4-DIMENSIONAL ZERO-HOPF BIFURCATION FOR POLYNOMIAL DIFFERENTIALS SYSTEMS WITH CUBIC HOMOGENEOUS NONLINEARITIES VIA AVERAGING THEORY

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ABSTRACT. The averaging theory of second order shows that for polynomial differential systems in $\mathbb{R}^4$ with cubic homogeneous nonlinearities at least nine limit cycles can be born in a zero-Hopf bifurcation.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Our goal is to study the periodic solutions which can bifurcate at a zero-Hopf bifurcation in a polynomial differential systems in $\mathbb{R}^4$ with cubic homogeneous nonlinearities by using the averaging theory of the second order.

In [7] the authors studied the zero-Hopf bifurcation in dimension $n > 2$, by using the first order averaging method. They proved that at least 2^{n-3} limit cycles can bifurcate from one singularity with eigenvalues $\pm bi$ and $n-2$ zeros.

In [5] (resp. [2]) the authors studied the zero-Hopf bifurcation in polynomial differential systems in $\mathbb{R}^3$ (resp. $\mathbb{R}^4$) with quadratic homogeneous nonlinearities. By applying the averaging theory of the second order to these systems, they show that at most 3 limit cycles can bifurcate from a singular point having eigenvalues of the form $\pm bi$ and one zero (resp. two zeros). The zero-Hopf bifurcation in polynomial differential systems in $\mathbb{R}^3$ with cubic homogeneous nonlinearities has been studied recently in [3].

In this paper we are interested on the existence of periodic solutions bifurcating from the origin of coordinates of a polynomial differential systems in $\mathbb{R}^4$ with cubic homogeneous nonlinearities having eigenvalues $\pm bi$ and two zeros, i.e for the differential systems

$$
\begin{aligned}
\dot{x} &= (a_1\varepsilon + a_2\varepsilon^2)x - (b + b_1\varepsilon + b_2\varepsilon^2)y + \sum_{j=0}^{2} \varepsilon^j X_j(x,y,z,w), \\
\dot{y} &= (b + b_1\varepsilon + b_2\varepsilon^2)x + (a_1\varepsilon + a_2\varepsilon^2)y + \sum_{j=0}^{2} \varepsilon^j Y_j(x,y,z,w),
\end{aligned} \tag{1}
$$

2010 *Mathematics Subject Classification.* Primary: 34C07, 34C05, 34C40.
Key words and phrases. Hopf bifurcation, averaging theory, cubic polynomial differential systems.

$$\dot{z} = (c_1\varepsilon + c_2\varepsilon^2)z + \sum_{j=0}^{2} \varepsilon^j Z_j(x,y,z,w),$$

$$\dot{w} = (d_1\varepsilon + d_2\varepsilon^2)w + \sum_{j=0}^{2} \varepsilon^j W_j(x,y,z,w),$$

where

$$\begin{aligned} X_j(x,y,z,w) = {} & a_{j0}x^3 + a_{j1}x^2y + a_{j2}x^2z + a_{j3}x^2w + a_{j4}xy^2 + a_{j5}xyz + a_{j6}xyw \\ & + a_{j7}xz^2 + a_{j8}xzw + a_{j9}xw^2 + a_{j10}y^3 + a_{j11}y^2z + a_{j12}y^2w + a_{j13} \\ & yz^2 + a_{j14}yzw + a_{j15}yw^2 + a_{j16}z^3 + a_{j17}z^2w + a_{j18}zw^2 + a_{j19}w^3, \end{aligned}$$

$Y_j(x,y,z,w)$, $Z_j(x,y,z,w)$ and $W_j(x,y,z,w)$ have the same expression as $X_j(x,y,z,w)$ by replacing a_{ji} by b_{ji}, c_{ji} and d_{ji} for $j = 0,1,2$ and $i = 0,1,\ldots,19$, respectively. The coefficients $a_{ij}, b_{ij}, c_{ij}, d_{ij}, a_1, a_2, b, b_1, b_2,\ c_1, c_2, d_1, d_2$ are real parameters with $b \neq 0$. Note that system (1) for $\varepsilon = 0$ at the origin has eigenvalues $\pm bi, 0, 0$. So for $\varepsilon = 0$ the origin is a *zero-Hopf equilibrium*.

Our main result is the following one.

