EFFECTIVENESS OF THE BENDIXSON–DULAC THEOREM

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Abstract. We illustrate with several new applications the power and elegance of the Bendixson–Dulac theorem to obtain upper bounds of the number of limit cycles for several families of planar vector fields. In some cases we propose to use a function related with the curvature of the orbits of the vector field as a Dulac function. We get some general results for Liénard type equations and for rigid planar systems. We also present a remarkable phenomenon: for each integer $m \geq 2$, we provide a simple 1-parametric differential system for which we prove that it has limit cycles only for the values of the parameter in a subset of an interval of length smaller that $3\sqrt{2}(3/m)^{m/2}$ that decreases exponentially when m grows. One of the strengths of the results presented in this work is that although they are obtained with simple calculations, that can be easily checked by hand, they improve and extend previous studies. Another one is that, for certain systems, it is possible to reduce the question of the number of limit cycles to the study of the shape of a planar curve and the sign of an associated function in one or two variables.

1. Introduction

Despite all the efforts dedicated to solve the second part of the Hilbert's 16th problem, it is yet a very difficult task to obtain criteria that give explicit upper bounds for many concrete families of planar smooth vector fields. Although there is no any universal approach, the aim of this paper is to present several families of planar systems for which the Bendixson–Dulac theorem allows to get, in a fast and elegant way, an upper bound of their number of limit cycles. We will avoid results based on cumbersome computations.

The families that we will consider include extensions of Liénard systems and rigid systems. As we will see, we obtain new results and we also present simple proofs of some recent results. They give explicit upper bounds for several families of planar vector fields. These upper bounds are also sharpened when we deal with more particular systems, obtaining results of at most two, one, or none limit cycles.

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