PARRONDO'S PARADOX FOR HOMEOMORPHISMS

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Abstract. We construct two planar homeomorphisms f and g for which the origin is a globally asymptotically stable fixed point whereas for $f \circ g$ and $g \circ f$ the origin is a global repeller. Furthermore, the origin remains a global repeller for the iterated function system generated by f and g where each of the maps appears with a certain probability. This planar construction is also extended to any dimension greater than 2 and proves for first time the appearance of the Parrondo's dynamical paradox in odd dimensions.

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1. Introduction and main results

The *Parrondo's paradox* is a well-known paradox in game theory, that in a few words affirms that *a combination of losing strategies can become a winning strategy,* see [9, 11]. In the dynamical context, when we study the stability of fixed points, the role of being a winning or a losing strategy can be replaced by being an attracting or repelling fixed point. A word of caution, throughout this note we use the term attracting (or attractor) and repelling (or repeller) as a synonym of asymptotically stable for a map and its inverse, respectively. Hence, for a fixed class of maps $\mathcal{C}$, from $\mathbb{R}^k$ into itself, we will say that a pair of maps $f, g \in \mathcal{C}$ exhibit a *dynamical Parrondo's paradox* if they have a common fixed point at which the maps are locally invertible and the fixed point is locally asymptotically stable for f and g but it is a repeller for the composite maps $g \circ f$ and $f \circ g$. Notice that, $g \circ f$ and $f \circ g$ are conjugate near the fixed point because, locally, $f \circ g = g^{-1} \circ g \circ f \circ g$.

As shown in [5], the dynamical Parrondo's paradox can arise when k is even and $\mathcal{C}$ is the class of polynomial maps. On the contrary, it is also proved in [5] that the paradox does not appear when $k = 1$ and $\mathcal{C}$ is the class of analytic maps. Notice also that the paradox is also impossible for any k when $\mathcal{C}$ is the class of maps for which the common fixed point is hyperbolic. Indeed, given two $k \times k$ matrices with all their eigenvalues with modulus smaller than 1, A and B, it holds that $|\det(A)| < 1, |\det(B)| < 1$ and hence $|\det(AB)| < 1$. As a consequence, when two maps share a common fixed point $\mathbf{x}$ which is asymptotically stable for both of them and the maps are of class $\mathcal{C}^1$ at $\mathbf{x}$, then, generically, for $g \circ f$ and $f \circ g$ the fixed point $\mathbf{x}$ is, in both cases, either locally asymptotically stable or of saddle type, but it can never be repeller. Examples of saddle type points for $g \circ f$, when f and g are linear maps, are given in [3] and [10, p. 8].

For the sake of completeness, and to compare it with our result, we recall the example in [5, Ex. 7] for $k = 2$,

$$f(x, y) = \big(-y + 2x^2 + 6xy, x - 3x^2 + 2xy + 3y^2 \big),$$
$$g(x, y) = \big(x/2 - \sqrt{3}y/2 - x\,(x^2 + y^2), \sqrt{3}x/2 + y/2 - y\,(x^2 + y^2) \big).$$

It can be proved that the origin is a locally asymptotic stable fixed point for f and g and the origin is a repelling fixed point for $g \circ f$ by computing the so called Birkhoff stability constants for the three maps. Notice that the dynamics near the fixed points is of rotation type. Taking the product of these maps j times with themselves we trivially obtain examples of pairs of maps exhibiting the dynamical Parrondo's paradox for all $k = 2j$.