A CHEBYSHEV CRITERION WITH APPLICATIONS

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Abstract. We show that a family of certain definite integrals forms a Chebyshev system if two families of associated functions appearing in their integrands are Chebyshev systems as well. We apply this criterion to several examples which appear in the context of perturbations of periodic non-autonomous ODEs to determine bounds on the number of isolated periodic solutions, as well as to persistence problems of periodic solutions for perturbed Hamiltonian systems.

1. Main results

Chebyshev systems (T-systems), complete Chebyshev systems (CT-systems) and extended complete Chebyshev systems (ECT-systems) are the natural extensions of polynomials of a given degree m to more general functions. Notice that degree m polynomials can be seen as elements of the vector space $\langle 1, x, \ldots, x^m \rangle$ of dimension $m+1$, for which each element has at most m roots, counting multiplicities, such that this bound is attained. In the next section we give the precise definition of T, CT and ECT-systems, which essentially introduce them as vector spaces of functions satisfying these properties.

When studying perturbations of Hamiltonian systems with a continuum of periodic orbits, the level sets of the periodic orbits that persist as limit cycles are given by the zeroes of a line integral, or *Abelian integral*, see for instance [8]. A commonly used method to control the number of zeroes of such integrals when the perturbation depends on parameters is to prove that they form a basis which is a Chebyshev system. With this aim, a criterion was developed in [6] which shows that if some functions constructed from the integrands of the Abelian integrals form a Chebyshev system, then the Abelian integrals that generate the complete Abelian integral itself are a Chebyshev system as well. This is a powerful result since proving the Chebyshev property for functions is usually easier than to do so for functions defined as line integrals.

In the same spirit, the goal of the present paper is to prove that some definite integrals form a Chebyshev system if families of functions given by the integrands of these integrals are also Chebyshev systems. Our main result is:

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Theorem 1.1. *Consider a family of integrals of the form*

$$I_i(h) = \int_a^b f_i(x)g(h,x)dx,$$

where $a, b \in \mathbb{R}$ and $h \in L$ for $L \subset \mathbb{R}$ open, under the following hypothesis:

- *$f_1, \ldots f_n$ are continuous functions on the open interval (a,b) such that $(f_1, \ldots f_n)$ is a CT-system on (a,b).*
- *g is a analytic function on $L \times (a,b)$ such that a for any fixed set of n elements $\{x_1, \ldots, x_n\} \in [a,b]$ such that $x_i \neq x_j$ when $i \neq j$, the functions*
$$g_i(h) := g(h, x_i) \quad \text{for } i = 1, \ldots n.$$
are an ECT-system on L.

Then $(I_1, \ldots, I_n)$ is an ECT-system on L.

By relaxing the above hypotheses, in Theorem 2.4 below we give a similar result but proving that $(I_1, \ldots, I_n)$ is a CT-system on L. The following is a corollary of Theorem 1.1, proved in Section 3, which we will use in all our applications.

Theorem 1.2. *Let $i \in \mathbb{N}$, $a, b \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \mathbb{N}$, and let p be a monotone and continuous function on $[a,b]$. Denote by L the open interval given by the connected component of the set $\{h \in \mathbb{R} : 1 + hp(x) > 0 \text{ for all } x \in [a,b]\}$ which contains the origin. For $h \in L$, we consider the analytic functions*

$$J_i(h) = \int_a^b f_i(x)(1 + hp(x))^\alpha dx, \tag{1}$$

where $f_0, \ldots f_n$ are analytic on (a,b) such that $(f_0, \ldots f_n)$ is a CT-system in (a,b). Then $J_0, \ldots, J_n$ is an ECT-system in L.

Theorem 1.2 complements the main result of [4] where the family of functions

$$K_i(h) = \int_a^b q^i(x)(1 + hq(x))^\alpha dx,$$

is studied and it is characterized under which conditions it forms a Chebyshev family. In that paper, nothing is assumed on the monotonicity of the function q but, the functions f_i appearing in (1) are simply powers of q. Moreover, the proof given in [4] differs from our present approach and is based on the fact that the Wronskians introduced in Lemma 2.3 to prove that the functions form an *ECT*-system can be related to Gram determinants.

In Section 3 we apply Theorem 1.2 to study the number of zeroes of two families of functions. The first one is a Melnikov type function that controls the periodic solutions that bifurcate from a one-dimensional non-autonomous periodic differential equation of Abel type. The second one controls the periodic orbits that persist in the rotary regions by perturbing several Hamiltonian potential planar systems. Recall that these persistent periodic orbits are given by zeroes of Abelian integrals, see [8]. In particular, we apply this result to study some periodic perturbations of the pendulum and of the whirling pendulum when the constant rotation rate is smaller that a given value.