Theorem 1. *By applying averaging theory of second order system* (1) *can exhibit at least* 9 *periodic solutions bifurcating from the origin when* $\varepsilon = 0$*, and this number of periodic solutions is reached if and only if the following condition is satisfied* $(3a_{00} + a_{04} + b_{01} + 3b_{010})b \neq 0$.

Theorem 1 is proved in section 3. In section 2 we recall the averaging theory of first and second order as it was stated in [1]. This will be the main tool for proving Theorem 1.

2. The averaging theory of first and second order

The aim of this section is to present the averaging theory of first and second order as it was developed in [1, 4, 6]. The following result is Theorem 4.2 of [1].

Theorem 2. *We consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. Assume that the following hypotheses* (i) *and* (ii) *hold. Assume:*

- (i) F_1, F_2, R *are locally Lipschitz with respect to* x, $F_1(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, *and* R *is differentiable with respect to* ε*. We define the averaging functions of* *first and second order* f_1, f_2*:* $D \longrightarrow \mathbb{R}^n$ *as*

$$
\begin{aligned}
f_1(z) = &\ \frac{1}{T}\int_0^T F_1(s,z)ds, \\
f_2(z) = &\ \frac{1}{T}\int_0^T \left[D_zF_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right]ds.
\end{aligned} \tag{3}
$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$*, there exists* $a \in V$ *such that* $f_1(a) + \varepsilon f_2(a) = 0$ *and* $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$*.*

Then for $|\varepsilon| > 0$ *sufficiently small there exists a* T*-periodic solution* $x(t, \varepsilon)$ *of the system* (2) *such that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

Where $d_B(f_1 + \varepsilon f_2, V, 0)$ denotes the Brouwer degree of the function $f_1 + \varepsilon f_2$ in the neighborhood V of zero. It is known that if the function $f_1 + \varepsilon f_2$ is C^1 then it is sufficient to check that $\det(D(f_1 + \varepsilon f_2(a_\varepsilon))) \neq 0$ in order to have that $d_B(f_1 + \varepsilon f_2, V, 0) \neq 0$, for more details see [8].

On the other hand if one of the real parts of the eigenvalues of the Jacobian matrix $D(f_1+\varepsilon f_2)(a_\varepsilon)$ is positive the periodic solution $x(t;\varepsilon)$ is unstable. If all the real parts of the eigenvalues of this matrix are negative the periodic solution is locally stable. For a proof see Theorem 11.6 of [10].

For a general information on the averaging theory see for instance the books [9, 10].

3. Proof of Theorem 1

First we scale the variables (x, y, z, w) doing the change of variables $(x, y, z, w) = (\varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon W)$, second we pass to cylindrical coordinates doing $(X, Y, Z, W) = (\rho\cos\theta, \rho\sin\theta, \eta, \xi)$, and third we take the angle θ as the new independent variable. Thus in the variables (ρ, η, ξ) system (1) writes

