DISCRETE MELNIKOV FUNCTIONS

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Abstract. We consider non-autonomous N-periodic discrete dynamical systems of the form $r_{n+1} = F_n(r_n, \varepsilon)$, having when $\varepsilon = 0$ an open continuum of initial conditions such that the corresponding sequences are N-periodic. From the study of some variational equations of low order we obtain successive maps, that we call discrete Melnikov functions, such that the simple zeroes of the first one that is not identically zero control the initial conditions that persist as N-periodic sequences of the perturbed discrete dynamical system. We apply these results to several examples, including some Abel-type discrete dynamical systems and some non-autonomous perturbed globally periodic difference equations.

1. Introduction and statement of the main results

The interest on the study non-autonomous periodic discrete dynamical systems has been increasing in the last years, among other reasons, because they are good models for describing the dynamics of biological and ecological systems that vary periodically, either due to external disturbances or for effects of seasonality, see for instance [2, 11, 12, 13, 15, 16, 17] and the references therein.

Consider non-autonomous discrete dynamical systems of the form

$$r_{n+1} = f_n(r_n), \qquad r_n \in \mathbb{R}^d,\ n \in \mathbb{N}, \tag{1}$$

where $d \in \mathbb{N}^+$, and f_n is an N-periodic sequence of real smooth invertible such that $f_m = f_{m+N}$ for all $m \in \mathbb{N}$. Here, $f_n : \mathcal{U} \subset \mathbb{R}^d \to \mathcal{U}$ being $\mathcal{U}$ an open set of $\mathbb{R}^d$. Given an initial condition $r_0 = \rho \in \mathcal{U}$ we will denote by $r_n = \varphi_n(\rho)$ the sequence defined by (1). For convenience, for $n > 0$, we write $f_{n,n-1,\ldots,1,0} = f_n \circ f_{n-1} \cdots \circ f_1 \circ f_0$. Then, for $n > 0$,

$$r_n = \varphi_n(\rho) = f_{n-1,n-2,\ldots,1,0}(\rho). \tag{2}$$

It is well-known that given an N-periodic discrete dynamical system (1), it can be understood via the so called *composition map* $f_{N-1,N-2,\ldots,1,0}$. For instance, if all maps share a common fixed point, the nature of this steady state point can be studied through the nature of this fixed point for $f_{N-1,N-2,\ldots,1,0}$, see [2, 6]. Similarly, the attractor of a periodic discrete dynamical system (1) is the union of attractors of some composition maps, see [13, Thms. 3 and 6].

To find N-periodic solutions of (1) is equivalent to find solutions of $\varphi_N(\rho) = \rho$, or equivalently of the equation $f_{N-1,N-2,\ldots,1,0}(\rho) = \rho$. Usually, is not easy to deal with it. The main goal of this paper is to give an alternative and indirect mechanism to study this problem for a special class of discrete dynamical systems. More specifically, it is

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