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Limit cycles of small amplitude in polynomial and piecewise polynomial planar vector fields

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December 2019

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Summary

David Hilbert in the year 1900, in the International Congress of Mathematics proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. Among these problems, one is linked with the study of ordinary differential equations. The 16th Hilbert problem, whose second part asking about the maximum number and the relative position of the isolated periodic orbits, also called limit cycles, of a planar polynomial system in function of its degree n. Until nowadays, the 16th Hilbert problem remain unsolved. Over the years and without one solution, weaker versions began to emerge to 16th Hilbert problem. We are interested here in one of them, that consist in to provide the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary center or an elementary weak-focus.

In order to help to solve this problem, our contribution in this thesis is offer a mechanism that simplifies the calculation of the Taylor developments of the Lyapunov constants and to present a theory that help us to use the constants obtained for classical differential system to the study of lower bounds for the value $M(n)$. We dedicate part of this work to study the same problem to piecewise systems. In this work, we consider fixed vector fields and we present the parallelization tool that will help us to calculate high order Taylor developments of Lyapunov constants near a center different from the linear one and get some results about how to obtain limit cycles using these developments. Moreover, we consider a family of vector fields and we present a result that allows us to get k extra limit cycles if the unperturbed system has a center having k free parameters. For piecewise systems, we consider again fixed vector fields and using parallelization, we were able to calculate the necessary Lyapunov constants for cubic and quartic systems to improve lower bounds of limit cycles. We prove that $M(3)$ and $M(4)$ are bigger than or equal to 12 and 21, respectively. Moreover, we prove that if an analytic piecewise system has weak-focus or order $2n+1$, we can unfold the total number of limit cycles perturbing in the analytic piecewise class. This result is a natural extension of the classical result showed by Andronov for analytic systems. Moreover, using the equivalence among Lyapunov constants and Melnikov functions, we improve also the lower bounds for the known values of the local cyclicity for sextic vector fields.

Introduction

From the moment that the human started to be aware of the natural events around him, the humanity sought to understand such events. In addition, it also find a way to predict them. Perhaps the most basic problem representing these phenomena is the rain cycle. Trying to understand the period of greatest rainfall would be useful to have a better and bigger planting. Taking this into account, the mathematics is, without any doubt, the basic language that describes natural events. For those who have faith, the mathematics is the language that God used to create the universe and its laws. There is evidence that mathematics started around 1900 BC and until nowadays there is no signs that it is near its end. In the 17th century, Isaac Newton and Gottfried Leibniz introduced the differential calculus. With this new approach, some phenomena of nature started to gain greater understanding, because it proved to be an important tool to model, in an abstract language, what occurs in the real world over of time. This created the pillars of what would be the study of ordinary differential equations.

We can write ordinary differential equations in the form

$$F(t, x, x', x'', \ldots, x^{(n)}) = 0, \tag{1}$$

where $x^{(n)}$ denote the $n-$th derivative of x with respect to t. When F not depends of t, we say that system is autonomous. If x is a vector instead of a real function, equation (1) is called a differential system. Many problems can be modeled by ordinary differential equations. We can cite the problem of n-bodies that was modeled by Newton, the problem prey-predator modeled by Vito Volterra and Alfred Lotka in 1925.

Almost two centuries later, the study of differential system gets a new approach with Henry Poincaré in "Mémoire sur les courbes définies par une équation différentielle". Here, Poincaré introduces a more qualitative study on ordinary differential equations. Using geometric and topological techniques, Poincaré was able to investigate qualitative properties of the solutions of a differential equation without such solutions having to be determined explicitly. Among the contributions of Poincaré, we can mention the concept of phase portrait, the concepts such as return map or the Annular Region Theorem, which are fundamental for classifying orbits with particular behaviors. These results would be the pillars of **Qualitative Theory of Differential Equations**.

The notion of limit cycle was also introduced by Poincaré that defines a limit cycle as a periodic orbit such that at least one trajectory of the vector field, approaches in positive or negative time. Usually, when the vector field is of class $\mathcal{C}^1$ an alternative definition is given. A limit cycle is a closed orbit isolated from the other periodic orbits. Years later, in the early twentieth century, a swedish mathematician named Ivar Otto Bendixson presents a result showing that the principal solutions are called singular or minimal sets (critical points, periodic orbits and separatrix) defined a differential equation on a compact set has the property that the other solution goes to a singular solution. This results would come to be known as Poincaré–Bendixon Theorem. Stimulate by this result, Lyapunov studied the behavior of solutions in a neighborhood of an equilibrium position. Because of his work, Lyapunov is will know as the founder the modern theory of stability of motion.

