a_{230}a_{310} + a_{121}a_{220}a_{230}^2a_{310} - a_{220}^2a_{230}^2a_{321} + \\
& a_{220}^3a_{230}a_{331} - a_{220}a_{221}a_{230}\omega^2 + 2a_{220}^2a_{231}\omega^2 - a_{131}a_{230}a_{310}\omega^2 + \\
& a_{220}a_{230}a_{331}\omega^2 + a_{231}\omega^4)\omega^2 u + a_{230}(a_{220}^2a_{221} - a_{121}a_{230}a_{310} + \\
& a_{220}a_{230}a_{321} + a_{221}\omega^2)\omega^3 v - (a_{220}^2+\omega^2)(-a_{220}^2\, a_{221}a_{230}a_{310} + \\
& a_{220}^3a_{231}a_{310} - a_{131}a_{220}a_{230}a_{310}^2 + a_{121}a_{230}^2a_{310}^2 - \\
& a_{220}a_{230}^2a_{310}a_{321} + a_{220}^2a_{230}a_{310}a_{331} - a_{211}a_{220}^2\omega^2 + \\
& a_{111}a_{230}a_{310}\omega^2 - a_{221}a_{230}a_{310}\omega^2 + a_{220}a_{231}a_{310}\omega^2 - \\
& a_{220}a_{230}a_{311}\omega^2 - a_{211}\omega^4)w - a_{220}a_{230}^2\omega^4u^2 - a_{230}(a_{220}^3 + \\
& a_{220}^2a_{230} + a_{220}\omega^2 - a_{230}\omega^2)\omega^3 uv + a_{230}^2a_{310}(a_{220}^2+\omega^2)\omega^2 uw + \\
& a_{220}a_{230}^2v^2\omega^4 + a_{220}a_{230}(a_{220}+a_{230})a_{310}(a_{220}^2+\omega^2)\omega vw\big) + O(\varepsilon^2), \\
\dot{w} = {} & \frac{\varepsilon}{\omega^2(a_{220}^2+\omega^2)}\big((a_{131}a_{220}^2 - a_{121}a_{220}a_{230} + a_{131}\omega^2)\omega^2 u + \\
& a_{121}a_{230}\omega^3 v - (a_{220}^2+\omega^2)(a_{131}a_{220}a_{310} - a_{121}a_{230}a_{310} - a_{111}\omega^2)w\big) \\
& + O(\varepsilon^2).
\end{aligned}
\tag{6}
$$

In the computations of the previous differential system we have obtained the expressions of $\dot{u}$, $\dot{v}$ and $\dot{w}$ until terms of $O(\varepsilon^4)$, but here we only present them until terms of order $O(\varepsilon^2)$, otherwise the expression of system (6) would need several pages. Using an algebraic manipulator as mathematica or mapple it is relatively easy to repeat our computations.

Now we write the differential system (6) in cylindrical coordinates (r, θ, w) where $u = r\cos\theta$ and $v = r\sin\theta$, and taking θ as the new independent variable of the differential system defined we get the new differential system

$$
\begin{aligned}
r' = {} & \varepsilon F_{11}(\theta, r, w) + \varepsilon^2 F_{21}(\theta, r, w) + \varepsilon^3 F_{31}(\theta, r, w) + O(\varepsilon^4), \\
w' = {} & \varepsilon F_{12}(\theta, r, w) + \varepsilon^2 F_{22}(\theta, r, w) + \varepsilon^3 F_{32}(\theta, r, w) + O(\varepsilon^4),
\end{aligned}
\tag{7}
$$

defined in in $r > 0$, where the prime denotes derivative with respect to the variable θ. Here we only provide the explicit expressions of $F_{11} = F_{11}(\theta, r, w)$ and $F_{12} = F_{12}(\theta, r, w)$ which are the shorter ones, but our next computations will use the expressions of F_{21}, F_{22}, F_{31} and F_{32}. Thus we have

