FLOW CURVATURE MANIFOLD AND ENERGY OF GENERALIZED LIÉNARD SYSTEMS

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ABSTRACT. In his famous book entitled *Theory of Oscillations*, Nicolas Minorsky wrote: "*each time the system absorbs energy the curvature of its trajectory decreases* and *vice versa*". According to the *Flow Curvature Method*, the location of the points where the *curvature of trajectory curve*, integral of such planar *singularly dynamical systems*, vanishes directly provides a first order approximation in ε of its *slow invariant manifold* equation. By using this method, we prove that, in the ε-vicinity of the *slow invariant manifold* of generalized Liénard systems, the *curvature of trajectory curve* increases while the *energy* of such systems decreases. Hence, we prove Minorsky's statement for the generalized Liénard systems. Then, we establish a relationship between *curvature* and *energy* for such systems. These results are then exemplified with the classical Van der Pol and generalized Liénard *singularly perturbed systems*.

1. INTRODUCTION

At the end of the 1930s, a general equation of *self-sustained oscillations* (1) was stated by the French engineer Alfred Liénard [34]. It encompassed the prototypical equation of the Dutch physicist Balthasar Van der Pol [45] modelling the so-called *relaxation oscillations*[1].

$$\frac{d^2x}{dt^2} + \omega f(x) \frac{dx}{dt} + \omega^2 x = 0. \tag{1}$$

Less than fifteen years later, a more general form was provided by the American mathematicians Norman Levinson and his former student Oliver K. Smith [32]:

$$\frac{d^2x}{dt^2} + \mu f(x) \frac{dx}{dt} + g(x) = 0. \tag{2}$$

At that time, the classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [44] and Levinson [33] stated that *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. These manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds*[2]. Then, Fenichel [5, 6, 7, 8]

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[1] For more details see J.-M. Ginoux [17].

[2] In other articles the *slow manifold* is the approximation of order $O(\varepsilon)$ of the *slow invariant manifold*.