ON THE POLYNOMIAL SOLUTIONS OF THE POLYNOMIAL DIFFERENTIAL EQUATIONS $y\,y' = a_0(x) + a_1(x)\,y + a_2(x)\,y^2 + \ldots + a_n(x)\,y^n$

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Abstract. In this paper we deal with differential equations of the form $yy' = P(x, y)$ where $y' = dy/dx$ and $P(x, y)$ is a polynomial in the variables x and y of degree n in the variable y. We provide the maximum number of polynomial solutions of this class of differential equations, and for some particular classes we study properties of their polynomial solutions.

1. Introduction and statement of the main results

The study of given solutions (as polynomial or rational solutions) of differential equations is of main interest for understanding the set of solutions of a differential equation. Rainville [14] in 1936 characterized all the Riccati differential equations of the form $y' = a_0(x) + a_1(x)y + y^2$, where a_0 and a_1 are polynomials in x, having polynomial solutions. He also gave an algebraic method for studying these polynomial solutions.

In 1954 Campbell and Golomb [6] gave an algorithm for computing all the polynomial solutions of the differential equation $a(x)y' = a_0(x) + a_1(x)y + a_2(x)y^2$, where a, a_0, a_1, a_2 are polynomials in x. In 2006 Behloul and Cheng [2] provided another algorithm for finding all the rational solutions of the equation $a(x)y' = \sum_{i=0}^{n} a_i(x)y^i$, where a, a_i are polynomials in x.

The differential equations $y' = a_0(x)+a_1(x)y+a_2(x)y^2+a_3(x)y^3$ are the Abel differential equations, which have been studied widely, either computing their periodic orbits (see for instance [8, 11]), or studying their centers (see [3, 4, 5]). More recently in [9] the authors studied the polynomial solutions of the differential equation $y' = \sum_{i=0}^{n} a_i(x)y^i$.

Also polynomial solutions of non-autonomous differential equations, or polynomial solutions of matrix differential equations have been studied, see for instance the articles [15] or [1] respectively, and the references quoted therein.

The Riccati-Abel equation $(a(x) + y(x))y'(x) = a_0(x) + a_1(x)\,y + a_2(x)\,y^2$ is studied in [16]. Moreover, in [10] (p. 28) the differential equation $(a(x) + b(x)y(x))y'(x) = a_0(x) + a_1(x)\,y + a_2(x)\,y^2 + a_3(x)\,y^3$ is considered. We deal in this paper with the generalization to degree n of these two equations with the restrictions $a(x) \equiv 0$ and $b(x) \equiv 1$. Indeed, we consider ordinary differential equations of the form

$$y\frac{dy}{dx} = a_0(x) + a_1(x)\,y + a_2(x)\,y^2 + \ldots + a_n(x)\,y^n, \qquad (1)$$

where x and y are complex variables, $a_i(x)$ are polynomials in $\mathbb{C}[x]$ for $i = 0, 1, 2, \ldots, n$ and $a_n(x) \not\equiv 0$, with n a nonnegative integer. We denote the derivative of y with respect to x by dy/dx or y'. We assume that $a_0 \not\equiv 0$, otherwise this differential equation becomes the one studied in [9]. We note that while the solutions of linear differential equations with constant coefficients admit relatively easily methods for solving them see for instance [12, 13], the solutions of nonlinear equations as equations (1) require special investigations.

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Differential equation (1) can be also written as the planar polynomial differential system

$$\dot{x} = y, \quad \dot{y} = a_0(x) + a_1(x)\,y + a_2(x)\,y^2 + \ldots + a_n(x)\,y^n, \tag{2}$$

where the dot denotes derivative with respect to an independent variable t.

We are interested in the *polynomial solutions* $y = p(x)$ of the differential equation (1), i.e. the solutions $y = p(x)$ of (1) where $p(x) \in \mathbb{C}[x]$. Notice that we can also consider the algebraic curves $p(x) - y = 0$ as invariant solutions of system (2).

Let $N = \max\{2, n\}$. Our four main results are the following.

Theorem 1. *The differential equation* (1) *has at most* $N2^{\deg a_0}$ *polynomials solutions, and this bound is sharp.*

The next two results concern special families of differential equations (1).

Theorem 2. *If the differential equation* (1)*, with* $n \geq 2$ *and* $a_i(x) \equiv 0$ *for all* $i = 1, \ldots, n-1$*, has more than one polynomial solution, then either it has exactly* n *solutions, all constant, or it has exactly two polynomial solutions* $y = \pm p(x)$*, for some polynomial* $p(x) \in \mathbb{C}[x]$.