2. Preliminary results and proof of the main results

We start by recalling the definitions of T, CT and ECT systems, the notions of continuous and discrete Wronskian and a useful characterization of CT and ECT-systems. This result and much more information on the subject can be found in the monographs [7, 10].

Definition 2.1. *Let $f_1, \dots f_n$ be functions on an open interval I.*

(a) $(f_1, \dots f_n)$ is a Chebyshev system *(T-system) on I if any nontrivial linear combination*

$$\alpha_1 f_1(x) + \cdots + \alpha_n f_n(x)$$

has at most $n-1$ isolated zeros on I.

(b) $(f_1, \dots f_n)$ is a complete Chebyshev system *(CT-system) on I if $(f_1, \dots f_k)$ is a T-system for all $k = 1, 2, \dots n$.*

(c) $(f_1, \dots f_n)$ is an extended complete Chebyshev system *(ECT-system) on I if the functions $f_1, \dots f_n$ are analytic and, for all $k = 1, 2, \dots n$, any nontrivial linear combination*

$$\alpha_1 f_1(x) + \cdots + \alpha_k f_k(x)$$

has at most $k-1$ isolated zeros on I counting multiplicity.

Definition 2.2. *Let $f_1, \dots f_k$ be functions on an open interval I. The* discrete Wronskian *of $(f_1, \dots f_k)$ at $(x_1, \dots, x_k) \in I^k$ is*

$$D[f_1, \dots f_k](x_1, \dots, x_k) = det\Big(f_j(x_i) \Big)_{1 \le i,j \le k}$$

If $f_1, \dots f_k$ are $\mathcal{C}^k$ on I, the continuous Wronskian *of $(f_1, \dots f_k)$ at $x \in I$ is*

$$W[f_1, \dots f_k](x) = det\Big(f_j^{(i)}(x) \Big)_{1 \le i,j \le k}$$

For the sake of brevity we use the shorthand $\mathbf{x_k} = x_1, \dots, x_k$.

Lemma 2.3. *The following equivalences hold:*

(a) $(f_1, \dots f_n)$ is a CT-system on I if and only if for all $k = 1, 2, \dots n$

$$D[\mathbf{f_k}](\mathbf{x_k}) \neq 0 \text{ for all } \mathbf{x_k} \in I^k \text{ such that } x_i \neq x_j \text{ for } i \neq j.$$

(b) $(f_1, \dots f_n)$ is an ECT-system on I if and only if for all $k = 1, 2, \dots n$

$$W[\mathbf{f_k}](x) \neq 0 \text{ for all } x \in I.$$

The proofs of the following two results are inspired by the proof of Proposition 3.3 in [6].

Theorem 2.4. *Consider a family of integrals of the form*

$$I_i(h) = \int_a^b f_i(x) g(h, x) dx,$$

for $i = 1, \dots, n$, where $a, b \in \mathbb{R}$ and $h \in L$ for $L \subset \mathbb{R}$ open, under the following hypotheses:

- *$f_1,\dots f_n$ are continuous functions on the open interval $[a,b]$ such that $(f_1,\dots f_n)$ is a CT-system on (a,b).*
- *g is a function on $L\times(a,b)$ such that a for any fixed set of n elements $\{h_1,\dots,h_n\}\in L$ such that $h_i\neq h_j$ when $i\neq j$, the functions*

$$g_i(x):=g(h_i,x)\quad \textit{for } i=1,\dots n.$$

are a CT-system on (a,b).

Then $(I_1,\dots,I_n)$ is a CT-system on L

Proof. Let $\mathbf{h_n}=(h_1,\dots h_n)\in L^n$ such that $h_i\neq h_j$ when $i\neq j$. We need to show that for all $k\leq n$, $D[\mathbf{I_k}](\mathbf{h_k})\neq 0$. Fix $k\in\{1,\dots,n\}$, let S_k be the symmetric group of k elements and denote by Δ_k the k-simplex defined by $\{\mathbf{x_k}\in[a,b]^n : x_1<\dots<x_k\}$. Taking into account the definition of the determinant we have that

$$\begin{aligned}
D[\mathbf{I_k}](\mathbf{h_k}) &= \det\left(I_i(h_j)\right)_{1\leq i,j\leq k}=\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{i=0}^{k}I_i(h_{\sigma(i)})\\
&=\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left[\int_a^b f_1(x_1)g_{\sigma(1)}(x_1)dx_1\cdots\int_a^b f_k(x_k)g_{\sigma(k)}(x_k)dx_k\right]\\
&=\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left[\int_{[a,b]^k} f_1(x_1)\dots f_k(x_k)\cdot g_{\sigma(1)}(x_1)\dots g_{\sigma(k)}(x_k)d\mathbf{x_k}\right]\\
&=\int_{[a,b]^k} f_1(x_1)\dots f_k(x_k)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(x_1)\dots g_{\sigma(k)}(x_k)d\mathbf{x_k}.
\end{aligned}$$