$$
\begin{aligned}
F_{11} = \; & \frac{1}{a_{230}\omega^5(a_{220}^2+\omega^2)}\Big((a_{230}(a_{131}a_{220}^3a_{310} - a_{121}a_{220}^2a_{230}a_{310} + \\
& a_{131}a_{220}a_{310}\omega^2 - a_{220}a_{230}a_{321}\omega^2 + a_{220}^2a_{331}\omega^2 + a_{331}\omega^4)\cos^2\theta + \\
& (a_{220}^4a_{231} - a_{220}^3a_{221}a_{230} - a_{131}a_{220}^2a_{230}a_{310} + 2a_{121}a_{220}a_{230}^2a_{310} - \\
& a_{220}^2a_{230}^2a_{321} + a_{220}^3a_{230}a_{331} - a_{220}a_{221}a_{230}\omega^2 + 2a_{220}^2a_{231}\omega^2 - \\
& a_{131}a_{230}a_{310}\omega^2 + a_{230}^2a_{321}\omega^2 + a_{220}a_{230}a_{331}\omega^2 + a_{231}\omega^4)\omega\cos\theta\sin\theta + \\
& a_{230}(a_{220}^2a_{221} - a_{121}a_{230}a_{310} + a_{220}a_{230}a_{321} + a_{221}\omega^2)\omega^2\sin^2\theta)\omega^2 r - \\
& (a_{220}^2+\omega^2)(a_{230}(a_{131}a_{220}^2a_{310}^2 - a_{121}a_{220}a_{230}a_{310}^2 - a_{111}a_{220}a_{310}\omega^2 - \\
& a_{230}a_{310}a_{321}\omega^2 + a_{220}a_{310}a_{331}\omega^2 - a_{311}\omega^4)\cos\theta + (a_{220}^3a_{231}a_{310} - \\
& a_{220}^2a_{221}a_{230}a_{310} - a_{131}a_{220}a_{230}a_{310}^2 + a_{121}a_{230}^2a_{310}^2 - a_{220}a_{230}^2a_{310}a_{321} + \\
& a_{220}^2a_{230}a_{310}a_{331} - a_{211}a_{220}^2\omega^2 + a_{111}a_{230}a_{310}\omega^2 - a_{221}a_{230}a_{310}\omega^2 + \\
& a_{220}a_{231}a_{310}\omega^2 - a_{220}a_{230}a_{311}\omega^2 - a_{211}\omega^4)\omega\sin\theta])w - (a_{230}\omega(a_{220}^2 + \\
& a_{220}a_{230} + \omega^2)\cos^2\theta\sin\theta + a_{230}(a_{220}^3 + a_{220}^2a_{230} + a_{220}\omega^2 - a_{230}\omega^2) \\
& \cos\theta\sin^2\theta - a_{220}a_{230}^2\omega\sin^3\theta)\omega^4 r^2 + (a_{230}(a_{220}+a_{230})a_{310}\omega^3(a_{220}^2 + \\
& \omega^2)\cos\theta\sin\theta + a_{220}a_{230}(a_{220}+a_{230})a_{310}\omega^2(a_{220}^2+\omega^2)\sin^2\theta)rw\Big), \\
F_{12} = \; & \frac{1}{\omega^3(a_{220}^2+\omega^2)}\Big(((a_{131}a_{220}^2 - a_{121}a_{220}a_{230} + a_{131}\omega^2)\cos\theta + \\
& a_{121}a_{230}\omega\sin\theta)\omega^2 r - (a_{220}^2 + \omega^2)(a_{131}a_{220}a_{310} - a_{121}a_{230}a_{310} - \\
& a_{111}\omega^2)w\Big),
\end{aligned}
$$

We note that the differential system (7) is written in the normal form (11) for applying the averaging theory of third order described in the appendix, where the variables t and x of the appendix are now θ and (r, w) respectively. Computing the averaged function of first order $f_1(r, w) = (f_{11}(r, w), f_{12}(r, w))$ defined in the appendix we get

$$f_{11}(r, w) = Ar, \qquad f_{12}(r, w) = Bw,$$

where

$$
\begin{aligned}
A &= \frac{(a_{131}a_{220} - a_{121}a_{230})a_{310} + (a_{221} + a_{331})\omega^2 + (a_{220} + a_{230})a_{310}a_{220}w}{2\omega^3}, \\
B &= \frac{(a_{121}a_{230} - a_{131}a_{220})a_{310} + a_{111}\omega^2}{\omega^3}.
\end{aligned}
$$

We look for the zeros (r^*, w^*) of $f_1(r, w)$ with $r > 0$, and since the unique zero of the function $f_1(r, w)$ is $(0, 0)$, or a continuum of zeros if the coefficient A or B is zero, the averaged function of first order does not give any information on the periodic solutions of system (7), see the appendix. Therefore we force that the averaged function of first order be identically zero and we shall use the averaged functions of higher order to obtain information on the periodic solutions of the differential system (7).