Theorem 3. *If the differential equation* (1)*, with* n *even and* $a_{2i-1}(x) \equiv 0$ *for* $i = 1, \ldots, n/2$*, with* $a_{2i}(x) \not\equiv 0$ *for some* i*, has more than one polynomial solution, then it has exactly two polynomial solutions* $y = \pm p(x)$*, for some polynomial* $p(x) \in \mathbb{C}[x]$.

The next theorem deals with the differential equation (1) for several small values of n.

Theorem 4. *Consider the differential equation* (1)*. Then the following statements hold.*

(a) *If* $n = 0$*, then either equation* (1) *has no polynomial solutions, or it has two non-constant polynomial solutions of the form* $y = \pm p(x)$*,* $p(x) \in \mathbb{C}[x]$.
(b) *If* $n = 1$*, then two polynomial solutions* $y = p_i(x)$*,* $i = 0, 1$*, of* (1) *always have a non-constant common factor. Moreover every factor of* $p_0(x) - p_1(x)$ *divides both* $p_0(x)$ *and* $p_1(x)$.
(c) *If* $n = 2$ *and there exist three polynomial solutions* $y = p_i(x)$*,* $i = 0, 1, 2$*, then* $p_i(x)/p_j(x)$ *is not a constant function, for* $i \neq j$.
(d) *If* $n = 3$ *and there exist four polynomial solutions* $y = p_i(x)$*,* $i = 0, 1, 2, 3$*, then* $p_i(x)/p_j(x)$ *is not a constant function, for* $i \neq j$*. Moreover, if* $y = p(x)$ *is a polynomial solution of* (1)*, then* $a_0(x)/p(x)$ *is a polynomial solution of the Abel differential equation* $a_0 y' = -a_0^2 a_3 + (a_0' - a_0 a_2)y - a_1 y^2 - y^3$.

The paper is organized as follows. In section 2 we prove Theorem 1. Theorem 2 is proved in section 3. Section 4 is devoted to the proof of Theorem 3. Finally in section 5 we prove Theorem 4.

We must mention that all the algebraic computations that appear in this paper have been done with the help of the algebraic manipulator Mathematica.

2. An upper bound for the number of polynomial solutions of (1)

The following lemma, despite that it has a trivial proof, provides important information on the polynomial solutions of the differential equation (1).

Lemma 5. *If* $y = p(x)$ *is a polynomial solution of the differential equation* (1)*, then* $p(x) | a_0(x)$.

The next lemma is a key-point in the proof of Theorem 1.

Lemma 6. *If* $y = p(x)$ *is a polynomial solution of* (1)*, then there exist at most* N *complex solutions of* (1) *of the form* $y = \kappa p(x)$*, with* $\kappa \in \mathbb{C}$.

Proof. Let $y = p(x)$ be a polynomial solution of (1). Suppose that $y = \kappa p(x)$, with $\kappa \in \mathbb{C} \setminus \{0, 1\}$, is another polynomial solution of (1). Then

$$\kappa^2 \sum_{i=0}^{n} a_i(x) p(x)^i = \kappa^2 p(x) p'(x) = \sum_{i=0}^{n} \kappa^i a_i(x) p(x)^i.$$

Here the first equality holds because $y = p(x)$ is a solution of (1) and the second one because $y = \kappa p(x)$ also is a solution. Hence we get

$$\sum_{i=0}^{n} (\kappa^i - \kappa^2) a_i(x) p(x)^i = 0. \tag{3}$$

This is a polynomial equation of degree exactly N in the variable κ for all x fixed, except perhaps for a finite number of values. Therefore this equation, for almost all fixed x, has exactly N complex solutions for κ, and of course this set of solutions includes $\kappa = 1$. □

Next we provide some definitions concerning integrability. They will be used later no. If $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ is a differential system, a non-constant $\mathcal{C}^1$-function $H(x, y)$ is a *first integral* of this system if it is constant on the solutions of the system; i.e., if it satisfies the equation

$$P(x, y) \frac{\partial H}{\partial x} + Q(x, y) \frac{\partial H}{\partial y} = 0. \tag{4}$$

An algebraic curve $f = 0$ is *invariant* under the flow of the differential system $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ if there exists a polynomial $k \in \mathbb{C}[x, y]$, called the *cofactor*, such that

$$P(x, y) \frac{\partial f}{\partial x} + Q(x, y) \frac{\partial f}{\partial y} = kf.$$

The Darboux Theory of Integrability relates the number of invariant algebraic curves of a differential system with the existence of a (Darboux) first integral, see [7]. The key point in the existence of such a Darboux first integral is whether a linear combination of the cofactors of such curves is identically zero.