Now we define for each permutation $\rho\in S_k$ the invertible mapping $\psi_\rho:\mathbb{R}^k\to\mathbb{R}^k$,

$$\psi_\rho(x_1,\dots,x_k)=(x_{\rho(1)},\dots,x_{\rho(k)}),$$

and note that

$$\bigcup_{\rho\in S_k}\psi_\rho(\Delta_k)=[a,b]^k\setminus\mathcal{R},$$

where $\mathcal{R}\subset\mathbb{R}^k$ is a set of Lebesgue measure zero. Therefore we can write

$$\begin{aligned}
D[\mathbf{I_k}](\mathbf{h_k}) &= \int_{[a,b]^k} f_1(x_1)\dots f_k(x_k)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(x_1)\dots g_{\sigma(k)}(x_k)d\mathbf{x_k}\\
&=\sum_{\rho\in S_k}\left[\int_{\psi_\rho(\Delta_k)} f_1(x_1)\dots f_k(x_k)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(x_1)\dots g_{\sigma(k)}(x_k)d\mathbf{x_k}\right].
\end{aligned}$$

The next step is to change coordinates in each integral of the above sum according to $\mathbf{x_k}=\psi_\rho(\mathbf{u_k})$, that is, $x_i=u_{\rho(i)}$ for $i=1,\dots,k$. In view of the fact that the absolute value of the determinant of the Jacobian of ψ_ρ is one, we find that

$$\begin{aligned}
D[\mathbf{I_k}](\mathbf{h_k}) &= \sum_{\rho\in S_k}\left[\int_{\psi_\rho(\Delta_k)} f_1(x_1)\dots f_k(x_k)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(x_1)\dots g_{\sigma(k)}(x_k)d\mathbf{x_k}\right]\\
&= \sum_{\rho\in S_k}\left[\int_{\Delta_k} f_1(u_{\rho(1)})\dots f_k(u_{\rho(k)})\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(u_{\rho(1)})\dots g_{\sigma(n)}(u_{\rho(k)})d\mathbf{u_k}\right]\\
&= \int_{\Delta_k}\sum_{\rho\in S_k}\Bigg[\operatorname{sgn}(\rho)f_1(u_{\rho(1)})\dots f_k(u_{\rho(k)})\times\\
&\qquad\qquad\times\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma\rho^{-1})g_{\sigma(1)}(u_{\rho(1)})\dots g_{\sigma(n)}(u_{\rho(k)})\Bigg]d\mathbf{u_k}\\
&= \int_{\Delta_k}\det\left(f_i(u_j)\right)_{1\le i,j\le k}\sum_{\rho,\sigma\in S_k}\operatorname{sgn}(\sigma\rho^{-1})g_{\sigma(\rho^{-1}(1))}(u_1)\dots g_{\sigma(\rho^{-1}(k))}(u_k)d\mathbf{u_k}\\
&= \int_{\Delta_k}\det\left(f_i(u_j)\right)_{1\le i,j\le k}\det\left(g_i(u_j)\right)_{1\le i,j\le k}d\mathbf{u_k},\\
&= \int_{\Delta_k} D[\mathbf{f_k}](\mathbf{u_k})D[\mathbf{g_k}](\mathbf{u_k})d\mathbf{u_k}.
\end{aligned}$$

Since both $(f_1,\dots,f_n)$ and $(g_1,\dots,g_n)$ are CT-systems on (a,b) the integrand in the last integral is different from zero. Since Δ_k is connected it follows that $\det(I_i(h_j))_{1\le i,j\le n}\neq 0$ and the proof is complete. □

Proof of Theorem 1.1. Now we need to prove that for any $k\le n$ and for any $h\in L$, $W[\mathbf{I_k}](h)\neq 0$. As in the previous proof fix $k\in\{1,\dots,n\}$, let S_k be the symmetric group of k elements and denote by Δ_k the k-simplex defined by $\{\mathbf{x_k}\in[a,b]^n : x_1<\dots<x_k\}$.Taking into account the definition of the determinant we have that

$$\begin{aligned}
W[\mathbf{I_k}](h) &= \det\left(I_i^{(j)}(h)\right)_{1\le i,j\le k} = \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{i=0}^{k} I_i^{(\sigma(i))}(h)\\
&= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left[\int_a^b f_1(x_1)g_{\sigma(1)}(h,x_1)dx_1\cdots\int_a^b f_k(x_k)g_{\sigma(n)}(h,x_k)dx_k\right]\\
&= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\int_{[a,b]^n} f_1(x_1)\dots f_k(x_k)\cdot g_{\sigma(1)}(h,x_1)\dots g_{\sigma(n)}(h,x_k)d\mathbf{x_k}\\
&= \int_{[a,b]^n} f_1(x_1)\dots f_k(x_k)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)g_{\sigma(1)}(h,x_1)\dots g_{\sigma(n)}(h,x_k)d\mathbf{x_k}.
\end{aligned}$$