Since the coefficient of rw in the function $f_{11}(r, w)$ is $(a_{220} + a_{230})a_{310}a_{220}$ we need to consider the following three cases in order that the averaged function of first order be identically zero:

Case 1: $a_{220} = -a_{230}$,

$a_{331} = (a_{121}a_{230}a_{310} + a_{131}a_{230}a_{310} - a_{221}\omega^2)/\omega^2$,

$a_{111} = (-a_{121}a_{230}a_{310} - a_{131}a_{230}a_{310})/\omega^2$.

Case 2: $a_{310}=0$, $a_{331}=-a_{221}$, $a_{111}=0$.

Case 3: $a_{220}=0$,
$a_{331}=(a_{121}a_{230}a_{310}-a_{221}\omega^2)/\omega^2$,
$a_{111}=-(a_{121}a_{230}a_{310})/\omega^2$.

Case 1. Since the averaged function of first order $f_1(r,w)$ is identically zero, we compute the averaged function of second order $f_2(r,w)=(f_{21}(r,w),\, f_{22}(r,w))$ and we obtain

$$f_{21}(r,w)=(Cw+D)r, \qquad f_{22}(r,w)=Ew,$$

where

$$\begin{aligned}
C = \;& -(-a_{121}a_{230}^2a_{310}^2-a_{131}a_{230}^2a_{310}^2+a_{121}a_{230}a_{310}\omega^2+a_{131}a_{230}a_{310}\omega^2+\\
& a_{221}a_{230}a_{310}\omega^2+a_{230}a_{231}a_{310}\omega^2+a_{211}\omega^4)/(2\omega),\\
D = \;& -(a_{121}^2a_{230}^4a_{310}^2+2a_{121}a_{131}a_{230}^4a_{310}^2+a_{131}^2a_{230}^4a_{310}^2-2a_{121}a_{221}a_{230}^3a_{310}\omega^2\\
& -2a_{131}a_{221}a_{230}^3a_{310}\omega^2-a_{121}a_{230}^3a_{231}a_{310}\omega^2-a_{131}a_{230}^3a_{231}a_{310}\omega^2+\\
& a_{121}a_{230}^3a_{310}a_{321}\omega^2+a_{131}a_{230}^3a_{310}a_{321}\omega^2-a_{121}a_{211}a_{230}^2\omega^4-\\
& a_{131}a_{211}a_{230}^2\omega^4-a_{131}a_{221}a_{230}a_{310}\omega^4+a_{122}a_{230}^2a_{310}\omega^4+a_{132}a_{230}^2a_{310}\omega^4\\
& -a_{131}a_{230}a_{231}a_{310}\omega^4+a_{121}a_{230}^2a_{311}\omega^4+a_{131}a_{230}^2a_{311}\omega^4-a_{131}a_{211}\omega^6-\\
& a_{222}a_{230}\omega^6-a_{230}a_{332}\omega^6)/(2a_{230}\omega^7),\\
E = \;& (a_{121}^2a_{230}^4a_{310}^2+2a_{121}a_{131}a_{230}^4a_{310}^2+a_{131}^2a_{230}^4a_{310}^2-2a_{121}a_{221}a_{230}^3a_{310}\omega^2\\
& -2a_{131}a_{221}a_{230}^3a_{310}\omega^2-a_{121}a_{230}^3a_{231}a_{310}\omega^2-a_{131}a_{230}^3a_{231}a_{310}\omega^2+\\
& a_{121}a_{230}^3a_{310}a_{321}\omega^2+a_{131}a_{230}^3a_{310}a_{321}\omega^2-a_{121}a_{211}a_{230}^2\omega^4-\\
& a_{131}a_{211}a_{230}^2\omega^4-a_{131}a_{221}a_{230}a_{310}\omega^4+a_{122}a_{230}^2a_{310}\omega^4+a_{132}a_{230}^2a_{310}\omega^4\\
& -a_{131}a_{230}a_{231}a_{310}\omega^4+a_{121}a_{230}^2a_{311}\omega^4+a_{131}a_{230}^2a_{311}\omega^4-a_{131}a_{211}\omega^6+\\
& a_{112}a_{230}\omega^6)/(a_{230}\omega^7).
\end{aligned}$$