Example 1. Consider the following differential equation (1) with $n = 1$:

$$yy' = -2x(2x^2 - 1) + 6xy. \tag{5}$$

This equation has four polynomial solutions given by $p_0(x) = x^2 - 1/2$, $p_1(x) = 2x^2 - 1$ and $y(x) = \sqrt{2}x(\sqrt{2}x \pm 1)$. Notice that all of them divide a_0 and have common factors. Moreover, we have $p_1(x) = 2p_0(x)$, hence equation (3) in this case and for $p(x) = p_0(x)$ has the maximum number of solutions κ, which is $N = \max\{1, 2\} = 2$.

Finally we note that the differential system associated to the differential equation (5) has the rational first integral

$$H(x, y) = \frac{(y - 2p_0(x))^2}{y - p_0(x)},$$

as is easy to check using (4). □

Proof of Theorem 1. We recall from Lemma 5 that a polynomial solution $y = p(x)$ must divide $a_0(x)$. This means that, up to a multiplicative constant, we can obtain at most $2^{\deg a_0}$ different polynomial solutions of (1). Now from Lemma 6 the first part of the theorem follows.

The differential equation $yy' = \prod_{i=1}^{n}(y - c_i)$, with $n \geq 2$ and $c_i \neq 0$ for all i, has n polynomial solutions given by $y = p_i(x) = c_i$, $i = 1, \ldots, n$. This is an example where the upper bound given by Theorem 1 is sharp, since it is $N 2^{\deg a_0} = n \cdot 2^0 = n$. This completes the proof of the theorem. □

3. Proof of Theorem 2

We consider in this section the subfamily of (1) given by the differential equation

$$y\frac{dy}{dx} = a_0(x) + a_n(x)\,y^n. \tag{6}$$

The case $n = 1$ will be studied in section 5. We assume here that $n \geq 2$.

Proof of Theorem 2. Suppose that the differential equation (6) has two polynomial solutions $y = p(x)$ and $y = q(x)$ such that $p(x)^n \not\equiv q(x)^n$. Substituting $p(x)$ and $q(x)$ into (6) we obtain two linear equations with unknowns $a_0(x), a_n(x)$, from which

$$a_0(x) = \frac{p(x)^n q(x)q'(x) - q(x)^n p(x)p'(x)}{p(x)^n - q(x)^n}, \quad a_n(x) = \frac{p(x)p'(x) - q(x)q'(x)}{p(x)^n - q(x)^n}.$$

Note that, since $n \geq 2$, $a_n(x)$ is not a polynomial, which is a contradiction. Hence this situation cannot happen. Therefore we must have $p(x)^n \equiv q(x)^n$. Then

$$\frac{pp' - a_0}{a_n} = p^n = q^n = \frac{qq' - a_0}{a_n}.$$

Hence $pp' = qq'$. We distinguish two cases. If $p' = q' = 0$ then p and q are constant, and therefore

$$p^n = q^n = -\frac{a_0}{a_n}$$

is a constant, meaning that we have n constant solutions given by

$$y = \sqrt[n]{-\frac{a_0}{a_n}}.$$

If $p', q' \neq 0$ then integrating $pp' = qq'$ we have $p^2 = q^2$ and hence $p = \pm q$. □

Example 2. The differential equation

$$yy' = x(1 - x)(3x - 4) + (3x - 1)y$$

has the polynomial solutions $y = \pm(x - 1)$. □

Example 3. The differential equation $yy' = 1 - y^4$ has four constant solutions: the fourth roots of unity. □

4. Proof of Theorem 3

We consider in this section the subfamily of (1) given by the differential equation

$$y\frac{dy}{dx} = \sum_{i=0}^{m} a_{2i}(x)y^{2i}, \tag{7}$$

with $m = n/2 \in \mathbb{N}$, with $a_0, a_{2j}, a_{2m} \not\equiv 0$, for some $0 < j < m$.