We consider again for each permutation $\rho \in S_k$ the mapping $\psi_\rho : \mathbb{R}^k \to \mathbb{R}^k$, $\psi_\rho(\mathbf{x_k}) = \mathbf{x}_{\rho(\mathbf{k})}$ to write the last integral as

$$W[\mathbf{I_k}](h) = \int_{[a,b]^n} f_1(x_1)\dots f_k(x_k) \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) g_{\sigma(1)}(h,x_1)\dots g_{\sigma(n)}(h,x_k) d\mathbf{x_k}$$

$$= \sum_{\rho\in S_k}\left[\int_{\psi_\rho(\Delta_k)} f_1(x_1)\dots f_k(x_k) \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) g_{\sigma(1)}(h,x_1)\dots g_{\sigma(n)}(h,x_k) d\mathbf{x_k}\right].$$

The change coordinates $\mathbf{x_k} = \psi_\rho(\mathbf{u_k})$ in each integral of the above sum and the fact that the absolute value of the determinant of the Jacobian of ψ_ρ is one, yields

$$W[\mathbf{I_k}](h) = \sum_{\rho\in S_k}\left[\int_{\psi_\rho(\Delta_k)} f_1(x_1)\dots f_k(x_k)\times \right.$$

$$\left. \times \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) g_{\sigma(1)}(h,x_1)\dots g_{\sigma(k)}(h,x_k) d\mathbf{x_k}\right]$$

$$= \sum_{\rho\in S_k}\left[\int_{\Delta_k} f_1(u_{\rho(1)})\dots f_k(u_{\rho(k)}) \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) g_{\sigma(1)}(h,u_{\rho(1)})\dots g_{\sigma(k)}(h,u_{\rho(k)}) d\mathbf{u_k}\right]$$

$$= \int_{\Delta_k} \sum_{\rho\in S_k}\left[\operatorname{sgn}(\rho) f_1(u_{\rho(1)})\dots f_k(u_{\rho(k)})\times \right.$$

$$\left. \times \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma\rho^{-1}) g_{\sigma(1)}(h,u_{\rho(1)})\dots g_{\sigma(k)}(h,u_{\rho(k)})\right] d\mathbf{u}_{\rho(\mathbf{k})}$$

$$= \int_{\Delta_k} \det\left(f_i(u_j)\right)_{1\le i,j\le k} \sum_{\sigma,\rho\in S_k} \operatorname{sgn}(\sigma\rho^{-1}) g_{\sigma(\rho^{-1}(1))}(h,u_1)\dots g_{\sigma(\rho^{-1}(k))}(h,u_k) d\mathbf{u_k}$$

$$= \int_{\Delta_k} \det\left(f_i(u_j)\right)_{1\le i,j\le k} \det\left(g^{(i)}(h,u_j)\right)_{1\le i,j\le k} d\mathbf{u_k}$$

$$= \int_{\Delta_k} D[\mathbf{f_k}](\mathbf{u_k}) W[g(h,u_1),\dots,g(h,u_k)](h) d\mathbf{u_k}.$$

Since $(f_1,\dots,f_n)$ is a CT-system on (a,b) and $(g(h,u_1),\dots,g(h,u_n))$ is an ECT-system on L for any ordered set $\{u_1,\dots,u_n\}\in(a,b)$, the integrand in the last integral is different from zero. Therefore, $\det(I_i(h_j))_{1\le i,j\le n}\neq 0$ and the proof is complete. □

Proof of Theorem 1.2. For any fixed set of $n+1$ elements $\{x_0,\dots\dots x_n\}\in(a,b)$ with $x_i\neq x_j$ when $i\neq j$ let $a_i := 1/p(x_i)$ for all $i=0,\dots,n$. Notice that all such a_i are distinct since p is monotone on $[a,b]$. Define

$$g_i(h) := g(h,x_i) := (1+hp(x_i))^\alpha.$$

By Theorem 1.1, the proof will follow if we prove that $(g_i(h))_{i=0}^n$ is an ECT-system on L. To prove this fact, note first that