Again the unique zero of the averaged function of second order $f_2(r,w)$ is the $(0,0)$ or a continuum of solutions in case that convenient coefficients C, D or E are zero. Therefore the averaging theory of second order does not provide any information on the periodic solutions of the differential system (7). Consequently we impose that the averaged function of second order $f_2(r,w)$ be identically zero, and we obtain that

$$\begin{aligned}
a_{211} = \;& \big(a_{121}a_{230}^2a_{310}^2+a_{131}a_{230}^2a_{310}^2-a_{121}a_{230}a_{310}\omega^2-a_{131}a_{230}a_{310}\omega^2-\\
& a_{221}a_{230}a_{310}\omega^2-a_{230}a_{231}a_{310}\omega^2\big)/\omega^4,\\
a_{332} = \;& \big(a_{121}^2a_{230}^2a_{310}+2a_{121}a_{131}a_{230}^2a_{310}+a_{131}^2a_{230}^2a_{310}-a_{121}a_{221}a_{230}^2a_{310}-\\
& a_{131}a_{221}a_{230}^2a_{310}-a_{121}a_{131}a_{230}a_{310}^2-a_{131}^2a_{230}a_{310}^2+a_{121}a_{230}^2a_{310}a_{321}+\\
& a_{131}a_{230}^2a_{310}a_{321}+a_{121}a_{131}a_{310}\omega^2+a_{131}^2a_{310}\omega^2+a_{122}a_{230}a_{310}\omega^2+\\
& a_{132}a_{230}a_{310}\omega^2+a_{121}a_{230}a_{311}\omega^2+a_{131}a_{230}a_{311}\omega^2-a_{222}\omega^4\big)/\omega^4,\\
a_{112} = \;& \big(-a_{121}^2a_{230}^2a_{310}-2a_{121}a_{131}a_{230}^2a_{310}-a_{131}^2a_{230}^2a_{310}+a_{121}a_{221}a_{230}^2a_{310}\\
& +a_{131}a_{221}a_{230}^2a_{310}+a_{121}a_{131}a_{230}a_{310}^2+a_{131}^2a_{230}a_{310}^2-a_{121}a_{230}^2a_{310}a_{321}\\
& -a_{131}a_{230}^2a_{310}a_{321}-a_{121}a_{131}a_{310}\omega^2-a_{131}^2a_{310}\omega^2-a_{122}a_{230}a_{310}\omega^2-\\
& a_{132}a_{230}a_{310}\omega^2-a_{121}a_{230}a_{311}\omega^2-a_{131}a_{230}a_{311}\omega^2\big)/\omega^4.
\end{aligned}$$

We compute the averaged function of third order $f_3(r,w) = (f_{31}(r,w), f_{32}(r,w))$ and we get

$$f_{31}(r,w) = \frac{a_0 r^4 + a_1 r^3 + a_2 r^2 w + a_3 r^2 + a_4 r w + a_5 w^2 + a_6 r + a_7 w}{384 a_{230}(a_{230}^2 + \omega^2)\omega^{13} r},$$

$$f_{32}(r,w) = \frac{b_0 r^3 + b_1 r^2 w + b_2 r^2 + b_3 r w + b_4 r + b_5 w}{24 a_{230}(a_{230}^2 + \omega^2)^2 \omega^9}.$$

We do not provide the explicit expressions of the coefficients a_j and b_j because we shall need approximately twenty pages for writing them.