Set $z = y^2/2$. Then $z' = yy'$, and hence the differential equation (7) can be written, in terms of z, as

$$\frac{dz}{dx} = \sum_{i=0}^{m} b_{2i}(x)z^i, \tag{8}$$

where $b_{2i} = 2^i a_{2i}$. The following result appears in [9].

Theorem 7. *The differential equation* (8) *has at most* m *polynomial solutions. The difference between two such polynomial solutions is a constant.*

Proof of Theorem 3. Let $z = r(x)/2$ be a solution of (8). Then $y = \pm\sqrt{r(x)}$ are two solutions of (7). They are polynomial if and only if $r(x)$ is a perfect square; that is, if there exists $p(x) \in \mathbb{C}[x]$ such that $r(x) = p(x)^2$. In such a case, $y = \pm p(x)$ are two polynomial solutions of (7). From Theorem 7, this implies that we have at most $n = 2m$ polynomial solutions for (7).

Now suppose that indeed $r(x) = p(x)^2$ is a perfect square. Suppose also that $z = s(x)/2$ is another polynomial solution of (8). Again from Theorem 7, we know that there exists a non-zero constant C such that $s(x) = p(x)^2 + C$, for all x. Hence $y = \pm\sqrt{p(x)^2 + C}$ are two solutions of (7). If $s(x)$ is a perfect square, that is $s(x) = q(x)^2$, then from the proof of Proposition 8 and from $q(x)^2 - p(x)^2 = C$ we get that $C = 0$, and therefore $r(x) = s(x)$. Thus only two polynomial solutions can exist for the differential equation (7), which are $y = \pm p(x)$, and the theorem follows. □

5. Proof of Theorem 4

5.1. **The differential equation** (1) **with** $n = 0$**.** The following proposition proves statement (a) of Theorem 4.

Proposition 8. *The differential equation* (1) *with* $n = 0$,

$$y\frac{dy}{dx} = a_0(x), \tag{9}$$

has either zero or two distinct polynomial solutions. They have the form $y = \pm p(x)$, $p(x) \in \mathbb{C}[x]$.

Proof. Equation (9) can be directly solved, providing the solutions

$$y(x) = \pm\sqrt{K + 2\int a_0(x)\,dx},$$

where K is an arbitrary constant. Now we must prove that, if there exists $K \in \mathbb{C}$ such that $y = p(x)$ is a polynomial solution, then this K is unique and $y = -p(x)$ is another polynomial solution. Indeed, if $y = q(x)$ is a solution of (9) different from $y = p(x)$, then $p^2 - q^2$ is a constant. We need to prove that this constant is zero, and hence $q(x) = -p(x)$.

Let $p(x) = \sum p_i x^i$ and $q(x) = \sum q_i x^i$. It is clear that $\deg p = \deg q$. Let $m \in \mathbb{N}$ be this degree. Thus

$$\begin{aligned} p^2 - q^2 &= (p_m^2 - q_m^2)x^{2m} \\ &\quad + 2(p_m p_{m-1} - q_m q_{m-1})x^{2m-1} + (p_{m-1}^2 - q_{m-1}^2 + 2(p_m p_{m-2} - q_m q_{m-2}))x^{2m-2} + \cdots . \end{aligned}$$

Since $p^2 - q^2$ is a constant, all the monomials of the previous polynomial except the constant one must be zero. In particular, $p_m^2 = q_m^2$. If $p_m = q_m \neq 0$, then from the monomial of x^{2m-1} we have $p_{m-1} = q_{m-1}$. Otherwise $p_m = -q_m \neq 0$ and then $p_{m-1} = -q_{m-1}$. From the next monomial, x^{2m-2}, we get either $p_{m-2} = q_{m-2}$ (in case $p_m = q_m$), or $p_{m-2} = -q_{m-2}$ (in case $p_m = -q_m$).

We can continue this argument by using the induction principle to have either $p_i = q_i$ or $p_i = -q_i$, for all $i = m, \ldots, 0$ from the monomials from x^{2m} to x^m: the coefficient of the monomial x^{m+i} is the sum of non-zero multiples of the expressions $p_m p_i - q_m q_i$ and $p_j p_k - q_j q_k$, with $j + k = m + i$, $i < j, k < m$. Since $p_j p_k = q_j q_k$, we have $p_m p_i = q_m q_i$. So again if $p_m = q_m \neq 0$, then $p_i = q_i$, and if $p_m = -q_m \neq 0$ then $p_i = -q_i$.