$$g_i(h) = (1+h/a_i)^\alpha = \frac{1}{a_i^\alpha}(a_i+h)^\alpha.$$

For $k \in \mathbb{N}$, we introduce the notation $[\alpha]_0 = 1$, $[\alpha]_k := \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$. Therefore, the Wronskian determinant

$$\det\left(g_i^{(j)}(h)\right)_{0\le i,j\le n} =$$

$$= \frac{1}{\Pi_{i=0}^n a_i^\alpha} \det \begin{pmatrix} (a_0+h)^\alpha & (a_1+h)^\alpha & \cdots & (a_n+h)^\alpha \\ [\alpha]_1(a_0+h)^{\alpha-1} & [\alpha]_1(a_1+h)^{\alpha-1} & \cdots & [\alpha]_1(a_n+h)^{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ [\alpha]_n(a_0+h)^{\alpha-n} & [\alpha]_n(a_1+h)^{\alpha-n} & \cdots & [\alpha]_n(a_n+h)^{\alpha-n} \end{pmatrix}$$

$$= \frac{\Pi_{i=0}^n [\alpha]_i}{\Pi_{i=0}^n a_i^\alpha} \det \begin{pmatrix} (a_0+h)^\alpha & (a_1+h)^\alpha & \cdots & (a_n+h)^\alpha \\ (a_0+h)^{\alpha-1} & (a_1+h)^{\alpha-1} & \cdots & (a_n+h)^{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ (a_0+h)^{\alpha-n} & (a_1+h)^{\alpha-n} & \cdots & (a_n+h)^{\alpha-n} \end{pmatrix}$$

$$= \frac{\Pi_{i=0}^n [\alpha]_i \left[\Pi_{i=0}^n (a_i+h)\right]^{n-\alpha}}{\Pi_{i=0}^n a_i^\alpha} \det \begin{pmatrix} (a_0+h)^n & (a_1+h)^n & \cdots & (a_n+h)^n \\ (a_0+h)^{n-1} & (a_1+h)^{n-1} & \cdots & (a_n+h)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

is the determinant of a Vandermonde matrix. Since all a_i are distinct, it is nonzero, and hence by Lemma 2.3, $(g_i(h))_{i=0}^n$ is an ECT-system on L as we wanted to prove. □

Notice that in the above proof we show in particular that the family of functions $(a_i+h)^\alpha$, with $i = 0, 1, \ldots, n$ and $\alpha \notin \mathbb{N}$ and all a_j distinct, is an ECT-system in a suitable interval. It is curious to observe that any permutation of this set of $n+1$ functions is also an ECT-system. This is an unusual property, which is not true for the ECT-system $(1, h, h^2, \ldots, h^n)$ for instance.

3. Applications

This section collects several applications of our main results.

3.1. **Perturbation of periodic non-autonomous ODEs.** Theorem 1.2 can be applied to determine upper bounds on the number of isolated periodic solutions obtained from the first order analysis for perturbations of certain 1-dimensional non-autonomous differential equations. Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y), \\ y(0) = h. \end{cases} \tag{2}$$

where f is real analytic and T-periodic with respect the first variable. Let $\varphi(x,h)$ be the solution of this problem and assume that it is T-periodic for all $h \in (h_1, h_2)$. Now for $g(x,y)$ also real analytic and T-periodic with respect the first variable we consider the perturbed problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) + \varepsilon g(x,y), \\ y(0) = h, \end{cases} \tag{3}$$

We will look for T-periodic orbits that *persist* after perturbation. Denote by $\psi(x,h,\varepsilon)$ the solution of the perturbed problem (3). Notice that $\psi(x,h,0) = \varphi(x,h)$. By similarity with the notation used when studying the perturbations of Hamiltonian systems, see [8], we will say that the periodic solution corresponding to $h = h^*$ persists if for ε small enough there exists h_ε such that $\psi(x,h_\varepsilon,\varepsilon)$ is T-periodic and $\lim_{\varepsilon\to 0} h_\varepsilon = h^*$. From the theorems on dependence of solutions on parameters we get that

$$\psi(x,h,\varepsilon) = \varphi(x,h) + m(x,h)\varepsilon + O(\varepsilon^2)$$

where

$$m(x,h) = e^{\int_0^x D_2(f(z,\varphi(z,h))dz} \times \int_0^x \frac{g(z,\varphi(z,h))}{e^{\int_0^z D_2(f(w,\varphi(w,h))dw}} dz.$$