In this work, let is consider a first-order autonomous planar differential systems in the form

$$\left\{\begin{array}{rcl}\dot{x} & = & X(x(t), y(t)),\\ \dot{y} & = & Y(x(t), y(t)),\end{array}\right. \tag{2}$$

where $x(t), y(t), X(x, y)$ and $Y(x, y)$ are real functions and the dot means the derivative with respect to the time t.

David Hilbert in the year 1900, in the International Congress of Mathematics proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. Among these problems, one is linked to the study of differential equations. The 16th Hilbert problem, whose second part asking about the maximum number (by convention this number is called $H(n)$) and the position of the limit cycles of a polynomial planar system in function of its degree, that is, a system like (2) with X and Y polynomials of degree n. Until nowadays, the 16th Hilbert problem remain unsolved, even for the simplest case $n = 2$.

Henri Dulac, in 1923 took the first steps in the direction of 16th Hilbert problem. His work goes in the direction of proving the finitude of the number of limit cycles in a polynomial vector field in the plane. In 1970, Yulij Ilyashenko observed that the proof given by Dulac was false. Some years later and independently, Ilyashenko and Écalle provided a correct proof. Although the proof given by Dulac was wrong, the ideas given by him were very fruitful and generated results like the classical Dulac Theorem and its generalization, known as the Bendixon–Dulac Theorem.

During the last decades many mathematicians have contributed to better understand 16th Hilbert problem. We highlight the works of A. Andronov, C. Christopher, F. Dumortier, J. Écalle, J.P. Françoise, A. Gasull, J. Giné, Y. Ilyashenko, J. Llibre C. Li, M. Peixoto, R. Roussarie, J. Sotomayor, J. Torregrosa, A. Varchenko, Y. Ye, Z. Zhang, H. Zoladek.

Over the years and without one solution, weaker versions began to emerge to 16th Hilbert problem. One of them is the so-called Arnold–Hilbert problem, however it is still unsolved. Arnold–Hilbert problem says that if H, P and Q be polynomials of degree n and V an inverse integrating factor, given $\Gamma(h)$ a level curve $\{H(x,y)=h\}$ of the system

$$\begin{cases} \dot{x} &= -\dfrac{\partial H}{\partial y} + \varepsilon P(x,y,\varepsilon,\lambda), \\ \dot{y} &= \dfrac{\partial H}{\partial x} + \varepsilon Q(x,y,\varepsilon,\lambda), \end{cases}$$

and given

$$\mathcal{M}(h) = \int_{\Gamma(h)} \frac{Q(x,y,0,\lambda)dx - P(x,y,0,\lambda)dy}{V(x,y)},$$

what is the number of zeros of $\mathcal{M}(h)$? The function $\mathcal{M}(h)$ is known as Abelian integral or Melnikov's function. The maximum number of simple zeros of $\mathcal{M}(h)$ is also closed to two related problems: the highest multiplicity of a weak-focus and the maximal cyclicity (the maximum number $M(n)$ of small limit that we get from an equilibrium point by a given polynomial perturbation) of an equilibrium point. Clearly $M(n) \leq H(n)$. In this work, we are interested in this version of the problem. For $n = 2$, Bautin proved that $M(2) = 3$. Sibirskii proved that for cubic systems without quadratic terms there are no more than five limit cycles bifurcating from one critical point. In fact these are the unique general families for which this local number is completely determined. The first evidence that $M(3) \geq 11$ was presented by Zoladek in 1995. Recently, Giné, conjectures that $M(n) = n^2 + 3n - 7$. This suggests a high value for $M(n)$ for polynomial vector fields of lower degree. For degree $n = 5, 7, 8, 9$ the best lower bounds for $M(n)$ until now were obtained by Liang and Torregrosa providing examples exhibiting 28, 54, 70, and 88 limit cycles of small amplitude, respectively.

For the reader to get an idea of the difficulty to solve 16th Hilbert problem, there is another version more restricted, which consists in determining the number $H(n)$ but for the Liénard family

$$\begin{cases} \dot{x} &= y - F(x), \\ \dot{y} &= -x, \end{cases}$$

where F is a real polynomial of degree n and $F(0) = 0$. This weaker version is still unsolved.

