THE LOCAL CYCLICITY PROBLEM. MELNIKOV METHOD USING LYAPUNOV CONSTANTS

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Abstract. In 1991, Chicone and Jacobs proved the equivalence between the computation of the first order Taylor developments of the Lyapunov constants and the developments of the first Melnikov function near a monodromic equilibrium point, in the study of limit cycles of small amplitude bifurcating from a quadratic center. We show that their proof is also valid for polynomial vector fields of any degree. This equivalence is used to provide a new lower bound for the local cyclicity of degree six polynomial vector fields, $M(6) \geq 44$. Moreover, we extend this equivalence to the piecewise polynomial class. Finally, we prove that $M_p^c(4) \geq 43$ and $M_p^c(5) \geq 65$.

1. Introduction

In the last century, Hilbert presented a list of problems that almost all of them are solved. One problem that remains open consists in determining the maximal number $H(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree n. This problem is known as the second part of the 16th Hilbert's problem. In the year of 1977, Arnol'd in [4] proposed a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonian systems.

In this work, we are interested in another local version, that consists in providing the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary center or an elementary focus, clearly $M(n) \leq H(n)$. In other words, $M(n)$ is an upper bound of the cyclicity of such equilibrium points. For more details, see [44]. For $n = 2$, Bautin proved that $M(2) = 3$, see [5]. For $n = 3$, the family of cubic systems without quadratic terms was studied in [6, 46]. The proof of $M_h(3) = 5$ was done in [50]. Żołądek in [51], show the first evidence that $M(3) \geq 11$. However, this problem was recently revisited by himself in [53]. The first proof of this fact was done by Christopher in [13], studying first order perturbations of another cubic center also provided by Żołądek in [52].

In 2012, Giné conjectures that $M(n) = n^2 + 3n - 7$, see [24, 25]. This suggests a higher value for $M(n)$ for polynomial vector fields of low degree. Gouveia and Torregrosa in [28] show that $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$ and $M(9) \geq 88$. The first evidence that this conjecture fails is given in [49]. Recently, a correct proof that $M(3) \geq 12$ is provided by Giné, Gouveia and Torregrosa in [26]. Moreover, in the same paper, they show that $M(4) \geq 21$. For $n = 6$, the best lower bound was given in [38], proving that $M(6) \geq 40$. The first main result of this paper updates this value.

Theorem 1.1. *The local cyclicity of monodromic equilibrium points for polynomial vector fields of degree $n = 6$ is greater or equal than 44. That is, $M(6) \geq 44$.*

In this work, we are also interested in piecewise polynomial vector fields. Andronov, in [3] was the first studying such class of systems. In the last years, they have been widely studied, since many problems of engineering, physics, and biology can be modeled

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