Hence we obtain either $p(x) = q(x)$ or $p(x) = -q(x)$, and the proposition follows. □

Example 4. The only polynomial solutions $y = p(x)$ of the differential equation (9) with $a_0(x) = 2(x-1)(x+1)(x-3)$ are $y(x) = \pm(x-3)(x+1)$. □

5.2. **The differential equation** (1) **with** $n = 1$**.** We recall that a polynomial is *square-free* if it has no multiple factors in this factorization.

Proposition 9. *Consider the differential equation* (1) *with* $n = 1$*:*

$$y\frac{dy}{dx} = a_0(x) + a_1(x)y. \tag{10}$$

Then two polynomial solutions of (10) *have a square-free common factor.*

Proof. Let $y = p(x)$ a polynomial solution of (10). Then, by Lemma 5, $p(x)|a_0(x)$, that is $a_0(x) = p(x)\tilde{a}_0(x)$, for some $\tilde{a}_0(x) \in \mathbb{C}[x]$. Hence $p(x)$ satisfies

$$p'(x) = \tilde{a}_0(x) + a_1(x).$$

Then $a_1(x) = p'(x) - \tilde{a}_0(x)$. Substituting the expressions of $a_0(x)$ and $a_1(x)$ into (10), we have

$$y\frac{dy}{dx} = p(x)\tilde{a}_0(x) + (p'(x) - \tilde{a}_0(x))y.$$

Let $y = q(x)$ be another solution of (10). Then

$$q(x)q'(x) = p(x)\tilde{a}_0(x) + (p'(x) - \tilde{a}_0(x))q(x),$$

and consequently

$$\tilde{a}_0(x) = -q(x)\frac{p'(x) - q'(x)}{p(x) - q(x)}.$$

This must be a non-zero polynomial. Since $\deg(p - q) > \deg(p' - q')$, the polynomials q and $p - q$ have a square-free common factor dividing also p. Therefore the proposition follows. □

We prove that the existence of $n + 1$ solutions determines completely the differential equation (1). We shall use this result later on.

Proposition 10. *Let* $y = p_i(x)$*, for* $i = 0, \ldots, n$*, be* $n + 1$ *distinct* $\mathcal{C}^1$*-solutions of the differential equation* (1)*. Then, for all* $i \in \{0, \ldots, n\}$*,* $a_i(x)$ *can be written as a function of* $p_0(x), \ldots, p_n(x)$ *and their first derivatives.*

Proof. For $j = 0, \ldots, n$, since $y = p_j(x)$ is a solution of (1), we have

$$p_j(x)p_j'(x) = \sum_{i=0}^{n} a_i(x)p_j(x)^i.$$

These $n + 1$ equations can be written altogether as a linear system of equations with unknowns the a_i:

$$\begin{pmatrix} 1 & p_0 & p_0^2 & \cdots & p_0^n \\ 1 & p_1 & p_1^2 & \cdots & p_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} p_0p_0' \\ p_1p_1' \\ \vdots \\ p_np_n' \end{pmatrix}. \tag{11}$$

The square matrix has non-zero determinant, since it is a Vandermonde matrix. Then there exist unique $a_0, \ldots, a_n$ that satisfy this linear system. In particular, the solutions a_i depend on the p_j's and their first derivatives. □

Remark 1. *When the* p_i*'s are polynomials, the linear system* (11) *provides rational solutions* $a_0, \ldots, a_n$ *with denominator* $\prod_{0\le i<j\le n}(p_i - p_j)$*. Since* $a_i(x)$ *is assumed to be polynomial for all* i*, some relations among the* p_j *may appear. See Proposition 9 for the case* $n = 1$*.*

Remark 2. *If equation* (10) *has two solutions* $y = p(x)$ *and* $y = q(x)$*, then by Proposition 10*

$$a_0(x) = -\frac{p(x)q(x)(p'(x) - q'(x))}{p(x) - q(x)}, \quad a_1(x) = \frac{p(x)p'(x) - q(x)q'(x)}{p(x) - q(x)}. \tag{12}$$

We recall that if $y = p(x)$ is a polynomial solution of (1), then $p(x)-y = 0$ is a polynomial solution of system (2). In such a case, there exists a polynomial $k(x, y)$, called the *cofactor*, such that

$$p'(x)y - \sum_{i=0}^{n} a_i(x)y^i = k(x, y)(p(x) - y). \tag{13}$$

This cofactor has degree at most $n-1$ in y. We note that equation (1) follows from equation (13) with $y = p(x)$.