Now we shall study the zeros of the function $f_3(r,w)$. Since the variable w appears linearly in the equation $f_{32}(r,w) = 0$, we isolate it and we get $w = W(r)$. Substituting $w = W(r)$ into the equation $f_{31}(r,w) = 0$, we obtain an equation in the variable r of the form

$$\frac{n(r)}{d(r)} = \frac{c_2 r^2 + c_3 r^3 + c_4 r^4 + c_5 r^5 + c_6 r^6 + c_7 r^7 + c_8 r^8}{(d_0 + d_1 r + d_2 r^2)^2} = 0. \tag{8}$$

The coefficients c_j and d_j are polynomials in some of the coefficients of the differential system (2), more precisely in the coefficients $a_{113}, a_{121}, a_{122}, a_{123}, a_{131}, a_{132}, a_{133}, a_{212}, a_{221}, a_{222}, a_{223}, a_{230}, a_{231}, a_{232}, a_{310}, a_{311}, a_{312}, a_{321}, a_{322}, a_{333}, \omega$. We have computed the rank of the Jacobian matrix of the function $(c_2, c_3, c_4, c_5, c_6, c_7, c_8)$ with respect to the 21 previous coefficients, it is the rank of a 7×23 matrix, and we get that this rank is 7. Therefore the seven coefficients of the polynomial $n(r)$ are independent, and consequently we can choose them in such a way that the polynomial $n(r)$ has six positive real roots. Moreover, we also can choose those coefficients in such a way that the resultant of the polynomials $n(r)$ and $d(r)$ is not zero, and consequently both polynomials do not have a common root. So equation (8) can have six positive solutions, r_j^* for $j = 1,2,3,4,5,6$.

In short, we have that $(r_j^*, W(r_j^*))$ for $j = 1,2,3,4,5,6$ are six zeros of the third averaged function $f_3(r,w)$. These zeros can be chosen simple, i.e. the Jacobian of the function $f_3(r,w)$ evaluated in such zeros is not zero. Consequently by the averaging theory (see the appendix) the differential system (7) has six periodic solutions $(r_j(\theta,\varepsilon), w_j(\theta,\varepsilon))$ such that $(r_j(0,\varepsilon), w_j(0,\varepsilon)) \to (r_j^*, W(r_j^*))$ when $\varepsilon \to 0$.

Going back to the differential system (6) we obtain for this system six periodic solutions $(u_j(t,\varepsilon), v_j(t,\varepsilon), w_j(t,\varepsilon))$ such that

$$(u_j(0,\varepsilon), v_j(0,\varepsilon), w_j(0,\varepsilon)) \to (r_j^*, 0, W(r_j^*)),$$

when $\varepsilon \to 0$. These periodic solutions provide six periodic solutions $(X_j(t,\varepsilon), Y_j(t,\varepsilon), Z_j(t,\varepsilon))$ for the differential system (5) such that

$$\begin{aligned} X_j(0,\varepsilon) &\to W(r_j^*), \\ Y_j(0,\varepsilon) &\to \frac{a_{230} a_{310} W(r_j^*)}{\omega^2} - \frac{a_{220} a_{230} r_j^*}{a_{220}^2 + \omega^2}, \\ Z_j(0,\varepsilon) &\to a_{230}\omega^2 r_j^* - a_{220} a_{230} a_{310} W(r_j^*), \end{aligned}$$

when $\varepsilon \to 0$. Finally going back to the differential system (2) we obtain six periodic solutions $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon))$ such that

$$
\begin{aligned}
x_j(0,\varepsilon) &= 1 + \varepsilon W(r_j^*) + O(\varepsilon^2), \\
y_j(0,\varepsilon) &= 1 + \varepsilon\Big(\frac{a_{230}a_{310}W(r_j^*}{\omega^2} - \frac{a_{220}a_{230}r_j^*}{a_{220}^2+\omega^2}\Big) + O(\varepsilon^2), \\
z_j(0,\varepsilon) &= 1 + \varepsilon\big(a_{230}\omega^2 r_j^* - a_{220}a_{230}a_{310}W(r_j^*)\big) + O(\varepsilon^2),
\end{aligned} \tag{9}
$$

when $\varepsilon \to 0$. Clearly from (9) these six periodic solutions $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon))$ tend to the equilibrium point $(1,1,1)$ of the differential system (2) when $\varepsilon \to 0$. Hence they bifurcate from that zero-Hopf equilibrium at $\varepsilon = 0$. This completes the proof of Theorem 1.