$$
\begin{aligned}
\frac{d\rho}{d\theta} &= \varepsilon F_{11}(\theta,\rho,\eta,\xi) + \varepsilon^2 F_{21}(\theta,\rho,\eta,\xi) + O(\varepsilon^3), \\
\frac{d\eta}{d\theta} &= \varepsilon F_{12}(\theta,\rho,\eta,\xi) + \varepsilon^2 F_{22}(\theta,\rho,\eta,\xi) + O(\varepsilon^3), \\
\frac{d\xi}{d\theta} &= \varepsilon F_{13}(\theta,\rho,\eta,\xi) + \varepsilon^2 F_{23}(\theta,\rho,\eta,\xi) + O(\varepsilon^3).
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
F_{11}(\theta,\rho,\eta,\xi) &= \frac{a_1\rho}{b},\\
F_{12}(\theta,\rho,\eta,\xi) &= \frac{c_1\eta}{b},\\
F_{13}(\theta,\rho,\eta,\xi) &= \frac{d_1\xi}{b},\\
F_{21}(\theta,\rho,\eta,\xi) &= \frac{1}{b^2}(-b_1\cos(\theta)^2-b_1\sin(\theta)^2)(a_1\rho\cos(\theta)^2+a_1\rho\sin(\theta)^2)+\frac{1}{b}((a_{016}\eta^3+\\
&\quad a_{017}\eta^2\xi+a_{018}\eta\xi^2+a_{019}\xi^3)\cos(\theta)+(a_2\rho+a_{07}\eta^2\rho+a_{08}\eta\xi\rho+a_{09}\xi^2\rho\\
&\quad)\cos(\theta)^2+(a_{02}\eta\rho^2+a_{03}\xi\rho^2)\cos(\theta)^3+a_{00}\rho^3\cos(\theta)^4+(b_{016}\eta^3+b_{017}\\
&\quad \eta^2\xi+b_{018}\eta\xi^2+b_{019}\xi^3)\sin(\theta)+((a_{013}+b_{07})\eta^2\rho+(a_{014}+b_{08})\eta\xi\rho+(\\
&\quad a_{015}+b_{09})\xi^2\rho)\cos(\theta)\sin(\theta)+((a_{05}+b_{02})\eta\rho^2+(a_{06}+b_{03})\xi\rho^2)\cos(\theta)^2\\
&\quad \sin(\theta)+(a_{01}+b_{00})\rho^3\cos(\theta)^3\sin(\theta)+(a_2\rho+b_{013}\eta^2\rho+b_{014}\eta\xi\rho+b_{015}\\
&\quad \xi^2\rho)\sin(\theta)^2+((a_{011}+b_{05})\eta\rho^2+(a_{012}+b_{06})\xi\rho^2)\cos(\theta)\sin(\theta)^2+(a_{04}\\
&\quad +b_{01})\rho^3\cos(\theta)^2\sin(\theta)^2+(b_{011}\eta\rho^2+b_{012}\xi\rho^2)\sin(\theta)^3+(a_{010}+b_{04})\rho^3\\
&\quad \cos(\theta)\sin(\theta)^3+b_{010}\rho^3\sin(\theta)^4),\\
F_{22}(\theta,\rho,\eta,\xi) &= \frac{1}{b^2}(bc_2\eta+bc_{016}\eta^3+bc_{017}\eta^2\xi+bc_{018}\eta\xi^2+bc_{019}\xi^3+b(c_{07}\eta^2\rho+c_{08}\eta\xi\rho\\
&\quad +c_{09}\xi^2\rho)\cos(\theta)+(-b_1c_1\eta+bc_{02}\eta\rho^2+bc_{03}\xi\rho^2)\cos(\theta)^2+b(c_{013}\eta^2\rho+\\
&\quad c_{014}\eta\xi\rho+c_{015}\xi^2\rho)\sin(\theta)+bc_{00}\rho^3\cos(\theta)^3+bc_{01}\rho^3\cos(\theta)^2\sin(\theta)+b(\\
&\quad c_{05}\eta\rho^2+c_{06}\xi\rho^2)\sin(\theta)\cos(\theta)+(-b_1c_1\eta+bc_{011}\eta\rho^2+bc_{012}\xi\rho^2)\sin(\theta)^2\\
&\quad +bc_{04}\rho^3\cos(\theta)\sin(\theta)^2+bc_{010}\rho^3\sin(\theta)^3),\\
F_{23}(\theta,\rho,\eta,\xi) &= \frac{1}{b^2}(bd_2\eta+bd_{016}\eta^3+bd_{017}\eta^2\xi+bd_{018}\eta\xi^2+bd_{019}\xi^3+b(d_{07}\eta^2\rho+d_{08}\eta\xi\rho\\
&\quad +d_{09}\xi^2\rho)\cos(\theta)+(-b_1d_1\eta+bd_{02}\eta\rho^2+bd_{03}\xi\rho^2)\cos(\theta)^2+b(d_{013}\eta^2\rho+\\
&\quad d_{014}\eta\xi\rho+d_{015}\xi^2\rho)\sin(\theta)+bd_{00}\rho^3\cos(\theta)^3+bd_{01}\rho^3\cos(\theta)^2\sin(\theta)+b(\\
&\quad d_{05}\eta\rho^2+d_{06}\xi\rho^2)\sin(\theta)\cos(\theta)+(-b_1d_1\eta+bd_{011}\eta\rho^2+bd_{012}\xi\rho^2)\sin(\theta)^2\\
&\quad +bd_{04}\rho^3\cos(\theta)\sin(\theta)^2+bd_{010}\rho^3\sin(\theta)^3).
\end{aligned}
$$

System (4) is written into the normal form (2) for applying the averaging theory taking

$$
\begin{aligned}
x=z &= (\rho,\eta,\xi),\\
t &= \theta,\\
F_1(t,x) &= (F_{11}(\theta,\rho,\eta,\xi),F_{12}(\theta,\rho,\eta,\xi),F_{13}(\theta,\rho,\eta,\xi)),\\
F_2(t,x) &= (F_{21}(\theta,\rho,\eta,\xi),F_{22}(\theta,\rho,\eta,\xi),F_{23}(\theta,\rho,\eta,\xi)),\\
T &= 2\pi.
\end{aligned}
$$

.