Proposition 11. *The following statements hold for the differential equation* (10)*.*

(a) If $(C + 1)a_0(x) + Cp(x)a_1(x) \equiv 0$ for some polynomial $p(x)$ and some constant $C \in \mathbb{C} \setminus \{-1, 0, 1\}$, then the differential equation has $y = p(x)$ and $y = Cp(x)$ as polynomial solutions.

(b) If the differential equation has two polynomial solutions given by $y = p(x)$ and $y = Cp(x)$, with $C \in \mathbb{C}\setminus\{-1, 0, 1\}$, then $a_0(x) = -Cp(x)p'(x)$ and $a_1(x) = (C+1)p'(x)$. Moreover the associated differential system (2) *has the first integral*

$$H(x, y) = \frac{y - p(x)}{(y - Cp(x))^C}.$$

Proof. The proof about the polynomial solutions and their relation with the differential equation follows directly after substituting $q(x)$ by $Cp(x)$ in the expressions of a_0 and a_1 provided in (12).

Concerning the first integral, we just need to note that the cofactor of the invariant algebraic curve $f_1(x, y) = p(x)-y = 0$ for the associated system (2) is $k_1(x, y) = Cp'(x)$ and the cofactor of $f_2(x, y) = Cp(x) - y = 0$ is $k_2(x, y) = p'(x)$. Hence $k_1(x, y) - Ck_2(x, y) \equiv 0$. Applying the Darboux Theory of Integrability the (Darboux) first integral f_1/f_2^C is obtained, for more details see Theorem 8.7 of [7]. □

Proposition 12. *If $y = p_i(x)$, $i = 0, 1$, are polynomial solutions of* (1) *with $n = 1$, then the factors of $p_0(x) - p_1(x)$ divide to $p_0(x)$ and $p_1(x)$.*

Proof. We have

$$p_0p_0' = a_0 + a_1p_0, \quad p_1p_1' = a_0 + a_1p_1.$$

The difference of these equations is

$$p_0p_0' - p_1p_1' = a_1(p_0 - p_1).$$

Let ω be a factor of $p_0 - p_1$. Then $p_0 - p_1 = \omega^k\gamma$, for some $k \in \mathbb{N}$ and some polynomial γ such that $\omega \nmid \gamma$. Thus $p_0' - p_1' = k\omega^{k-1}\omega'\gamma + \omega^k\gamma'$. Substituting p_0 and p_0' from these equalities in the previous equation we obtain

$$(p_1 + \omega^k\gamma)(p_1' + k\omega^{k-1}\omega'\gamma + \omega^k\gamma') - p_1p_1' = a_1\omega^k\gamma.$$

After some simplifications we have

$$kp_1\omega'\gamma + \omega(p_1\gamma' + p_1'\gamma - a_1\gamma) = 0.$$

Hence $\omega|p_1$. From $p_0 - p_1 = \omega^k\gamma$ we also have that $\omega|p_0$. □

Note that Propositions 9, 11 and 12 prove statement (b) of Theorem 4. We needed to omit the case $C = -1$ in Proposition 11 because in that case we have $a_1 \equiv 0$, which is not possible because we are working with $n = 1$.

Remark 3. *Statement (b) of Proposition 11 shows that, given a polynomial $p(x)$ and a constant C, we can construct a differential equation* (10) *having the polynomial solutions $y = p(x)$ and $y = Cp(x)$.*

Remark 4. *Regarding equation* (3) *for* $n = 1$*, we note that it has degree* 2 *in* κ*. So if* $(C+1)a_0(x) + Ca_1(x)p(x) = 0$*, then it writes always as* $\kappa^2 - (C+1)\kappa + C = 0$ *and has the solutions* $\kappa = 1, C$*.*

The next lemma provides some information about the degrees of a_0 and a_1 when the differential equation (10) has a polynomial solution. Its proof is trivial and we omit it.