Case 2. Again since the averaged function of first order $f_1(r,w)$ is identically zero, we compute the averaged function of second order $f_2(r,w) = (f_{21}(r,w), f_{22}(r,w))$ and we obtain

$$
f_{21}(r,w) = (Cw + D)r, \qquad f_{22}(r,w) = Ew,
$$

where

$$
\begin{aligned}
C =& \ \frac{a_{220}(a_{211}a_{220}^2 + a_{211}a_{220}a_{230} + a_{220}a_{230}a_{311} + a_{230}^2a_{311} + a_{211}\omega^2)}{2a_{230}\omega^3}, \\
D =& \ \frac{1}{2a_{230}\omega^3}\big(a_{121}a_{211}a_{220}a_{230} - a_{131}a_{211}a_{220}^2 - a_{131}a_{220}a_{230}a_{311} + \\
& a_{121}a_{230}^2a_{311} - a_{131}a_{211}\omega^2 - a_{222}a_{230}\omega^2 - a_{230}a_{332}\omega^2\big), \\
E =& \ \frac{1}{a_{230}\omega^3}\big(a_{121}a_{211}a_{220}a_{230} - a_{131}a_{211}a_{220}^2 - a_{131}a_{220}a_{230}a_{311} + \\
& a_{121}a_{230}^2a_{311} - a_{131}a_{211}\omega^2 + a_{112}a_{230}\omega^2\big).
\end{aligned}
$$

As in the previous case the unique zero of the averaged function of second order $f_2(r,w)$ is the $(0,0)$ or a continuum of solutions in case that convenient coefficients C, D or E are zero. Consequently we impose that the averaged function of second order $f_2(r,w)$ be identically zero, but since the coefficient of rw in the function $f_{21}(r,w)$ is a product of two factors we have two consider two subcases.

Subcase 2.1: $a_{220} = 0$. Then in order that the averaged function of second order $f_2(r,w)$ be identically zero we take

$$
\begin{aligned}
a_{332} &= \frac{a_{121}a_{230}^2a_{311} - a_{131}a_{211}\omega^2 - a_{222}a_{230}\omega^2}{a_{230}\omega^2}, \\
a_{112} &= \frac{a_{131}a_{211}\omega^2 - a_{121}a_{230}^2a_{311}}{a_{230}\omega^2}.
\end{aligned}
$$

We compute the averaged function of third order $f_3(r,w) = (f_{31}(r,w), f_{32}(r,w))$ and we get

$$
\begin{aligned}
f_{31}(r,w) &= \frac{a_0r^3 + a_1r^2 + a_2rw + a_3w^2 + a_4r + a_5w}{384a_{230}^2\omega^5 r}, \\
f_{32}(r,w) &= \frac{b_0r^3 + b_1r^2 + b_2rw + b_3r + b_4w}{24a_{230}^2\omega^5}.
\end{aligned} \tag{10}
$$

Here the expressions of the coefficients a_j's and b_j's are relatively short, but we do not need them explicitly.

We shall study the zeros of the function $f_3(r,w)$. Since the variable w appears linearly in the equation $f_{32}(r,w)=0$, we isolate it and we get $w=W(r)$. Substituting $w=W(r)$ into the equation $f_{31}(r,w)=0$, we obtain an equation in the variable r of the form

$$\frac{c_2r^2+c_3r^3+c_4r^4+c_5r^5+c_6r^6}{(d_0+d_1r+d_2r^2)^2}=0.$$

So at most we have four positive solutions for the variable r, and consequently at most four zeros for the averaged function of third order $f_3(r,w)$. In any case less than the six obtained in Case 1.