From (3) we have that the first averaging function $f_1 = (f_{11}, f_{12}, f_{13})$ is

$$f_{1i}(\rho, \eta, \xi) = \frac{1}{2\pi} \int_0^{2\pi} F_{1i}(\theta, \rho, \eta, \xi) d\theta.$$

Doing these computations we get that

$$f_{11}(\rho, \eta, \xi) = \frac{a_1 \rho}{b}, \quad f_{12}(\rho, \eta, \xi) = \frac{c_1 \eta}{b}, \quad f_{13}(\rho, \eta, \xi) = \frac{d_1 \xi}{b}.$$

Since we look for solutions (ρ^*, η^*, ξ^*) of $f_1(\rho, \eta, \xi) = 0$ with $\rho^* > 0$, if $a_1 \neq 0$ the first averaging function does not provide any information on the periodic solutions of the differential system (3). In order that the second averaging function can give information on the periodic solutions of the differential system (3) the first averaging function must be identically zero. So we take $a_1 = c_1 = d_1 = 0$, and compute the second averaging function.

Then from (3) we have that $f_2 = (f_{21}, f_{22}, f_{23}) = (f_{21}(\rho, \eta, \xi), f_{22}(\rho, \eta, \xi), f_{23}(\rho, \eta, \xi)$ is given by

$$\begin{aligned} f_{21} &= \frac{\rho}{8b}(4(a_{07} + b_{013})\eta^2 + 4(a_{08} + b_{014})\eta\xi + (3a_{00} + a_{04} + b_{01} + 3b_{010})\rho^2 + \\ &\quad 4(a_{09} + b_{015})\xi^2 + 8a_2), \\ f_{22} &= \frac{1}{2b}(2c_{016}\eta^3 + 2c_2\eta + 2\xi(c_{017}\eta^2 + c_{018}\eta\xi + c_{019}\xi^2) + (c_{02} + c_{011})\eta\rho^2 + \\ &\quad (c_{03} + c_{012})\rho^2\xi), \\ f_{23} &= \frac{1}{2b}(2d_{016}\eta^3 + 2d_2\eta + 2\xi(d_{017}\eta^2 + d_{018}\eta\xi + d_{019}\xi^2) + (d_{02} + d_{011})\eta\rho^2 + \\ &\quad (d_{03} + d_{012})\rho^2\xi). \end{aligned} \tag{5}$$

We isolate ρ^2 from the equation$f_{21}(\rho, \eta, \xi) = 0$, and we substitute it in $f_{2i}(\rho, \eta, \xi) = 0$ for $i = 2, 3$. Then we get two polynomials $(g_{22}, g_{23}) = (g_{22}(\eta, \xi), g_{23}(\eta, \xi))$ given by

$$\begin{aligned} g_{22} &= \frac{1}{(3a_{00} + a_{04} + b_{01} + 3b_{010})b}(C_1\eta + C_2\xi + C_3\eta\xi^2 + C_4\eta^2\xi + C_5\eta^3 + C_6\xi^3) = 0, \\ g_{23} &= \frac{1}{(3a_{00} + a_{04} + b_{01} + 3b_{010})b}(D_1\eta + D_2\xi + D_3\eta\xi^2 + D_4\eta^2\xi + D_5\eta^3 + D_6\xi^3) = 0, \end{aligned}$$

where

$$\begin{aligned} C_1 &= -4a_2c_{02} - 4a_2c_{011} + 3a_{00}c_2 + a_{04}c_2 + b_{01}c_2 + 3b_{010}c_2, \\ C_2 &= -4a_2(c_{03} + c_{012}), \\ C_3 &= -2(a_{08}c_{03} + a_{08}c_{012} + a_{09}c_{02} + a_{09}c_{011} + b_{014}c_{03} + b_{014}c_{012} + b_{015}c_{02} + b_{015}c_{011}) + \\ &\qquad 3a_{00}c_{018} + a_{04}c_{018} + b_{01}c_{018} + 3b_{010}c_{018}, \\ C_4 &= -2(a_{07}c_{03} + a_{07}c_{012} + a_{08}c_{02} + a_{08}c_{011} + b_{013}c_{03} + b_{013}c_{012} + b_{014}c_{02} + b_{014}c_{011}) + \\ &\qquad 3a_{00}c_{017} + a_{04}c_{017} + b_{01}c_{017} + 3b_{010}c_{017}, \\ C_5 &= -2(c_{02} + c_{011})(a_{07} + b_{013}) + 3a_{00}c_{016} + a_{04}c_{016} + b_{01}c_{016} + 3b_{010}c_{016}, \end{aligned}$$