Lemma 13. *Consider the differential equation* (10) *having a polynomial solution* $y = p(x)$*. Then*

(a) *If* $\deg p < (\deg a_0 + 1)/2$*, then* $\deg a_1 = \deg a_0 - \deg p$*.*
(b) *If* $\deg p = (\deg a_0 + 1)/2$*, then* $\deg a_1 \le \deg a_0 - \deg p$*.*
(c) *If* $\deg p > (\deg a_0 + 1)/2$*, then* $\deg a_1 = \deg p - 1$*.*

5.3. **The differential equation** (1) **with** $n = 2$**.** When $n = 2$, the proof of Proposition 10 gives

$$\begin{aligned} a_0(x) =& \frac{p_0p_1p_2\left[p_0'(p_1-p_2)+p_1'(p_2-p_0)+p_2'(p_0-p_1)\right]}{\prod_{i<j}(p_i-p_j)}, \\ a_1(x) =& \frac{p_0p_0'(p_2^2-p_1^2)+p_1p_1'(p_0^2-p_2^2)+p_2p_2'(p_1^2-p_0^2)}{\prod_{i<j}(p_i-p_j)}, \\ a_2(x) =& \frac{p_0p_0'(p_1-p_2)+p_1p_1'(p_2-p_0)+p_2p_2'(p_0-p_1)}{\prod_{i<j}(p_i-p_j)}, \end{aligned}$$

if $y = p_i(x)$, for $i = 0, 1, 2$, are solutions of the differential equation (1) with $n = 2$.

Next we provide an example of the differential equation (1) with $n = 2$ having three polynomial solutions.

Example 5. The differential equation $yy' = x^2(x^2-1) - (2x^2-2x-1)y + y^2$ has three polynomial solutions $p_0(x) = x^2$; $p_1(x) = x^2 - 1$; and $p_2(x) = x(x+1)$. □

Proposition 14. *Suppose that the differential equation* (1) *with* $n = 2$ *has three distinct solutions* $y = p_i(x)$*,* $i = 0, 1, 2$*. Then there does not exist* $C_1 \in \mathbb{C} \setminus \{0, 1\}$ *such that* $p_1(x) = C_1p_0(x)$*.*

Proof. Suppose that we can write $p_1(x) = C_1p_0(x)$, for some $C_1 \in \mathbb{C}\setminus\{0,1\}$. The associated differential system (2) has a first integral given by

$$H(x,y) = \frac{(y-p_0(x))(C_1p_0(x)-p_2(x))^{C_1}}{(y-C_1p_0(x))^{C_1}(p_0(x)-p_2(x))}.$$

Since the curves $p_i(x) - p_2(x) = 0$, $i = 0, 1$, are not invariant by the system because they do not divide $\dot{x} = y$, the quotient $(C_1p_0 - p_2)^{C_1}/(p_0 - p_2)$ must be a constant, say $1/(C(1-C_1))$. If we let $q(x) = C_1p_0(x) - p_2(x)$, then

$$q(x)^{C_1} = \frac{p_0(x)}{C} + \frac{q(x)}{C(1-C_1)},$$

or

$$p_0(x) = Cq(x)^{C_1} - \frac{q(x)}{1-C_1}, \quad p_2(x) = CC_1q(x)^{C_1} - \frac{q(x)}{1-C_1}.$$

Substituting $p_0(x)$ and $p_2(x)$ into the expression of a_2 we get $a_2 \equiv 0$, which contradicts the initial hypotheses on (1). Hence no such C_1 may exist. □

Remark 5. *If, besides* $p_1(x) = kp_0(x)$*, we have* $p_2(x) = C_2p_0(x)$*, then direct computations show that* $a_0(x) \equiv 0$*.*

Proposition 14 proves statement (c) of Theorem 4.

5.4. **The differential equation** (1) **with** $n = 3$**.** If $y = p_i(x)$, for $i = 0, 1, 2, 3$, are solutions of the differential equation (1), then from Proposition 10 with $n = 3$ we can write a_i as functions of these p_j. We do not write the expression of the a_i because they are too long.

Statement (d) of Theorem 4 follows from the next two propositions.

Proposition 15. *Suppose that the differential equation* (1) *with* $n = 3$ *has four distinct solutions* $y = p_i(x)$, $i = 0, 1, 2, 3$. *Then there do not exist* $C_1, C_2 \in \mathbb{C} \setminus \{0, 1\}$ *such that* $p_i(x) = C_i p_0(x)$, $i = 1, 2$.