Subcase 2.2: $a_{211}a_{220}^2+a_{211}a_{220}a_{230}+a_{220}a_{230}a_{311}+a_{230}^2a_{311}+a_{211}\omega^2=0$. Then in order that the averaged function of second order $f_2(r,w)$ be identically zero we take

$$\begin{aligned}
a_{311} &= -\frac{a_{211}a_{220}^2+a_{211}a_{220}a_{230}+a_{211}\omega^2}{a_{230}(a_{220}+a_{230})},\\
a_{332} &= -\frac{a_{121}a_{211}+a_{131}a_{211}+a_{220}a_{222}+a_{222}a_{230}}{a_{220}+a_{230}},\\
a_{112} &= \frac{a_{121}a_{211}+a_{131}a_{211}}{a_{220}+a_{230}}.
\end{aligned}$$

We compute the averaged function of third order $f_3(r,w)=(f_{31}(r,w),f_{32}(r,w))$ and we get again the expressions given in (10), of course the coefficients a_j's and b_j's are now different. Repeating the arguments of the previous subcase we obtain at most four zeros for the averaged function of third order $f_3(r,w)$.

Case 3. Again since the averaged function of first order $f_1(r,w)$ is identically zero, we compute the averaged function of second order $f_2(r,w)=(f_{21}(r,w),f_{22}(r,w))$ and we obtain

$$f_{21}(r,w)=(Cw+D)r, \qquad f_{22}(r,w)=Ew,$$

where

$$\begin{aligned}
C = & -\frac{a_{310}(a_{121}-a_{221})a_{230}}{2\,\omega^3},\\
D = & -\frac{1}{2a_{230}\omega^5}\big(a_{121}a_{230}^3a_{310}a_{321}-a_{131}a_{221}a_{230}a_{310}\omega^2+a_{122}a_{230}^2a_{310}\omega^2+\\
& a_{121}a_{230}^2a_{311}\omega^2-a_{131}a_{211}\omega^4-a_{222}a_{230}\omega^4-a_{230}a_{332}\omega^4\big),\\
E = & \frac{1}{a_{230}\omega^5}\big(a_{121}a_{230}^3a_{310}a_{321}-a_{131}a_{221}a_{230}a_{310}\omega^2+a_{122}a_{230}^2a_{310}\omega^2+\\
& a_{121}a_{230}^2a_{311}\omega^2-a_{131}a_{211}\omega^4+a_{112}a_{230}\omega^4\big).
\end{aligned}$$

As in the previous case the unique zero of the averaged function of second order $f_2(r,w)$ is the $(0,0)$ or a continuum of solutions in case that convenient coefficients C, D or E are zero. Consequently we impose that the averaged function of second order $f_2(r,w)$ be identically zero, but since the coefficient of rw in the function $f_{21}(r,w)$ is a product of two factors which can be zero, namely $a_{310}(a_{121}-a_{221})$, we have two consider two subcases.

Subcase 3.1: $a_{310} = 0$. Then in order that the averaged function of second order $f_2(r, w)$ be identically zero we take

$$a_{332} = \frac{a_{121}a_{230}^2 a_{311} - a_{131}a_{211}\omega^2 - a_{222}a_{230}\omega^2}{a_{230}\omega^2},$$
$$a_{112} = \frac{-a_{121}a_{230}^2 a_{311} + a_{131}a_{211}\omega^2}{a_{230}\omega^2}.$$

We compute the averaged function of third order $f_3(r, w) = (f_{31}(r, w), f_{32}(r, w))$ and we get again the expression given in (10), consequently at most four solutions.

Subcase 3.2: $a_{221} = a_{121}$. Then in order that the averaged function of second order $f_2(r, w)$ be identically zero we take

$$\begin{aligned} a_{332} = {} & \frac{1}{a_{230}\omega^2}\big(a_{121}a_{230}^3 a_{310}a_{321} - a_{121}a_{131}a_{230}a_{310}\omega^2 + a_{122}a_{230}^2 a_{310}\omega^2 + \\ & a_{121}a_{230}^2 a_{311}\omega^2 - a_{131}a_{211}\omega^4 - a_{222}a_{230}\omega^4\big), \\ a_{112} = {} & \frac{1}{a_{230}\omega^4}\big(a_{121}a_{131}a_{230}a_{310}\omega^2 - a_{121}a_{230}^3 a_{310}a_{321} - a_{122}a_{230}^2 a_{310}\omega^2 - \\ & a_{121}a_{230}^2 a_{311}\omega^2 + a_{131}a_{211}\omega^4\big). \end{aligned}$$

