A NEW APPROACH FOR THE STUDY OF LIMIT CYCLES

J.D. GARCÍA-SALDAÑA, A. GASULL, AND H. GIACOMINI

Abstract. We prove that star-like limit cycles of any planar polynomial system can also be seen either as solutions defined on a given interval of a new associated planar non-autonomous polynomial system or as heteroclinic solutions of a 3-dimensional polynomial system. We illustrate these points of view with several examples. One of the key ideas in our approach is to decompose the periodic solutions as the sum of two suitable functions. As a first application we use these new approaches to prove that all star-like reversible limit cycles are algebraic. As a second application we introduce a function whose zeroes control the periodic orbits that persist as limit cycles when we perturb a star-like reversible center. As far as we know this is the first time that this question is solved in full generality. Somehow, this function plays a similar role that an Abelian integral for studying perturbations of Hamiltonian systems.

1. Introduction and results

Consider a planar differential system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y), \tag{1}$$

where X and Y are polynomial functions vanishing at the origin. The most elusive problem about this system is to know its number of limit cycles. Many efforts have been dedicated to this objective and the reader can consult several books, and their references, where this question is addressed, see [5, 7, 14, 15]. Of course, it is also very related with the celebrated XVIth Hilbert's problem, see [6, 10].

The goal of this work is to present a new approach to study the so called star-like limit cycles. To give a first description of our approach we start introducing some notation.

By applying the polar coordinates transformation $x = r\cos\theta$, $y = r\sin\theta$ to system (1) and by eliminating the variable t, we obtain the associated non-autonomous differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{R\big(r\cos\theta, r\sin\theta\big)}{T\big(r\cos\theta, r\sin\theta\big)}, \tag{2}$$

where

$$R(x, y) = \big(xX(x, y) + yY(x, y)\big)/r \quad \text{and} \quad T(x, y) = \big(xY(x, y) - yX(x, y)\big)/r^2.$$

Notice that (2) is well defined in the region $\mathbb{R}^2 \setminus \mathcal{T}$, where $\mathcal{T} = \{(x, y) : T(x, y) = 0\}$. Moreover, the periodic solutions of (1) that do not intersect $\mathcal{T}$ correspond to smooth 2π-periodic solutions of (2). These periodic orbits are usually called *star-like* periodic orbits. In particular when these 2π-periodic solutions are isolated in the set of periodic solutions they will correspond to star-like limit cycles.

2010 *Mathematics Subject Classification.* Primary 34C25, 34C07; Secondary 34C37.

Key words and phrases. Periodic orbits, Limit cycle, Abelian integral, Heteroclinic solution, Reversible center, Algebraic limit cycle.