So we have that

$$\frac{\psi(T,h,\varepsilon) - h}{\varepsilon} = m(T,h) + O(\varepsilon).$$

Therefore, by the implicit function theorem, simple zeros of $M(h) := m(T,h)$ give the values h^* for which one periodic orbit persists after bifurcation. Moreover, using Weierstrass's Preparation Theorem ([1, Thm 69]) it follows that if h^* is a zero of $M(h)$ with multiplicity k, then at most k periodic orbits bifurcate from the periodic orbit corresponding to $h = h^*$. So if $M(h)$ is not identically zero, its number of zeroes in (h_1,h_2), counted with multiplicity bounds the number of isolated periodic orbits that bifurcate from the continuum of periodic orbits of (2). Observe that from the theorem on dependence of initial conditions we have that

$$\frac{\partial\varphi(x,h)}{\partial h} = e^{\int_0^x D_2(f(z,\varphi(z,h))dz},$$

and since $\varphi(T,h) = h$ we get that $e^{\int_0^T D_2(f(z,\varphi(z,h))dz} = 1$ and

$$M(h) = \int_0^T \frac{g(x,\varphi(x,h))}{e^{\int_0^x D_2(f(z,\varphi(z,h))dz}} dx = \int_0^T \frac{g(x,\varphi(x,h))}{\partial\varphi(x,h)/\partial h}\, dx. \tag{4}$$

As a more concrete example, consider the problem

$$\begin{cases} \dfrac{dy}{dx} = \lambda(x)y^k + \varepsilon F(x)G(y), \\ y(0) = h, \end{cases} \tag{5}$$

which is of the form (3) with $k \in \mathbb{N}$, λ and F T-periodic and $\Lambda(x) = \int_0^x \lambda(s)ds$ such that $\Lambda(T) = 0$. Direct computations give that

$$\varphi(x,h) = h\left(1 + (1-k)\Lambda(x)h^{k-1}\right)^{\frac{1}{1-k}}. \tag{6}$$

Notice that the condition $\Lambda(T) = 0$ implies that a neighborhood of $y = 0$ is full of T-periodic solutions. Moreover,

$$\frac{\partial \varphi}{\partial h}(x,h) = \left(1 + (1-k)\Lambda(x)h^{k-1}\right)^{\frac{k}{1-k}} = \left(\frac{\varphi(x,h)}{h}\right)^k$$

and from (4) we get that

$$M(h) = h^k \int_0^T \frac{F(x)G(\varphi(x,h))}{\varphi^k(x,h)}dx. \tag{7}$$

Example 3.1. In our first example we revisit some results of [4], in which the number of isolated periodic solutions obtained from first order perturbations of some generalized Abel equations were obtained. The equations under consideration,

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{\cos x}{k-1} y^k + \varepsilon P_n(\cos x, \sin x) y^p, \\ y(0) = h, \end{cases}$$

are of the form (5) with $T = 2\pi$, $1 < k \in \mathbb{N}$, $k < p \in \mathbb{N}$ and P_n a degree n polynomial. Using (6) and (7), simple computations show that

$$M(h) = h^p \sum_{i+j=n} a_{i,j} \int_0^{2\pi} \sin^i x \cos^j x (1 - h^{k-1}\sin x)^{\frac{p-k}{1-k}} dx.$$

for some $a_{i,j} \in \mathbb{R}$. Since $\int_0^{2\pi} \sin^i x \cos^j x (1 - h^{k-1}\sin x)^{\frac{p-k}{1-k}} = 0$ when j is odd and

$$\int_0^{2\pi} \sin^i x \cos^j x (1 - h^{k-1}\sin x)^{\frac{p-k}{1-k}} = \sum_{l=i}^{i+j} c_l \int_0^{2\pi} \sin^l x (1 - h^{k-1}\sin x)^{\frac{p-k}{1-k}}$$

for some $c_l \in \mathbb{R}$ when j is even, we get that

$$M(h) = h^p \sum_{i=0}^n a_i I_i(h^{k-1}),$$

where

$$I_i(h) = \int_0^{2\pi} \frac{\sin^i x}{(1 - h\sin x)^\alpha} dx, \text{ with } \alpha = \frac{p-k}{k-1}.$$

This is exactly the type of analytic functions studied in [4], see equations (1) and (5) of that paper. Notice that

$$\begin{aligned} I_i(h) &= \int_0^{2\pi} \frac{\sin^i x}{(1 - h\sin x)^\alpha} dx = \int_{\pi/2}^{5\pi/2} \frac{\sin^i x}{(1 - h\sin x)^\alpha} dx \\ &= \int_{\pi/2}^{3\pi/2} \frac{\sin^i x}{(1 - h\sin x)^\alpha} dx + \int_{3\pi/2}^{5\pi/2} \frac{\sin^i x}{(1 - h\sin x)^\alpha} dx \\ &= 2\int_{\pi/2}^{3\pi/2} \frac{\sin^i x}{(1 - h\sin(x))^\alpha} dx. \end{aligned}$$

Thus, we are under the hypotheses of Theorem 1.2 with $[a,b]=[\pi/2,3\pi/2]$, $f_i(x)=\sin^i x$ and $p(x)=-\sin(x)$. Therefore, when $\frac{p-k}{1-k}\notin\mathbb{N}$ the family $(I_0(h),I_1(h),\ldots,I_n(h))$ is an ECT-family on $(0,1)$.

