MEROMORPHIC FIRST INTEGRALS OF ANALYTIC DIFFEOMORPHISMS

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ABSTRACT. We give an upper bound for the number of functionally independent meromorphic first integrals that a discrete dynamical system generated by an analytic map f can have in a neighborhood of one of its fixed points. This bound is obtained in terms of the resonances among the eigenvalues of the differential of f at this point. Our approach is inspired on similar Poincaré type results for ordinary differential equations. We also apply our results to several examples, some of them motivated by the study of several difference equations.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the first steps to study the dynamics of a discrete dynamical system (DDS) is to determine the number m of functionally independent first integrals that it has. It is clear that each new first integral reduces the region where any orbit can lie, so the bigger is m, the simpler will be the dynamics. For instance, if this DDS is n-dimensional, this number m is at most n, and in the case $m = n$ the DDS is called integrable and it has extremely simple dynamics: in most cases it is globally periodic, that is, there exists $p \in \mathbb{N}$, such that $f^p = \mathrm{Id}$, where f is the invertible map that generates it, see [4]. Similarly, DDS having $m = n-1$ are such that all their orbits lie in one-dimensional manifolds, see some examples in [5].

The aim of this paper is to give an upper bound of the number of meromorphic first integrals that a DDS generated by an invertible analytic map can have in a neighborhood of a fixed point. We follow the approach of Poincaré for studying the same problem for continuous dynamical systems given by analytic ordinary differential equations. It is based on the study of the resonances among the eigenvalues of the differential of the vector field at one of its critical points, see for instance [9] and their references. We will use similar tools that the ones introduced in that paper.

Consider analytic diffeomorphisms in $(\mathbb{C}^n, 0)$, a neighborhood of the origin,

$$y = f(x), \qquad x \in (\mathbb{C}^n, 0), \tag{1}$$

with $f(0) = 0$. A function $R(x) = G(x)/H(x)$ with G and H analytic functions in $(\mathbb{C}^n, 0)$ is a *meromorphic first integral* of the diffeomorphism (1) if

$$G(f(x))H(x) = G(x)H(f(x)), \qquad \text{for all } x \in (\mathbb{C}^n, 0).$$

Notice that the above condition implies that

$$R(f(x)) = R(x),$$

for all $x \in (\mathbb{C}^n, 0)$ where both functions are well defined. Specially if G and H are polynomial functions, then $R(x)$ is a *rational first integral* of (1). If H is a non–zero constant, then $R(x)$ is an *analytic first integral* of (1). So meromorphic first integrals include rational and analytic first integrals as particular cases.

Denote by $A = \mathrm{D}f(0)$ the Jacobian matrix of $f(x)$ at $x = 0$. Let $\mu = (\mu_1, \ldots, \mu_n)$ be the n–tuple of eigenvalues of A. Notice that since f is a diffeomorphism at 0, we have $\mu_1\mu_2\cdots\mu_n \neq 0$.

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