Proof. Suppose that we can write $p_i(x) = C_i p_0(x)$, $i = 1, 2$, for some $C_i \in \mathbb{C} \setminus \{0, 1\}$, $C_1 \neq C_2$. From Proposition 10 we can compute $a_i(x)$ in terms of $p_j(x)$. We note that it is not possible to have, in addition, $p_3(x) = C_3 p_0(x)$, otherwise $a_0 \equiv 0$. From the expression of a_0 and a_1 we have the relation

$$(C_1 + C_2 + C_1C_2)a_0(x) + C_1C_2p_0(x)a_1(x) = 0.$$

Moreover it is easy to check that the associated system (2) has a first integral given by

$$H(x, y) = \frac{(y - p_0(x))^{C_1 - C_2}(y - C_1p_0(x))^{C_1(C_2-1)}(C_2p_0(x) - p_3(x))^{C_2(C_1-1)}}{(y - C_2p_0(x))^{C_2(C_1-1)}(C_1p_0(x) - p_3(x))^{C_1(C_2-1)}(p_0(x) - p_3(x))^{C_1-C_2}}.$$

Since the curves $p_i(x) - p_3(x) = 0$, $i = 0, 1, 2$, are not invariant by the system because they do not divide $\dot{x} = y$, it must happen that

$$\frac{(C_2p_0(x) - p_3(x))^{C_2(C_1-1)}}{(C_1p_0(x) - p_3(x))^{C_1(C_2-1)}(p_0(x) - p_3(x))^{C_1-C_2}}$$

is a constant. Let $q(x) = C_2p_0(x) - p_3(x)$. Thus

$$\frac{q(x)^{C_2(C_1-1)}}{((C_1 - C_2)p_0(x) + q(x))^{C_1(C_2-1)}((1 - C_2)p_0(x) + q(x))^{C_1-C_2}} = C \in \mathbb{C}. \tag{14}$$

We claim that $q(x)/p_0(x) = C_q$ is a constant. Assuming the claim, we obtain $p_3(x) = (C_2 - C_q)p_0(x)$, and thus as before $a_0 \equiv 0$. Hence no such C_1, C_2 exist and the proposition will follow once the claim is proved.

So it remains to prove the claim. From (14) we have

$$q(x)^{C_2(C_1-1)} = C\big((C_1 - C_2)p_0(x) + q(x)\big)^{C_1(C_2-1)}\big((1 - C_2)p_0(x) + q(x)\big)^{C_1-C_2}.$$

Or, equivalently,

$$1 = C\left((C_1 - C_2)\frac{p_0(x)}{q(x)} + 1\right)^{C_1(C_2-1)}\left((1 - C_2)\frac{p_0(x)}{q(x)} + 1\right)^{C_1-C_2}.$$

Thus

$$\left((1 - C_2)\frac{p_0(x)}{q(x)} + 1\right) = \tilde{C}\left((C_1 - C_2)\frac{p_0(x)}{q(x)} + 1\right)^{\frac{C_1(C_2-1)}{C_2-C_1}},$$

where $\tilde{C}$ is another constant. From this last equality we get that either p_0/q is a constant, or $C_1(C_2 - 1) = C_2 - C_1$. In the second case we get $C_1 = 1$, which is a contradiction. Therefore the claim is proved. □

Proposition 16. *We have that* $y = p(x)$ *is a polynomial solution of the differential system* (1) *with* $n = 3$ *if and only if* $y = q(x) = a_0(x)/p(x)$ *is a polynomial solution of the Abel differential equation*

$$a_0y' = -a_0^2a_3 + (a_0' - a_0a_2)y - a_1y^2 - y^3. \tag{15}$$

Proof. It is clear from Lemma 5 that if $y = p(x)$ is a polynomial solution of (1) then $q(x)$ is a polynomial. If $y = q(x)$ is a polynomial solution of (15), then we have

$$a_0 \frac{a_0' p - a_0 p'}{p^2} = -a_0^2 a_3 + (a_0' - a_0 a_2)\frac{a_0}{p} - a_1 \frac{a_0^2}{p^2} - \frac{a_0^3}{p^3}.$$

This is equivalent to have

$$a_0 a_0' p^2 - a_0 p p' = -a_0^2 a_3 p^3 + a_0(a_0' - a_0 a_2)p^2 - a_0^2 a_1 p - a_0^3,$$

which is equivalent to

$$pp' = a_3 p^3 + a_2 p^2 + a_1 p + a_0.$$

And this last equation means that the polynomial $y = p(x)$ is a solution of (1). □

Example 6. The differential equation

$$yy' = 5(x-2)(x-3)(3x^2 - 9x + 5) - (17x^3 - 97x^2 + 168x - 80)y + x^2 y^2 + (x-1)y^3$$

has the polynomial solutions $y = x - 2$ and $y = -5(x-2)$. Direct computations show that it has no more polynomial solutions. □

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