Example 3.2. Rigid planar systems are frequently studied because they encompass all of the difficulties of Hilbert's XVIth problem in a more tangible context: the system has a unique critical point at the origin and can be globally reduced to a one-dimensional non-autonomous differential equation. Moreover, the solution of the center-focus problem is equivalent to determining the isochronous centers, see [5, 12] or [2], where these systems are called uniformly isochronous centers. Polynomial rigid systems with a center or focus at the origin are of the form

$$\begin{cases} x'=-y+xP(x,y),\\ y'=x+yP(x,y),\end{cases}$$

where P is an arbitrary polynomial. In polar coordinates they can be written as

$$\frac{dr}{d\theta}=rP(r\cos\theta,r\sin\theta),\tag{8}$$

because $\theta'=1$, the property which gives rise to their name. In the particular case $P=P_{k-1}+\varepsilon P_{p-1}$, where P_m are homogeneous polynomials of degree m and $k,p\in\mathbb{N}$, equation (8) reads

$$\frac{dr}{d\theta}=r^kP_{k-1}(\cos\theta,\sin\theta)+\varepsilon r^pP_{p-1}(\cos\theta,\sin\theta),\tag{9}$$

which is of the form (5). Hence, adding some additional hypotheses on P_{k-1} and P_{p-1} we have found new families of non-autonomous differential equations for which Theorem 1.2 can be applied to obtain upper bounds on the number of isolated periodic solutions by studying the zeroes of the corresponding $M(h)$ given in (7). We skip the details.

3.2. **Persistence of periodic orbits for double potentials.** Consider the double potential

$$H(x,y)=V(x)+\frac{y^{2s}}{2s},\quad s\in\mathbb{N},$$

on the cylinder $[0,2T]\times\mathbb{R}$, where V is a $2T$-periodic even function with a unique minimum in $[0,2T)$ at $x=0$ and a unique maximum at $x=T$. Denote $\bar h=H(T,0)=V(T)$. This energy level separates the oscillatory region $\mathcal{R}^0$ from the rotary regions $\mathcal{R}^\pm$: for $h<\bar h$ there exist periodic orbits inside the region enclosed by the heteroclinic connections between the saddle points at $(T,0)$, while for $h>\bar h$ the periodic orbits encircle the cylinder with period $[0,2T]$ above and below the heteroclinic connections, see Figure 1. In what follows we will focus on periodic orbits in the rotary regions $\mathcal{R}^\pm$.

We consider the following perturbation of the double potential

$$\begin{cases} x'=y^{2s-1}\\ y'=-V'(x)+\varepsilon y^p\sum_{i=0}^n a_if_i(x)\end{cases}\tag{10}$$

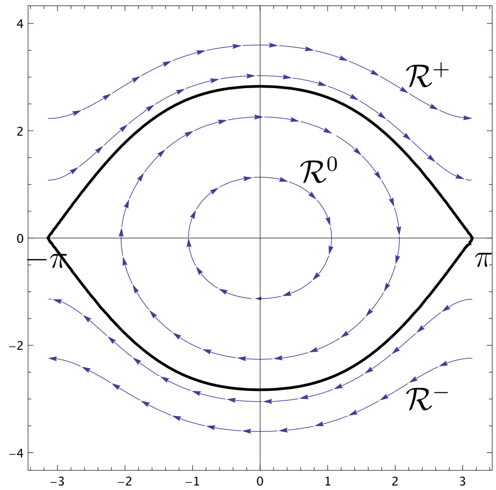

FIGURE 1. Phase portrait showing the rotary regions $\mathcal{R}^{\pm}$ and the oscillatory region $\mathcal{R}^0$ for a double potential.