$C_6 = 3a_{00}c_{019} + a_{04}c_{019} - 2a_{09}c_{03} - 2a_{09}c_{012} + b_{01}c_{019} + 3b_{010}c_{019} - 2b_{015}c_{03} - 2b_{015}c_{012}$,
$D_1 = -4a_2(d_{02} + d_{011}) + 3a_{00}d_2 + a_{04}d_2 + b_{01}d_2 + 3b_{010}d_2$,
$D_2 = -4a_2(d_{03} + d_{012})$,
$D_3 = -2(a_{08}d_{03} + a_{08}d_{012} + a_{09}d_{02} + a_{09}d_{011} + b_{014}d_{03} + b_{014}d_{012} + b_{015}d_{02} + b_{015}d_{011}) + 3a_{00}d_{018} + a_{04}d_{018} + b_{01}d_{018} + 3b_{010}d_{018}$,
$D_4 = -2(a_{07}d_{03} + a_{07}d_{012} + a_{08}d_{02} + a_{08}d_{011} + b_{013}d_{03} + b_{013}d_{012} + b_{014}d_{02} + b_{014}d_{011}) + 3a_{00}d_{017} + a_{04}d_{017} + b_{01}d_{017} + 3b_{010}d_{017}$,
$D_5 = -2(d_{02} + d_{011})(a_{07} + b_{013}) + 3a_{00}d_{016} + a_{04}d_{016} + b_{01}d_{016} + 3b_{010}d_{016}$,
$D_6 = 3a_{00}d_{019} + a_{04}d_{019} - 2a_{09}d_{03} - 2a_{09}d_{012} + b_{01}d_{019} + 3b_{010}d_{019} - 2b_{015}d_{03} - 2b_{015}d_{012}$.

We suppose that $3a_{00} + a_{04} + b_{01} + 3b_{010} \neq 0$.

Looking only at the coefficients of system (1) which appear in C_j and D_j we see that C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4, D_5, D_6 are all independent because the rank of the Jacobian matrix of the functions C_1 , C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4, D_5, D_6 with respect to the variables $a_{00,}$ a_{04}, a_{07}, a_{08}, a_{09}, a_2, b_{01}, b_{010}, b_{013}, b_{014}, b_{015}, c_{02}, c_{03}, c_{011}, c_{012}, c_{016}, c_{017}, c_{018}, c_{019}, c_2, d_{02}, d_{03}, d_{011}, d_{012}, d_{016}, d_{017}, d_{018}, d_{019}, d_2 is 12, as it can be easily checked using maple or mathematica.

In short, since all coefficients of the polynomials $g_{22}(\eta, \xi) = 0$ and $g_{23}(\eta, \xi) = 0$ are independent they can be chosen arbitrary. By Bezout Theorem, we know that at most system $g_{22}(\eta, \xi) = 0, g_{23}(\eta, \xi) = 0$ has 9 solutions. We give an example of polynomial differential system with nine limit cycles

$$
\begin{aligned}
\dot{x} &= 3w^2x + xz^2 + a_{013}yz^2 + a_{016}z^3 + \frac{x\varepsilon^2}{2} - y, \\
\dot{y} &= 4w^2y + (-4)x^2y + 2yz^2 + \frac{y\varepsilon^2}{2} + x, \\
\dot{z} &= -4w^3 + wy^2 + \frac{21w^2z}{4} - \frac{3x^2z}{2} - \frac{3wz^2}{2} + 2z^3 + z\varepsilon^2, \\
\dot{w} &= -7w^2z + x^2z + y^2z - \frac{3wz^2}{4} - \frac{15z^3}{4} - \frac{z\varepsilon^2}{4}.
\end{aligned} \tag{6}
$$

From system (5) we have for system (6) that

$$
\begin{aligned}
f_{21}(\rho, \eta, \xi) &= \frac{1}{2}\rho(1 + 3\eta^2 + 7\xi^2 - \rho^2), \\
f_{22}(\rho, \eta, \xi) &= \frac{1}{4}(8\eta^3 - 6\eta^2\xi + \eta(4 + 21\xi^2 - 3\rho^2) + 2\xi(-8\xi^2 + \rho^2)), \\
f_{23}(\rho, \eta, \xi) &= -\frac{1}{4}\eta(1 + 15\eta^2 + 3\eta\xi + 28\xi^2 - 4\rho^2).
\end{aligned} \tag{7}
$$

