SOLVING POLYNOMIALS WITH ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this work we consider a given root of a family of n-degree polynomials as a one-variable function that depends only on the independent term. Then we prove that this function satisfies several ordinary differential equations (ODE). More concretely, it satisfies several simple separated variables ODE, a first order generalized Abel ODE of degree $n-1$ and an $(n-1)$-th order linear ODE. Although some of our results are not new, our approach is simple and self-contained. For $n = 2, 3$ and 4 we recover, from these ODE, the classical formulas for solving these polynomials.

1. Introduction and main results

It is known that although general polynomial equations of degree $n \geq 5$ can not be solved by radicals, their roots can be obtained in terms of elliptic or hyperelliptic functions, their inverses or other trascendental functions, like hypergeometric or theta functions. This is a classical subject which starts with results of Hermite, Kronecker and Brioschi and continues with contributions of many others authors, see for instance [1, 7, 9, 12] and the references therein. We will not try to survey all the different points of view from which the question of solving polynomials is addressed.

Because we usually work on ordinary differential equations (ODE) we simply decided to explore which kind of results about polynomial equations can be obtained by using ODE as a main tool. As we will see, our results are self-contained and recover some of the known results on the subject. Before we state our contributions and compare them with these known results, we present a brief survey of the most relevant results that we have found on this subject that also use ODE as a main tool. To the best of our knowledge this approach started with the contributions of Betti ([2]) in 1854 and the ones around 1860 of Cockle and Hartley ([4, 10]). In fact, Enrico Betti proved that the solutions of general polynomial equations satisfy a separated variables ODE and using this fact that he proved that the solutions of these equations can be obtained in terms of hyperelliptic functions and their inverses. He also proved that for quintic equations it suffices to consider elliptic functions and their inverses. On the other hand, James Cockle and Robert Harley showed explicit linear ODE satisfied for a solution of an arbitrary trinomial polynomial equation in terms of its coefficients. For instance, they found a linear homogeneous ODE of 4-th order for a solution $x(q)$ of the quintic polynomial equation in the Bring-Jerrard form $x^5 - x + q = 0$. These results are presented and extended a litle in the 1865 Boole's book [3, pp. 190–199]. In his Thesis ("première thèse"), published as a book

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