where f_i are $2T$-periodic even functions satisfying that $(f_0, \ldots, f_n)$ is a CT-system on $[0, T]$. To find out how many periodic orbits in the rotary region persist under the perturbation we study the zeros of the Abelian integrals $M^{\pm}(h)$, which for $h > \bar{h}$ are given by

$$
\begin{aligned}
M^{\pm}(h) &= \sum_{i=0}^{n} a_i \int_{H(x,y)=h} f_i(x) y^p dx = \sum_{i=0}^{n} a_i \int_{-T}^{T} f_i(x)\big(2sh - 2sV(x)\big)^{\frac{p}{2s}} dx, \\
&= \sum_{i=0}^{n} 2a_i \int_0^T f_i(x)\big(2sh - 2sV(x)\big)^{\frac{p}{2s}} dx \qquad (11)\\
&= 2(2sh)^{\frac{p}{2s}} \sum_{i=0}^{n} a_i \int_0^T f_i(x)\Big(1 - \frac{1}{h}V(x)\Big)^{\frac{p}{2s}} dx,
\end{aligned}
$$

see [8], Chapter 7. The superscript $^{\pm}$ denotes the positive and negative rotary regions, respectively. If we denote $I_i(h) = \int_0^T f_i(x)(1 - \frac{1}{h}V(x))^{\frac{p}{2s}} dx$ we can apply Theorem 1.2 to obtain that $(I_0(h), \ldots I_n(h))$ are an ECT-system on $(\bar{h}, \infty)$ when $\frac{p}{2s} \notin \mathbb{N}$.

In the following we will study two examples to show how the formula (11) for the Abelian integrals $M^{\pm}(h)$ can be used to obtain bounds for the number of periodic orbits that persist in perturbed systems of the form (10).

Example 3.3 (Pendulum)**.** In [3] the authors study the number of limit cycles for perturbed pendulum equations of the form (10) with $V(x) = 1 - \cos x$, $s = 1$ and $f_i(x) = \cos^i x$ on the cylinder $[-\pi, \pi] \times \mathbb{R}$. Let $h \in L = (2, \infty)$ and let us focus on the rotary region $\mathcal{R}^+$. Applying formula (11) to this example yields that the Abelian integral for $\mathcal{R}^+$ is

$$M^+(h) = 2^{\frac{p+2}{2}} \sum_{i=1}^{n} a_i I_{i,p}(h),$$

where

$$I_{i,p}(h) = \int_0^\pi \cos^i x\,(h - 1 + \cos x)^{\frac{p}{2}}\,dx,$$

which is exactly of the form found in [3]. Note that

$$I_{i,p}(h) = (h-1)^{\frac{p}{2}} \int_0^\pi \cos^i x \left(1 + \frac{1}{h-1}\cos x\right)^{\frac{p}{2}} dx.$$

Therefore, if we denote

$$J_{i,p}(h) = \int_0^\pi \cos^i x\,(1 + h\cos x)^{\frac{p}{2}}\,dx,$$

then $(I_{1,p}(h),\ldots,I_{n,p}(h))$ is an ECT-system on $(2,\infty)$ if and only if $(J_{1,p}(h),\ldots,J_{n,p}(h))$ is an ECT-system on $(0,1)$. This last property holds by Theorem 1.2 provided that $p/2 \notin \mathbb{N}$. Therefore, the Abelian integral $M^+(h)$ has at most $n-1$ isolated zeros, and consequently the maximum number of limit cycles appearing in the upper rotary region $\mathcal{R}^+$ by the first order analysis is $n-1$.

Example 3.4 (The whirling pendulum)**.** Another example of a perturbed system with double potential of the form (10) is the so-called *whirling pendulum*,

$$\begin{cases} x' = y \\ y' = \sin x(\cos x - \gamma), \qquad \gamma > 0. \end{cases} \tag{12}$$

with $V(x) = -\gamma\cos x + \frac{1}{2}\cos^2 x + \gamma - \frac{1}{2}$ and $s = 1$ on $[-\pi,\pi]\times\mathbb{R}$, cf. [9, 11]. Let us focus on the case $\gamma \geq 1$, meaning that the constant rotation rate is smaller than a given value γ, for which the phase portrait is as in Figure 1. As in the previous example, we consider the persistence problem of periodic orbits in the rotary regions $\mathcal{R}^\pm$ under perturbations of the form (10) with f_i analytic, even and 2π periodic functions and $(f_1,\ldots,f_n)$ a CT-system on $(0,\pi)$. The formula (11) for this example yields that the Abelian integral for the rotary region $\mathcal{R}^+$ is given by

$$M^+(h) = 2(2h)^{\frac{p}{2}} \sum_{i=1}^{n} a_i J_{i,p}(h),$$

where

$$J_{i,p}(h) = \int_0^\pi f_i(x)\left(1 - \frac{V(x)}{h}\right)^{\frac{p}{2}} dx,$$

for $h \in L = (2\gamma,\infty)$. Applying Theorem 1.2 we obtain that $(I_{0,p}(h),\ldots,I_{n,p}(h))$ is an ECT-system on $(2\gamma,\infty)$ provided that $p/2 \notin \mathbb{N}$. Therefore, the Abelian integral has at most $n-1$ isolated zeros, and consequently the maximum number of limit cycles appearing in the upper rotary region $\mathcal{R}^+$ by the first order analysis is $n-1$.

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