We compute the averaged function of third order $f_3(r, w) = (f_{31}(r, w), f_{32}(r, w))$ and we get

$$f_{31}(r, w) = \frac{a_0 r^4 w + a_1 r^4 + a_2 r^2 w^2 + a_3 r^3 + a_4 r^2 w + a_5 w^3 + a_6 r^2 + a_7 r w + a_8 w^2}{384 a_{230}^2 \omega^9 r},$$
$$f_{32}(r, w) = \frac{b_0 r^3 + b_1 r^2 w + b_2 r^2 + b_3 r w + b_4 w^2 + b_5 r + b_6 w}{24 a_{230}^2 \omega^7}.$$

Here the explicit expressions of the coefficients a_j's and b_j's only should need approximately three pages for writing them. But unfortunately in this case we do not know how to control the zeros (r^*, w^*) of the function $f_3(r, w)$ with $r^* > 0$. We think that in this subcase it is possible that more than six simple zeros can be obtained, but for the moment this is an open problem.

Appendix: The averaging theory of first, second and third order

The averaging theory of third order for studying periodic orbits was developed [2] and in [19] at any order. It can be summarized as follows.

Consider the differential system

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 R(t, x, \varepsilon), \tag{11}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t, \cdot) \in C^2(D)$, $F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1$, $D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2, 3$ as

$$\begin{aligned} f_1(x) = &\ \frac{1}{T}\int_0^T F_1(s,x)ds,\\ f_2(x) = &\ \frac{1}{T}\int_0^T \left[D_x F_1(s,x)\cdot y_1(s,x) + F_2(s,x)\right] ds,\\ f_3(x) = &\ \frac{1}{T}\int_0^T \Big[\frac{1}{2} y_1(s,x)^T \frac{\partial^2 F_1}{\partial x^2}(s,x) y_1(s,x) + \frac{1}{2}\frac{\partial F_1}{\partial x}(s,x) y_2(s,x)\\ &+\frac{\partial F_2}{\partial x}(s,x) y_1(s,x) + F_3(s,x)\Big] ds, \end{aligned}$$

where

$$\begin{aligned} y_1(s,x) &= \int_0^s F_1(t,x)dt,\\ y_2(s,x) &= \int_0^s \left[\frac{\partial F_1}{\partial x}(t,x)\int_0^t F_1(r,x)dr + F_2(t,x)\right] dt. \end{aligned}$$

(ii) For an open and bounded set $V \subset D$ and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\setminus\{0\}$, there exists $a \in V$ such that $f_1(a) + \varepsilon f_2(a) + \varepsilon^2 f_3(a) = 0$ and $d_B(f_1 + \varepsilon f_2 + \varepsilon^2 f_3, V, a_\varepsilon) \neq 0$ (i.e. the Brouwer degree of the function $f_1 + \varepsilon f_2 + \varepsilon^2 f_3$ at the point a is not zero).

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $x(t, \varepsilon)$ of system (11) such that $x(0, \varepsilon) \to a$ when $\varepsilon \to 0$.

A sufficient condition in order that $d_B(f_1 + \varepsilon f_2 + \varepsilon^2 f_3, V, a_\varepsilon) \neq 0$ is that the Jacobian of the function $f_1 + \varepsilon f_2 + \varepsilon^2 f_3$ at a is not zero, see for details [21].

The *averaging theory of first order* takes place when f_1 is not identically zero. Therefore the zeros of $f_1 + \varepsilon f_2 + \varepsilon^2 f_3$ are mainly the zeros of f_1 for ε sufficiently small.

The *averaging theory of second order* takes place when f_1 is identically zero and f_2 is not identically zero. Then the zeros of $f_1 + \varepsilon f_2 + \varepsilon^2 f_3$ are mainly the zeros of f_2 for ε sufficiently small.

Finally the *averaging theory of third order* takes place when f_1 and f_2 are identically zero and f_3 is not identically zero. Therefore the zeros of $f_1 + \varepsilon f_2 + \varepsilon^2 f_3$ are mainly the zeros of f_3 for ε sufficiently small.

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E-mail address: mahan@shnu.edu.cn

E-mail address: jllibre@mat.uab.cat

E-mail address: ytian22@shnu.edu.cn