One way to approach Arnold–Hilbert problem is using Lyapunov constants. From the study of the return map, Liapunov consider the importance of the terms of the series expansion of this application. The problem with this approach, is the difficult of calculations. In order to help to solve this problem, our contribution in this thesis is offer a mechanism that simplifies the calculation of the Taylor developments of the Lyapunov constants and to present a theory that help us to use the constants obtained for classical differential system to study lower bounds for $M(n)$. Moreover, we improve the known values of $M(n)$ for $3 \le n \le 9$.

We dedicate a part of this work to study the Arnold–Hilbert problem to piecewise systems. The study of piecewise linear systems started by Andronov and has been widely studied in the last years, since many problems of engineering, physics, economy and biology can be modeled by such systems. One of the most studied problem is given by two vector fields defined in two hapf-planes separated by a straight line. Moreover, a large set of classical theorems are not satisfied by the piecewise systems. Among others, we can cite the Existence and Uniqueness Theorem and the Poincaré–Bendixson Theorem.

In this work, we are interested in the study of limit cycles of small amplitude bifurcating from the origin, for piecewise differential equations of the form

$$\begin{cases} (x', y') = (P^+(x, y, \lambda), Q^+(x, y, \lambda)), \text{ when } y \ge 0, \\ (x', y') = (P^-(x, y, \lambda), Q^-(x, y, \lambda)), \text{ when } y < 0, \end{cases}$$

with $P^{\pm}(x, y, \lambda)$ and $Q^{\pm}(x, y, \lambda)$ are polynomials. The straight line $\Sigma = \{y = 0\}$ divides the plane in two half-planes $\Sigma^{\pm} = \{(x, y) : \pm y > 0\}$ and the trajectories on Σ are defined following the Filippov convention. We call of $M_p^c(n)$ the maximum number of limit cycles bifurcating from a monodromic singular point and $H_p^c(n)$ the maximum number of limit cycles of polynomial piecewise systems of degree n. Clearly $M_p^c(n) \le H_p^c(n)$. It is well-know that linear systems have no limit cycles, so $H(1) = M(1) = 0$. This is not the case for piecewise linear systems defined in two zones separated by a straight line. There are works showing $H_p^c(1) \ge 3$. For quadratic vector fields is also well known that $H(2) \ge 4$. But for piecewise quadratic systems there are few works providing good lower bounds. Using averaging theory of order five, and perturbing the linear center, Llibre and Tang in proved that $H_p^c(2) \ge M_p^c(2) \ge 8$. Recently, da Cruz, Novaes and Torregrosa provide a better lower bound, $H_p^c(2) \ge M_p^c(2) \ge 16$. The best known lower bound for the number of limit cycles in cubic systems is $H(3) \ge 13$,. For piecewise cubics a recent work provides $H_p^c(3) \ge 18$ in two nests of nine limit cycles each.

The work has been developed in collaboration with Joan Torregrosa, and it is structured in an introduction and then four chapters where the results and proofs are developed. As it is explained in the title, the main results are concerning to limit cycles of small amplitude for differential and piecewise differential systems in the plane.

In Chapter 1, considering fixed vector fields, we present the concept of Lyapunov constants, the Parallelization tool that will help us to calculate high order Taylor developments of Lyapunov constants near a center different from the linear one and get some results about how to obtain limit cycles using these equations. With these tools, we present a new cubic system having also 11 limit cycles of small amplitude and we have improved the valuee for $M(n)$ for $4, 5, 7, 8$ and 9.

In Chapter 2, considering a family of vector fields, we present a theorem that allows us to get k extra limit cycles if the unperturbed system has a center having k free parameters. Using this result, we show that $M(3) \geq 12$ and $M(4) \geq 21$. We present also two new families of cubic vector fields such that $M(3) \geq 11$. This chapter has been done in collaboration with Jaume Giné.

In Chapter 3, again considering fixed vector fields, we dedicate our effort to cyclicity but in piecewise systems. Using also Parallelization, we were able to calculate the necessary Lyapunov constants for cubic and quartic systems to show that $M_p^c(3) \geq 26$ and $M_p^c(4) \geq 40$. Moreover, we prove that if an analytic piecewise system has weak-focus or order $2n+1$, we can unfold the total number of limit cycles perturbing in the analytic piecewise class. This result is a natural extension of the classical result showed by Andronov for polynomial systems.

In Chapter 4, using the equivalence among Lyapunov constants and Melnikov functions, we improve that $M(n)$ for $n = 6$. Moreover, we also extend this result to piecewise systems.

Finally, we dedicate the last chapter to conclusions of this work and future works.

We notice that all our calculations were made using the Computer Algebra System MAPLE on a cluster with 9 machines that have 128 CPUs with 725 MB of ram memory.