Solving system (7) there are nine solutions $z_i = (\rho_i^*, \eta_i^{**}, \xi_i^*)$ with $\rho_i^* > 0$ for $i = 1, \ldots, 9$ given by

$$
\begin{aligned}
z_1 &= (2, -1, 0),\\
z_2 &= (2\sqrt{2}, 0, -1),\\
z_3 &= (1, 0, 0),\\
z_4 &= (2\sqrt{2}, 0, 1),\\
z_5 &= (2, 1, 0),\\
z_6 &= \left(\sqrt{\frac{1}{2}(55 - 13\sqrt{7})}, -\sqrt{3-\sqrt{7}}, \frac{1}{2}\left(-4\sqrt{3-\sqrt{7}} + (3-\sqrt{7})^{\frac{3}{2}}\right)\right),\\
z_7 &= \left(\sqrt{\frac{1}{2}(55 - 13\sqrt{7})}, \sqrt{3-\sqrt{7}}, \frac{1}{2}\left(4\sqrt{3-\sqrt{7}} - (3-\sqrt{7})^{\frac{3}{2}}\right)\right),\\
z_8 &= \left(\sqrt{\frac{1}{2}(55 + 13\sqrt{7})}, -\sqrt{3+\sqrt{7}}, \frac{1}{2}\left(-4\sqrt{3+\sqrt{7}} + (3+\sqrt{7})^{\frac{3}{2}}\right)\right),\\
z_9 &= \left(\sqrt{\frac{1}{2}(55 + 13\sqrt{7})}, \sqrt{3+\sqrt{7}}, \frac{1}{2}\left(4\sqrt{3+\sqrt{7}} - (3+\sqrt{7})^{\frac{3}{2}}\right)\right).
\end{aligned}
$$

Since the determinant

$$
\det\left(\left.\frac{\partial(f_{21}, f_{22}, f_{23})}{\partial(\rho, \eta, \xi)}\right|_{(\rho,\eta,\xi)=(\rho^*,\eta^*,\xi^*)}\right) \tag{8}
$$

for these nine solutions $z_i = (\rho_i^*, \eta_i^*, \xi_i^*)$ are $-\frac{9}{2}$, -6, $\frac{3}{8}$, -6, $-\frac{9}{2}$, $-\frac{9}{8}(-189 + 67\sqrt{7})$, $-\frac{9}{8}$, $(-189 + 67\sqrt{7})$, $\frac{9}{8}(189 + 67\sqrt{7})$, $\frac{9}{8}(189 + 67\sqrt{7})$ respectively, we obtain, using the averaging theory of second order (see Theorem 2), 9 periodic solutions $(\rho_i(\theta, \varepsilon), \eta_i(\theta, \varepsilon), \xi_i(\theta, \varepsilon))$ of system (4) such that $(\rho_i(0, \varepsilon), \eta_i(0, \varepsilon), \xi_i(0, \varepsilon)) = z_i$. In fact, these periodic soutions are limit cycles because the solutions z_i are isolated solutions of system (7) because the determinants (8) are non-zero.

Going back through the changes of variables these 9 limit cycles provide 9 limit cycles $(x_i(t, \varepsilon), y_i(t, \varepsilon), z_i(t, \varepsilon), w_i(t, \varepsilon)$ of the differential system (1) such that

$$
(x_i(0, \varepsilon), y_i(0, \varepsilon), z_i(0, \varepsilon), w_i(0, \varepsilon) = \varepsilon(\rho_i^* \cos(bt), \rho_i^* \sin(bt), \eta_i^*, \xi_i^*) + O(\varepsilon^2).
$$

Since these initial conditions then to the origin of $\mathbb{R}^4$ when $\varepsilon \to 0$, the corresponding 9 limit cycles tend to the zero-Hopf singular point localized at the origin of $\mathbb{R}^4$. In short, the differential system (1) can exhibit a zero-Hopf bifurcation at the origin of system (6) with at least 9 limit cycles. This completes the proof of Theorem 1.

Acknowledgements

This work is supported by the Ministerio de Economìa, Industria y competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajusts Universitaris i de Recerca grant 2017SGR1617, and the European project Dynamics-H2020-MSCA-RISE-2017-777911.

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